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Endurance Limited Lines of Approach for a Sprinting Submarine

Matthew R. MacLeod

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ABSTRACT

Limiting lines of approach (LLA) are the standard tactical construct for bounding the area from which a submarine that is slower than a ship or convoy can reach a firing position. This paper addresses two main limitations of LLA: they do not take into account that many conventional submarines are able to sprint faster than a ship for short periods of time, and they do not reflect that the submarine’s area of approach will in many cases be limited by endurance rather than speed. The paper introduces formulas for endurance limited lines of approach (ELLA) that bound the area based on both speed and endurance. It is shown that for a notional submarine whose endurance decreases worse than linearly with speed, the worst-case speed it can adopt varies with approach angle, and an efficient computational approach is developed to combine those ELLAs across a range of speeds to present an overall worst-case curve for a given adversary.

INTRODUCTION

A major concern for ships or convoys during a conflict is their potential vulnerability to attack by submarines. Although ships and their embarked aircraft can use sonar to search the water for submarines, this is resource intensive. As early as World War I, however, tacticians realized that the relatively slower speed of conventionally powered submarines limited the angles from which they could approach a ship or convoy, and therefore the area that needed to be actively searched for threats—the geometry of this problem is well known and incorporated into standard tactics and planning software. Nuclear-powered submarines can often travel faster than surface ships and have much greater submerged endurance, but their proliferation remains limited, and associated tactics are generally sensitive. This paper addresses what appears to be a gap in the literature with respect to the increasing ability of conventionally powered submarines to sprint at speeds faster than a ship for short but significant periods of time.

This issue was brought to the author’s attention via a draft tactical note submitted by a naval officer (Delcourt, 2017) to the Canadian Forces Maritime Warfare Centre, where the author was embedded as an operational
Figure 1. Traditional limiting lines of approach. Ship is located at point O at the center of the circular torpedo danger zone, with direction of advance indicated by the arrow.

Tacticians have understood for more than a century (Navy Department, 1918) that the area from which a submarine—or really any threat slower than a ship—can reach an attack position can be bounded by limiting lines whose angles are determined by a simple formula:

$$\omega = \sin^{-1} \frac{k}{s},$$

with $s$ being ship speed and $k$ the threat speed. When these lines are drawn at tangents to the torpedo danger zone (TDZ) around a ship, they are known as limiting lines of approach (LLA), which are depicted in Figure 1.

The tactical benefits provided by understanding and applying these lines are:

- A submarine within the lines can be denied the opportunity to attack the ship if it can change its direction of advance such that the submarine is moved outside of the LLA.
- Similarly, the submarine can be denied if the ship is able to speed up to narrow the angle between the lines, putting the submarine outside of them.
- The lines can be used to bound the area that is necessary for forward screening assets (e.g., helicopters or other ships) to actively search for a submerged threat.

Previous work has looked at extending the concept to cases where the threat is limited in time (Koopman, 1980), or where the threat is willing to pursue the threat only so far in a given direction (Funk and Dickinson, 1985). The author’s literature review did not uncover any existing work combining these two cases. Even relatively recent work continues to refer to standard LLA constructions (see, e.g., Unlu (2015), Zaman Khan (2017)). As noted in the introduction, this paper seeks to extend the concept of the LLA to consider cases where the adversary is able to trade off endurance for speed, and may be able to sprint faster than the ship for short distances. With respect to the severity of the trade-off, prior work has claimed that a diesel-electric submarine may
be able to maintain its top speed for as little as 15–30 minutes (Armo, 2000). Given this context, traditional LLA based on a typical cruising speed may be too optimistic, but a line extending out to infinity in front of the ship based on some higher speed will be far too pessimistic given the limited endurance.

Figure 2. Region of approach under time constraints for a circular danger zone, based on Figure 2-9 in (Koopman, 1980). Note: In each case, the TDZ is the smaller circle centered on the ship at O, whereas the isochron is the larger circle (or arc) centered at G. The ship is moving in the positive y direction (or up the page). (a) Case where the threat speed exceeds the ship speed. (b) Case where the threat speed equals the ship speed. The classical limiting line is a straight line perpendicular to the direction of ship advance, tangent to the rear of the danger zone. (c) Case where the threat speed is less than the ship speed. The classical limiting lines have angle $\sin^{-1}\frac{1}{2}$ to the direction of ship advance and are plotted tangent to the danger zone, meeting at point L.
This paper will not consider any specific real-world diesel-electric submarine. Given the algebraic and computational nature of the approach, the interested reader can insert real values. Both allies and potential adversaries closely guard their submarine capabilities, and their speed and endurance values tend to be quoted at a limited number of values, if at all. That said, it is clear that battery technology for nonnuclear submarines continues to improve, so it is likely that the capability of some submarines to sprint will improve as well. For instance, the latest models of the Japanese Soryu-class are set to improve their endurance by some unknown amount by replacing lead-acid batteries with lithium-ion (Axe, 2019). Unmanned underwater vehicles (UUVs) are also not constrained by the need for cabin space, or to provide oxygen for crew members, so can use a higher percentage of their volume for batteries and propulsion. It is difficult to find good estimates of the endurance of large UUVs in the open literature, but a report from 2005 lists a long distance UUV as being able to sprint up to 12 knots with unspecified endurance, while having an endurance of five hours at six knots (National Research Council, 2005).

The primary assumption necessary for this study is that the tactician can obtain a reasonable estimate of the speed versus endurance curve of a potential adversary submarine (a notional curve will be used here); in reality this may not be an easily fit function due to the many nonlinearities present in drag, power generation, and power conversion. For much of the paper we will assume a circular TDZ, but the final section will address how this limitation can be removed.

**Regions of Approach under Time Constraints**

Koopman’s classic *Search and Screening* (Koopman, 1980) contains a brief section on regions of approach under time constraints (chapter 2, section 3). The approach in this paper builds on Koopman’s constructions, as limited endurance can be closely related to a time constraint.

Koopman defined the concept of an isochron as a curve of constant approach time (this is different from the definition used in dynamical systems). For a threat with a hard time limit to accomplish its mission, the isochron corresponding to that time limit defines the worst-case area from which a threat can reach the ship’s TDZ (the area from which a torpedo can be launched at the ship with a reasonable chance of success). Following Koopman, for a submarine with speed $k$ and ship with speed $s$, for a circular danger zone the isochron is circular and has the following features:

- It surrounds the danger zone when $k > s$ (as in Figure 2(a)).
- It is tangent to it in the rear when $k = s$ (as in Figure 2(b)).
- It consists of a forward circular arc, sections of the traditional limiting lines, and an arc forming a section of the danger zone, when $k < s$ (as in Figure 2(c)).

The center of the circle (or circular arc) of the isochron is shifted forward from the current position of the ship by $sT$, where $T$ is the constant time constraint of the threat. This can be physically interpreted as follows: a threat currently on the isochron must plot a course directly toward where the ship will be at the end of time $T$, in order to just reach the boundary of the TDZ at time $T$.

For the first two cases, the isochron fully describes the time-limited region of approach; the limiting line in Figure 2(b) serving to demonstrate that a threat directly behind the ship can only reach the danger zone if it starts on its boundary. The most complicated case to consider is where the threat is slower than the ship, as in Figure 2(c); here, both the limiting lines and the isochron need to be considered. Where the threat starts directly behind the danger zone, it cannot catch the ship, and the rear arc of the danger zone between points $M'$ and $N'$ therefore forms part of the boundary of the region of approach. The classical limiting lines also form part of the boundary, specifically line segments $MM'$ and $NN'$. Finally, the arc of the isochron from $M$ to $N$ completes the boundary for approach angles in which the time constraint is more constraining than the speed-based limiting lines.

To adapt this construction to the problem of a sprinting, endurance limited threat, we replace $T$ with $D/K$, where $D$ is the endurance in distance of the threat at speed $K$. However, there are
several further extensions we must make to fully capture the problem; specifically, a submarine
has the option to stop and wait in front of the danger zone, and it will have trade-offs with respect
to endurance and speed. As we develop these extensions over the course of this paper, instead of
treating the time constraint as a constant $T$, we will treat it as a variable $t$ corresponding to a vari-
able speed $k$.

**LIMITING LINES OF TACTICAL APPROACH**

A naïve approach to considering the issue of threats that are faster than a ship is to consider
that they may approach from any angle. However, a threat with a small speed advantage may for
practical reasons not be able to chase a ship long enough to catch it, or may be unwilling to use its
full speed due to concerns of counterdetection. Canadian operational researchers extensively
explored versions of this problem in an earlier internal document (Funk and Dickinson, 1985).
Funk and Dickinson primarily considered a threat limited to only pursuing a certain distance in
the direction of ship travel, but which was not constrained with respect to travel perpendicular to
the ship’s direction; the general idea being that a threat vessel may be tactically constrained to op-
erate in a box of some limited length, rather than being limited by physical constraints—and that
this box would be oriented along the axis of ship advance. Analysis of this constraint resulted in
what they called limiting lines of tactical approach, which took the form of hyperbolæ, rather than
the classical wedge shape of LLA, as depicted in Figure 1. In their method, any threat ahead of a
parameterized hyperbola can reach the ship’s danger zone without leaving the area in which it is
constrained to remain.

Funk and Dickinson did acknowledge, however, that when a threat is physically rather than
tactically limited in its endurance, the distance $d$ in the direction traveled by the threat rather than
that of the ship would be the hard limit. They called this an alternate formulation, in which they also
assumed the threat could stop and wait indefinitely once it had reached a given position (i.e., if the
submarine could get into the path of the ship it can linger indefinitely without surfacing). If the
ship is considered as a single point rather than including a danger zone around it, this results in a
bound made up of the back half of a circle of radius $d$, which is then extended into a channel of
width $2d$ reaching indefinitely ahead of the ship. They rejected pursuing this formulation further,
as for their tactical situations of interest they felt it more appropriate to use the more conservative
hyperbolic limiting lines. As the sprinting submarines considered in this paper do have a physical
limit in endurance, it is more appropriate to build on their alternate formulation. The author uses
this formulation to extend the Koopman approach by considering the ability of the threat to stop
and loiter ahead of the ship, while maintaining Koopman’s formulation of the TDZ around the
ship. In the final section, a computational method will be presented for build limiting lines for
arbitrarily shaped danger zones through repeated application of a point-based formula, to avoid
being limited by regular geometric shapes.

**REGIONS OF APPROACH UNDER ENDURANCE CONSTRAINTS**

Combining the approaches of the two previous sections, it is relatively straightforward to con-
struct a formula for the half circle making up the rear of an isochron that considers endurance con-
straints. As in Koopman’s formulation in Figure 2, we have:

- $O$: the ship, at the origin $(0, 0)$.
- $G$: the center defining the isochron, at $(0, st)$ — the position of the ship at time $t$.
- $R$: the radius of the danger zone.
- $R + kt$: the radius of the isochron (recalling that the threat on the isochron must drive directly
  at $G$ to reach the boundary of the danger zone at time $t$).

Using the Pythagorean theorem to relate $x$ and $y$ along the rear half of the isochron circle, we
obtain:
The $y$ values in Equation (2) are negated as when the $y$ coordinate of a point on the circle is in the negative plane it should result in a positive distance, and when it is in the positive plane should result in a negative distance (i.e., when the $y$ coordinate is between that of $O$ (the origin) and $G$). Rearranging for $x$ and considering the extension into a channel ahead of the half circle, we have the following, as plotted in Figure 3:

$$x = \begin{cases} 
\pm \sqrt{(R + kt)^2 - (-y + st)^2}, & y < st \\
\pm(R + kt), & y \geq st.
\end{cases} \quad (3)$$

One can use Equation (3) directly to calculate and plot the region of approach endurance constraints for a given combination of threat speed and endurance, along with a given ship speed, given a circular TDZ centered on the ship. The author defines this curve as an *endurance limited line of approach* (ELLA). It should be noted that both the positive and negative values of the square root function are valid values for $x$ and should both be plotted to create the symmetric curve.

For practical purposes of plotting, it can be convenient to instead calculate $y$ in terms of $x$, particularly in cases where the slope of $y$ is very shallow (note that this construction is not a function, as $y$ will take on multiple values for some values of $x$; note the set notation in the second line of

![Figure 3. Approach region for an endurance limited threat.](image-url)
Rearranging Equations (2) for $y$ rather than $x$, and continuing the extension into a channel as in Figure 3, we have:

$$y = \begin{cases} 
-\sqrt{(R + kt)^2 - x^2} + st, |x| < R + kt \\
|st, \infty), |x| = R + kt.
\end{cases}$$

(4)

These equations are simplifications of the reality. By drawing a danger zone of equal distance in all directions, Equations (3) and (4) essentially assume that no matter the angle from which the torpedo is fired, the ship can instantly turn to the optimal evasion angle and commence its optimal evasive manoeuvre, which is not realistic. Adaptation to arbitrary danger zone shapes and sizes will be discussed in the Extensions section.

Finally, it should be noted that the previous constructions can only be applied directly if the threat is purely endurance limited. If the threat is approaching more slowly than the ship, the traditional LLA must be combined with the pure ELLA, as endurance or speed may be the limiting factor depending on the approach angle.

CONSIDERING VARYING ENDURANCE VALUES

Throughout the previous discussion, submarine speed and endurance have been treated as variables. Although it could be left to the tactician to calculate every possible set of values for their threat and plot the resulting curves, this is impractical. The initial proposal that motivated this work (Delcourt, 2017) took a further step by (a) fitting a well-known function to the relationship between endurance and speed for a particular vessel, (b) substituting that function into the equation for the limiting boundary, and (c) zeroing the derivative of the resulting equation to find the worst-case $x$ value over all speed—endurance combinations. The resulting formula was used to plot the worst-case boundary over all possible speeds.
Although this approach produces an elegant result, it presents several practical challenges. First, one must have a good set of estimates for the endurance of the threat at varying speeds. Second, the relationship between them must be amenable to fitting with a standard function. Lastly, someone with the requisite skills and knowledge must follow through on finding the derivative of the function and calculating the boundary in a way that can be imported into the tactical system. Inevitably, there will also be some degree of error in the fitted function, which is particularly problematic if the estimate of endurance is too low for certain speeds. The author instead recommends a computational approach, as presented in this section.

Important design targets in submarine hull design are to minimize required power for a given speed, or conversely to maximize speed for a given power (Bertram, 2013). There are many nonlinear interactions between drag and propulsive power. For illustrative purposes, the following simple nonlinear equation (see Equation (5) and Figure 4) is used to represent a notional submarine’s endurance, the key features being (a) that endurance decreases worse than linearly with increasing speed, and (b) the submarine has the ability to sprint faster than a ship traveling at 10–12 knots for a short but nonnegligible time:

$$t = \frac{2,000}{k^{2.5}}$$  \hspace{1cm} (5)

As noted in the Background section, reasonable estimates of speed and endurance across a broad range are hard to come by for operational submarines. For perspective, according to publicly available estimates, the Russian-produced Kilo-class diesel-electric submarine’s speed has
been given at two point values: 400 miles submerged at three knots (~130 hours) and 12.7 miles submerged at 21 knots (roughly 0.6 hours) (Federation of American Scientists, 2000); the previous formula gives slightly lower values. The author compared the results to apparently empirically derived values for an unnamed submarine in a Naval Postgraduate Studies thesis (Akbori, 2004), and the shape of the notional curve and a piece-wise connection of the empirical values are similar (see Figure 4).

Figure 5 shows combined ELLA and LLA curves for a set of notional submarine speeds from four to 20 knots, a ship speed of 10 knots, and a TDZ radius of five nautical miles (the discussion of how the LLA and ELLA are computationally combined to find the worst-case curve for a given speed will be deferred a few paragraphs). Given the proximity of the curves near the origin, a zoomed-in plot is provided in Figure 6. The curves for submarine speeds of four, eight, and 17 knots are highlighted to enable discussion of three types of cases, where the submarine proceeds significantly slower than the ship, where it approaches close to but below the speed of the ship, and where it sprints much faster than the ship.

The case where the submarine approaches the speed of the ship (eight knots), shows a transition around 50 nautical miles ahead of the ship from speed being the limiting factor (the straight line classical LLA portion), to endurance being the limiting factor (an arc length of the outer circle as defined in Figure 2(c), extended into a channel in front of the ship). As can be seen in the crossover of the plots for four knots and eight knots, the notional submarine can approach from further away at higher speeds at lower $y$ values, but as $y$ increases (further down range from the ship), the submarine is better off adopting lower and lower speeds to maximize endurance $kt$; for this to hold true in general, the endurance of the threat must decrease worse than linearly with speed. This also becomes unrealistic at very low speeds, as (a) endurance does not become infinite (and indeed, propulsion may become less efficient at low speeds), (b) the distance the submarine must
be ahead of the ship becomes unrealistic, and (c) the concept of resting in the channel assumes the submarine can idle indefinitely on zero battery charge.

For higher speeds, endurance is purely the limiting factor—the curve for 17 knots is highlighted in Figures 5 and 6 as a representative of this, and Equations (3) or (4) can be applied directly. At approach angles close to directly behind the ship, higher speeds become the most dangerous (indeed, to approach from behind the TDZ at all, the threat must be faster than the ship; it is only these curves that do not overlap at all with the rear of the TDZ), although they bunch together, making it difficult to visually determine the worst-case curve. That said, straightforward reasoning can be applied to the case where the threat is directly behind the ship (where $x = 0$); in the frame of reference of the ship, the submarine is closing at $(k - s)$, and has time $t$ to make progress, meaning it can close at most $(k - s)t$ nautical miles for a given combination of the three variables. By substituting into Equation (4), then taking the derivative and setting it to 0, one can locate the worst case for this example at $k = 50/3 ≈ 16.67$ knots for approximately $t = 1.76$ hours (see Figure 7). It can be seen in Figures 5 and 6 that there are several curves with narrower channels than that for 17 knots—these are the curves for 18, 19, and 20 knots. These speeds above the critical value therefore do not allow the conceptual submarine to threaten the ship from further away at any approach angle, and therefore will not form any part of the overall worst-case boundary; as in the previous paragraph, for this to hold true in general, the endurance of the threat must decrease worse than linearly with speed. It should also be noted that for an empirical table of estimates one will not be able to reliably calculate an exact worse-case value, but will have to interpolate.

The next step is to combine the curves into an overall worst-case curve. Although this could be done analytically in this artificial case where the speed versus endurance curve is generated by a known function (as in Delcourt (2017)), that will not generally be possible (note the irregular nature of the empirical curve from Akbori (2004) in Figure 4). Instead, the author recommends computationally combining the curves using a convex hull algorithm (see, e.g., Chan (1996)), which efficiently identifies the vertices of the smallest polygon that would enclose the entire set of points. The author uses this same approach to combine the LLA and ELLA for each speed to create a single worst-case curve for a given speed, considering both the speed and endurance limitation. The
Figure 8. Overall worst-case curves of approach under endurance constraints, at three different levels of granularity for the submarine speed: (a) overall plot, (b) zoomed-in segment. Note: The zoomed-in area in (b) is indicated by the dashed box in (a).
precision of these operations depends on the resolution at which points on each LLA or ELLA is calculated. Figure 8 shows plots of the worst-case curve for varying granularities of threat speed and \(x\) and \(y\) values. Notably the lower resolution plots are pessimistic, so the lower precision errs on the side of caution. The transition between worst-case speeds gets further apart at lower speeds, so a variable resolution could be used if computational complexity is a concern. In Figure 8(a), the difference between the curves is barely noticeable at lower ranges; a zoomed-in view showing a difference of a few nautical miles at longer range is shown in Figure 8(b). That said, realistically one will likely only have empirical estimates of adversary capabilities, so increasing the resolution may rely on interpolation, with its own potential imprecision. For completeness, the author would note a concave hull algorithm (see, e.g., Moreira and Santos (2007)) could be used to more tightly bound the lines, but in addition to potentially introducing risk by excluding areas from which a submarine at a fractional speed could approach the ship, these algorithms have varying parameters and are not uniquely defined. The author therefore rejected the concave hull approach in favor of the convex hull approach.

This example demonstrated plotting a completely general worst-case region of approach for even a single ship speed presents a challenge. It reinforced the motivating observation from Delcourt (2017) that there is no single worst-case speed that the submarine can adopt, but rather that it is approach angle dependent. This suggests an important limitation of classical LLA. Given that calculating the set of points on each individual curve is now trivial from a computational perspective, as is combining them into a worst-case curve, it is now possible to move beyond those limitations. The computational approach has the advantage of not relying on any structure in the speed versus endurance relationship; indeed, an arbitrary table based on the available intelligence on the threat could be fed to the algorithm. Such an approach also enables various extensions, such as considering different danger zone shapes, including those surrounding a convoy. Such an approach will be explored in the next section.

EXTENSIONS

Both Koopman (1980) and Funk and Dickinson (1985) considered only either a circular TDZ around the ship, or the ship as a single point. Neither is particular realistic. Work considering different or arbitrary convoy sizes exists (e.g., Cooper and Hughes (1964) and Bertsche et al. (2001)), but generally with straight LLA simply being drawn tangent to a given danger region plotted around a hand- or computer-drawn convoy shape. The convex hull approach allows one to
directly compute a combined worst-case approach curve for arbitrary convoy and associated danger zone shapes. The key building block for such an approach is to calculate the relevant ELLA (or combined LLA and ELLA) for a single point, which can be done by setting \( R = 0 \) in Equations (3) and (4). One could rewrite these equations to be centered on an arbitrary \((x,y)\) pair, and then calculate the ELLA for each of these points on the outer boundary of the arbitrarily shaped danger zone. It is less computationally complicated to instead generate a single curve centered on the origin, and then create a copy for each \((x,y)\) pair with every value in that copy translated by that amount (i.e., if the points on the ELLA are stored in a vector of points, simply doing an element-wise addition to that vector). By then taking the convex hull over the entire set of translated ELLAs, one creates the ELLA for that arbitrary boundary shape. The author verified that taking this approach to create an ELLA using \((x,y)\) points from a circle of a given radius and taking the convex hull produced the same result as using Equations (3) with \( R \) set to the same radius, albeit with a small increase in computational time.

Similarly, to calculate the worst-case boundary over a set of speeds as well as an arbitrary danger zone shape, it is computationally efficient to first calculate a single worst-case boundary for the desired set of speed centered on the origin, and then create translated copies of that around the edge of the boundary, rather than recalculate the worst-case boundary for each individual point. Figure 9 shows the differences in overall worst-case boundary for a single ship centered on the origin with a circular danger zone and a square danger zone, as well as for a convoy of three ships with a combined danger zone constructed from overlapping circles. Slight differences are noticeable, in that the “three ships” case has a more distant danger area directly to the rear of the convoy, but the corners of the square danger zone create a slightly larger boundary further off-axis.

The only limits to the danger zone shape that could be considered are the practical issues of data entry, or user interface design if this were adapted for use in a tactical decision aid. For instance, assumptions about a given torpedo and the convoy’s reaction may require a more oblong or ellipsoid shape for the TDZ. Differing evasion capabilities within the convoy may mean the danger zone is not symmetric. Working through each custom situation algebraically would be at best tedious, but are easily handled with the previous approach combining translated copies of a single ELLA or worst-case boundary with an efficient convex hull algorithm (the author used the chull function built into the R programming language on a standard laptop, and the most complicated cases at higher resolutions took a few seconds to calculate).

A further extension that could be considered is the issue of nonhomogeneous threats. More than five decades ago, researchers had already considered that a convoy may be simultaneously concerned with two or more classes of submarines (Cooper and Hughes, 1964, pp. 23-24), although their specific work was only fully declassified in 2010. Those authors approached the problem analytically and were able to generate some of their results with the aid of computer, but noted that their “model is too elementary to warrant any conclusions as to the optimal policy of convoy escort disposition. It does, however, demonstrate the utility of [risk] as a measure of effectiveness for the assignment of forces. . . . It remains to construct a more detailed model, retaining this attribute, but including the other variables which vitally affect the solution.” With the computational approach described in this paper, there is no reason why worst-case curves for multiple threat types cannot be easily combined via the convex hull method, which is particularly relevant if there is no one single dominant threat that can approach from the greatest range from every approach angle.

CONCLUSION

This paper has introduced the concept of an endurance limited line of approach (ELLA), which can be applied directly in cases where a threat is assumed to be able to sprint faster than the ship or convoy, or in combination with a traditional limiting line of approach (LLA) in cases where it is sprinting but remains slower than the ship. For a modest computational burden, it allows much
greater flexibility to combine multiple threat speeds and type than a purely analytical or geometric approach, and can handle arbitrarily shaped danger zones.

This work was primarily motivated by the diesel-electric submarine threat. Whereas in the example shown, the danger area does not greatly increase behind the ship, this may not apply in all real scenarios. This also must be considered in light of improving battery technology, and the potential for further developments of larger UUVs with sprint capabilities. Ultimately, it draws attention to the limitations of constructing and using LLA for only a single assumed threat speed, and provides the tactician additional considerations for which threat behaviors may be the most challenging.

AUTHOR STATEMENT

The author produced this work as an employee of the Government of Canada.

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REFERENCES


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<td>External Literature (P)</td>
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13. ABSTRACT/RÉSUMÉ (When available in the document, the French version of the abstract must be included here.)

Limiting lines of approach (LLA) are the standard tactical construct for bounding the area from which a submarine that is slower than a ship or convoy can reach a firing position. This paper addresses two main limitations of LLA: they do not take into account that many conventional submarines are able to sprint faster than a ship for short periods of time, and they do not reflect that the submarine’s area of approach will in many cases be limited by endurance rather than speed. The paper introduces formulas for endurance limited lines of approach (ELLA) that bound the area based on both speed and endurance. It is shown that for a notional submarine whose endurance decreases worse than linearly with speed, the worstcase speed it can adopt varies with approach angle, and an efficient computational approach is developed to combine those ELLAs across a range of speeds to present an overall worst-case curve for a given adversary.