Estimates of shape by eye, or, The little invariant that could

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Abstract

Five experiments were conducted to test the hypothesis that observers apprehend specific constancies under change in perspective. The constancies were projective properties of ellipses pictured to slant and tilt in depth. Observers were asked to reproduce the static upright view of a moving pair of ellipses, using a computer graphics display and interface. Projective invariants for pairs of conics were computed on the observers' productions. A few experimental conditions revealed near-perfect performance. When pairs of coplanar ellipses were viewed under dynamic transformation in perspective, then invariants calculated on the observers' productions were a match—in value on average—to the invariants of the transforming ellipse pairs. It is proposed that measures of projective properties afford a family of techniques that can be applied to gauge acuity for complex shapes in the study of visual form perception. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Objects in rigid motion do not look to change their shape, despite the effects of perspective. Yet sometimes under unusual conditions of foreshortening, an object's
shape may not be judged very accurately, or else an object in rigid motion may be mistaken for an object in non-rigid motion. Foreshortening and other geometric characteristics of light – i.e., the geometry of the medium by which we see – can affect the estimates of shape which we make with the help of vision. Sometimes observers have difficulty in separating the medium from the message in that respect. There are some properties of shape which are unaffected by the geometrical characteristics of the medium, that is, they are unaffected by the uniform propagation of light. The measure of those properties can be used to gauge how well observers judge the changing or unchanging shape of an object despite the effects of perspective. Such measures can be said to abstract from any transformation in shape that is imposed by the propagation of light. The properties in question are called projective invariants. The present article extends the use of these measures, to include the projective invariants of conics. Though descriptive tools for the measurement of projective invariants are well understood – not only for conics, but also for cubics, quadric surfaces and more general classes of shapes – these mathematical tools have not been applied in experiment to problems of visual shape constancy.

The experiments that follow will concentrate on the projective invariants of conics, but why study the projective invariants of conics? They may seem to represent an obscure detail of geometry – a detail not easily generalized or not readily applied to practical constructions. Yet beyond their applications in computer vision (Herman, 1989), the projective properties of conics find application in the industrial modelling of complex shape. They find interpretation in the geometry standards of curve and surface design, as in the design of aircraft fuselage and automobile bodies. "Conics may be pieced together to form a more complicated curve, whose shape is too complex to be captured by a single conic. Such composite curves are called conic splines" (Farin, 1995, p. 85). The projective invariants of conics have immediate application to the business of measuring and describing complex stimuli for purposes of the study of vision, when the geometry of the viewing situation is unspecified. "Projective geometry in computer vision is primarily applied to the study of uncalibrated cameras, for which the linear intrinsic parameters are unknown and the non-linear effects are assumed negligible" (Collins, 1997, p. 434). This is only one way to proceed, though; another is to apply a still more subtle mathematics (for instance, to assess the absolute invariants of projective differential geometry: Weiss, 1992; Mundy and Zisserman, 1992; Wileczynski, 1906). What follows is a first demonstration or any case, an effective demonstration of shape constancy for the projective properties of conics under change in perspective.

2. Experiment one

2.1. Performance under continuous change in perspective

On first consideration, the notion that a geometric quantity unchanged under all perspective transformations can be used to gauge the accuracy of visual form perception may seem implausible. There has been little to demonstrate that observers'
estimates are stable and reliable in terms of these measures (though a recent effort began with Cutting, 1986; see Niall, 1987, for a critical review of that effort). Although perceptual psychologists have continued to outline their intuitions on the topic of invariance, little has been done to measure projective invariants in the context of experiment (Foster and Wagemans, 1993, provide an exception). Perhaps a classical approach to geometry has been thought too rigorous. Or else it may have seemed implausible that observers compute the measures of quantitative invariants, which consideration may have sparked study of ‘qualitative invariants’ in the study of human form perception (Kukkonen et al., 1996). A classical and quantitative approach to the application of projective invariants to human vision has not yet been exploited to its potential. In this study we begin to assess accuracy of shape perception by measuring the projective invariants of conics.

2.2. Method

2.2.1. Observers

Seven students and employees of DCEIEM (the Defence and Civil Institute of Environmental Medicine, North York, Ont., Canada) were tested, of whom four were women and three were men. All had normal uncorrected acuity. None were aware of the specific purpose of the study, apart from the experimental instructions given. Each observer signed a document indicating informed consent in the study, as did all observers in the other experiments that will be reported. Observers were given a few dollars per diem allowance for their participation.

2.2.2. Display

The basic stimuli for the first two experiments consist of 18 pairs of ellipses. These differ in the eccentricity of the ellipses, and in the relative orientation of the major axis of one ellipse to the major axis of the other. The pairs of ellipses differ in the values of their projective invariants, as a consequence. The ellipses of each pair have a common centre, that is, a common midpoint between their foci. The 18 pairs of ellipses are formed in this way: three ellipses are chosen that differ in the ratio of the length of their minor axis \( b \) to the length of their major axis \( a \). That ratio is 5:1 for the first ellipse, 9:1 for the second ellipse, and 9:5 for the third ellipse.

Each of these ellipses is paired with another, with the restriction that all pairs should be distinct under rigid rotations of the pair. Six pairs are made of the three ellipses, as expressed in six pairs of ratios: (5:1,5:1), (5:1,9:1), (5:1,9:5), (9:1,9:1), (9:1,9:5), (9:5,9:5). The major axis of one of these ellipses (the second of each pair) is rotated with respect to the major axis of the first. The difference between their major axes is either 15°, 45°, or 75° of angle. These three rotations, applied to the six distinct pairs of ellipses, produce 18 pairs of ellipses. The 18 pairs are displayed as Fig. 1. Each pair is characterised by two invariants (that differ by the order in which the ellipses of the pair are named).

Each day of testing consisted of many trials, that is, many pairs of shapes. A standard figure and a comparison figure are present on each trial. The standard figure is a pair of ellipses, seen under perspective in continuous motion. One of these
Fig. 1. The basic figures of the experiments are pairs of ellipses. In each pair the major axis of one ellipse makes an angle with the major axis of another ellipse. Six pairs of ellipses are formed at first: three ellipses that differ in eccentricity are used to form the pairs. None of these pairs can be rotated in the plane to coincide in shape with any other of the pairs. These pairs of ellipses have the same centre, in that the midpoint of their foci is in the same location. For the first group of six pairs, the angle between the major axes of the ellipses is 15°. The angle between the major axes of the ellipses is 45° for a second group of six pairs, and 75° for a third group of six pairs. These 18 pairs of shapes were used in all the experiments. (Other shapes were also included in Experiment two.)
Can such a sparse display be considered 'ecologically valid'? In Gibson's words (Gibson, 1979, p. 305): "The experimenter should not hope, and does not need, to display all the information in an ambient optic array. He is not trying to simulate reality...What is required is only that the essential invariant be isolated and set forth."

Fig. 2. A reproduction of the display gives some impression of the task given to observers. The left-hand side of the display shows a pair of ellipses which may be in motion; a textured surface lies beneath them, and appears to recede in perspective. The right-hand side of the display shows a static pair of ellipses whose dimensions may be changed by the observer. The observer uses a sliding scale to alter the eccentricity, size, and orientation of each of the two comparison ellipses in an independent manner. The observer may also change the positions of the ellipses on the right-hand side of the display.
2.2.3. Dependent measure

A projective invariant can be computed on pairs of ellipses from the numeric coefficients of the standard equation:

\[ Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0. \]

These coefficients are represented by the letters \( A \) to \( F \). Not all these coefficients will be different from zero, when ellipses are in question. There will be two such equations, one for each ellipse. (Call the first ellipse the one horizontal to the page in Fig. 1. This convention will be maintained throughout.)

\[
A_1x^2 + 2B_1xy + C_1y^2 + 2D_1x + 2E_1y + F_1 = 0, \\
A_2x^2 + 2B_2xy + C_2y^2 + 2D_2x + 2E_2y + F_2 = 0.
\]

The coefficients are arranged in two separate matrices.

\[
X = \begin{bmatrix} A_1 & B_1 & D_1 \\ B_1 & C_1 & E_1 \\ D_1 & E_1 & F_1 \end{bmatrix}, \quad Y = \begin{bmatrix} A_2 & B_2 & D_2 \\ B_2 & C_2 & E_2 \\ D_2 & E_2 & F_2 \end{bmatrix}
\]

Invariants can be formed from the determinants of these matrices, and their powers (including negative powers, like the simple inverse). Matrix operations can be performed on the two matrices that have been formed. The inverse of one matrix, the product of that inverse with the other matrix, and the trace of that product can be computed. Call these 'invariant one' and 'invariant two'.

\[
\text{Inv}_1 = \text{Trace} [X^{-1}Y], \quad \text{Inv}_2 = \text{Trace} [Y^{-1}X].
\]

These are two versions of the invariant in which we are interested. They are absolute projective invariants of pairs of conics; here we are interested in a subspecies of pairs of conics, namely pairs of ellipses. Such invariants are not found for individual conics, since any conic may be transformed into any other by an operation of projection. Heisterkamp and Bhattacharya (1997) show how the enumeration of invariants may be extended. They list independent absolute invariants of conics, from a pair of conics up to and including seven conics at once. Quan (1995) extends the computation of such invariants to pairs of non-coplanar conics in space.

The principal dependent measures of the experiments are the log ratio of invariant one, and the log ratio of invariant two. By 'log ratio', I mean the natural logarithm of the invariant as calculated on the observers' production (that is, on the comparison ellipses), minus the natural logarithm of the invariant as calculated on the standard ellipses. The logarithm of these invariants is a quantity with an elegant property: it has all the axiomatic properties of a distance measure. It makes sense to speak of adding, multiplying, or even dividing such quantities. The quantity defines an absolute scale, that is, it is a ratio scale measure with an absolute zero point. (The logarithmic function of cross ratio, that stands in direct relation to the projective invariants of conics, has been described in the psychological literature: eg. Julesz, 1971, pp. 291-292.)
2.2.4. Procedure

A software program was designed to run the experiments and to present the stimuli on a Personal IRIS graphics computer. (The software configuration was as follows: IRIX operating system version 4.0, GL graphics library, and X/Motif library.) A number of parameters characterize the motions of the standard figure and the background; the values of these parameters are unchanging in the course of this experiment. Their names are: Viewpoint distance, Viewpoint excursion, Viewpoint limit, Viewpoint interval, Tilt excursion, and Slant excursion. (Definitions of these terms and default values for their settings are listed in Appendix A.)

Eighteen standard pairs of ellipses were presented in continuous motion on each day of testing; these 18 pairs were all distinct configurations. Each ellipse pair was presented three times to each observer on each day; the order of presentation of ellipse pairs was randomized within these Blocks of trials. A standard pair and a static comparison pair were displayed on each trial in separate 'windows' of the graphics display (Fig. 2). To manipulate the comparison shapes, observers adjusted the tabs of six 'slider' scales by lateral movement (a 'click and drag' operation) of a mouse device. Movements along three of the scales transformed the red ellipse of the comparison pair, while movements along the other three scales transformed the blue ellipse of the comparison pair. A movement along one scale changed the size of the ellipse; a movement along a second scale changed the eccentricity of the ellipse; and movement along a third scale changed the orientation of the major axis of the ellipse about the centre of the ellipse. In addition, the two ellipses could be repositioned independently, using a click and drag operation of the mouse device. No fixed order of execution was specified for operations on the ellipses by the observer. The observer was asked to indicate with a key press when her or his adjustments to the comparison ellipses constituted a satisfactory match. After each trial observers were given opportunity to pause in the sequence of trials. Usually the entire session lasted less than ninety minutes. The time that was required to complete a sequence of trials changed from day to day, and decreased with practise.

Observers were asked to recover the non-foreshortened shape of the standard pair of ellipses. They were asked to reconstruct the pair in the picture plane as it would appear when seen face-on in static view: to reconstruct its actual fixed shape despite the variable distortion induced by the changing angle of the line of sight to the flat surface. They were given the following instructions:

Your task is to reconstruct the face-on view of the image presented on the left-hand side of the computer screen, as accurately as possible. You must show what this object would look like if you viewed it face onto the picture plane; that is, you should imagine how the screen image appeared in its original and undistorted form. Consider what the screen image would be like if the object were on the ground, and you were looking directly down at it; or else try to capture what the object would look like if it were rotated around so that it appeared in the picture plane. Construct the image as best you can, within the workspace on the right. You may use the mouse and slider switches to adjust the size, rotation, and 'stretch' of the image you are creating. You may also 'click and drag' the objects
on the right-hand side of the screen in order to adjust their placement within the workspace.

The values of two parameters are independent variables in this experiment: the names of these factors are *Tilt limit*, and *Update interval*. The standard pair of ellipses lies in a plane, which can tilt with respect to the picture plane (the $xy$ plane). A bound can be set on the degree of this tilt, to limit the degree of foreshortening that can be applied to the standard figure. In this experiment, the degree of *Tilt limit* takes the values 20°, 40°, 60°, or 80°. While the pair of ellipses slants and tilts in motion, it does not tilt beyond this depicted angle to the picture plane. The tilt of the ellipse pair changes in steps of five degrees; it changes five degrees in slant at the same time it changes five degrees in tilt. The direction of these changes is randomly assigned at each step. In other words their motion was ragged: the pair was always in motion, but the direction of this motion could reverse at any time.

These changes occur with fixed frequency. In other words, we may speak of an interval within which the tilt and slant of the pair are changed: this is the update interval. In this experiment the update interval takes values of 100, 200, 400, and 800 ms. A longer update interval indicates that the ellipse pair transforms or moves more slowly. The accuracy of the *Update interval* is limited by the refresh rate (60 Hz) of the graphics console screen; the update intervals can be considered to be composed of several refresh times for single frames, each of which lasts approximately 20 ms.

Each combination of *Tilt limit* and *Update interval* is presented on a different day of testing; since there are four levels of tilt limit and four levels of update interval, there were 16 days of testing for each observer. Combinations of *Tilt limit* and *Update interval* were presented in a different random order across days for each observer. Each ellipse pair was presented three times to each observer on each day, the order of presentation of ellipse pairs was randomized within these *Blocks* of trials. Eighteen distinct pairs of ellipses were presented in three *Blocks* of trials each day, over a period of 16 (not necessarily contiguous) *Days*, which means that 864 trials in all were presented to each observer.

### 2.3. Results

The results of this experiment may be presented in several ways to express one point: that observers reproduce at least one of the invariants very accurately. Let us begin by applying a simple linear regression to the results of the entire experiment. The estimated (ln) values of the magnitude of the invariant are plotted against the actual (In) values of the magnitude of the invariant (i.e., $y$ values against $x$ values), for all conditions and all observers of the experiment. Natural logarithms are computed for reasons of geometry, to establish the ‘distance’ metric, and not for reasons of statistics or psychophysics. The values of the invariants for the standard ellipses will be called the actual value of the invariants; the values of the invariants for the comparison figures under the observers' control will be called the estimated values of the invariants. The values of these invariants are usually normalized by setting the determinants of the matrices of coefficients to have a value of one: the present values were unnormalized. Results based on unnormalized values will de-
pend on the stability of the method used to generate the coefficients of the conics. Values of invariant one are plotted separately from values of invariant two. The scatterplot displays a total of 6048 observations on each invariant, comprised of 336 observations (from seven observers) at each of 18 different values of the invariant. Then consider a simple regression line for the (ln) estimated values of invariant one, versus the (ln) actual values of invariant one (y values versus x values). The regression coefficient for the best-fitting line by least-squares technique is 0.96, with a slope of 0.96 and an intercept of 0.08. Similarly for a simple regression for the (ln)

![Graph showing the relationship between ln invariant one and ln invariant two.](image)

Fig. 3. Mean values of observers' estimates of two invariants (y-axis) are plotted against the actual values of the invariants (x-axis), presented in experiment one. Both the observers' estimates and actual values have been log-transformed; this transformation reflects the comparison of invariants in geometry, and is not an alteration applied for statistical or psychophysical reasons. Mean values of estimates for invariant one are marked by filled circles; mean values of estimates for invariant two are marked by open circles. Error bars mark the variance of estimate, that is, the square of the standard deviation.
estimated values of invariant two, versus the (ln) actual values of invariant two: then
the regression coefficient is 0.91, with a slope of 0.95 and an intercept of 0.06 for the
regression line (see Fig. 3). In general, observers in this experiment reproduce the
invariants very accurately.

Then let us examine the data more closely, observer by observer. The results of
simple regression analyses are presented separately for the seven observers, and for
the two invariants (Table 1). Each regression equation is based on 864 observations
(there are no missing observations). Regression coefficients as well as the slopes and
intercepts of the best-fitting regression lines are shown. Clearly, individual observers
in this experiment reproduce both invariants with accuracy.

Several analyses of variance were performed on the results: two ANOVAs of the
same form were executed on different dependent measures (on the log ratio of in-
variant one – that is, the log of the estimated value of invariant one, minus the log of
the actual value of invariant one – and on the log ratio of invariant two). These
analyses were two-way factorial ANOVAs, with all measures repeated; the effects of

<table>
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<th>Table 1</th>
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<tr>
<td>Regression coefficients and equations are tabled for individuals in Experiment one.</td>
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<table>
<thead>
<tr>
<th>Observer one</th>
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<tr>
<td>Invariant one</td>
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<td>Invariant two</td>
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<tbody>
<tr>
<td>Invariant one</td>
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<td>Invariant two</td>
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<th>Observer three</th>
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<tbody>
<tr>
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<td>Invariant two</td>
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<tr>
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<td>Invariant two</td>
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<tr>
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<td>Invariant two</td>
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<th>Observer six</th>
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<tr>
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<td>Invariant two</td>
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<th>Observer seven</th>
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<tbody>
<tr>
<td>Invariant one</td>
</tr>
<tr>
<td>Invariant two</td>
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* The equations relate the log values of invariants for the standard figures (x values), to the log values of
invariants in the comparison figures which are adjusted by the observers in experiment (y values). Results
for invariant one and invariant two are expressed separately. Each equation is based on 864 observations.
In several instances the; data are well fit by a line of unit slope and zero intercept.
Table 2
Mean error in the reproduction of projective properties is tabulated by levels of tilt limit and update interval for Experiment one.  

<table>
<thead>
<tr>
<th>Tilt</th>
<th>20°</th>
<th>40°</th>
<th>60°</th>
<th>goo</th>
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<tbody>
<tr>
<td>Dependent measure: In (estimate of invariant one) – In (invariant one)</td>
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<tr>
<td>100 ms</td>
<td>-0.01</td>
<td>-0.03</td>
<td>0.00</td>
<td>0.04</td>
</tr>
<tr>
<td>200 ms</td>
<td>-0.01</td>
<td>-0.05</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td>400 ms</td>
<td>0.00</td>
<td>0.02</td>
<td>0.00</td>
<td>0.06</td>
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<tr>
<td>800 ms</td>
<td>0.02</td>
<td>0.01</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>Dependent measure: In (estimate of invariant two) – In (invariant two)</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 ms</td>
<td>-0.18</td>
<td>-0.20</td>
<td>-0.08</td>
<td>-0.07</td>
</tr>
<tr>
<td>200 ms</td>
<td>-0.10</td>
<td>-0.17</td>
<td>0.09</td>
<td>-0.09</td>
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<tr>
<td>400 ms</td>
<td>-0.09</td>
<td>-0.12</td>
<td>-0.17</td>
<td>-0.06</td>
</tr>
<tr>
<td>800 ms</td>
<td>-0.11</td>
<td>0.13</td>
<td>-0.15</td>
<td>-0.14</td>
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</table>

‘Tilt limit’ expresses the maximum pictured slant or tilt to the picture plane that a moving ellipse pair may be subjected to in the display, while ‘update interval’ expresses the period within which the ellipse pair changes its tilt and slant. Error is entered as the mean log ratio of invariant one in the first half of the table, and as the mean log ratio of invariant two in the second half of the table.

Tilt limit, Update interval, and their interaction were evaluated. No significant differences were found in any of these analyses. Of course, the log ratios of the invariants might not differ across conditions of the ANOVA, and yet the log ratios of the invariants could be consistently different from zero. The mean values of the invariants have been tabulated for the conditions of the experiment (Table 2). At least for invariant one, observers reproduce the precise value of the invariant on average: not so for invariant two. This bias in estimate of invariant two may not be evidence of a psychological effect: though the data are scaled by a distance measure, restrictions on the stimulus set may have led to the weighting of one sign or direction of error over another. The meaning of such a bias should be considered in terms of geometrical statistics, before deciding to impute the bias to the output of a so-called psychological mechanism. For the present, a bias in absolute value on invariant two will be described but left unexplained.

2.4. Discussion

Observers can discern a projective invariant, for these shapes under these conditions. Observers reproduce the absolute magnitude of invariant one, in a manner reminiscent of the psychophysical estimation of line length (or perhaps better: Norman et al., 1996, outline some vagaries of distance estimation). The invariants afford ‘perspective-free’ measures of shape. Do observers ‘resonate’ to these invariants? Not necessarily: for other shapes under other conditions, performance is less than ideal in the estimation of projective invariants (Niall, 1992; Niall and Macnamara, 1989, 1990). Is the perception of invariants a matter of human competence in vision? No; more likely this experiment shows accurate performance in the face of certain physical limiting conditions defined by the action of light. Such limiting
conditions or constraints are optical constraints, not psychological constraints. But one can say that under these conditions, the measure of interest can be used to show accurate performance. Other factors may intervene, of course: an inattentive, tired, or astigmatic observer may not always perform so well. Then what positive statement can be made of these results? We may say that a robust measure has been applied, and that observers' performance is precise according to one of these measures. I regard this as a conclusion largely drained of psychology: observers' performance has been shown to accord to a standard (a standard of ecological optics, one could say), leaving little room for psychological explanation. If the notion of 'direct perception' were less confused (Austin, 1962), one might claim that observers perceive this invariant directly. The more sensible conclusion is a simpler and milder one – observers can make judgments by eye that match for this invariant. This is a good thing: if observers never achieved matches on projective invariants under non-trivial conditions of perspective, then the usefulness of measuring those invariants in psychophysical experiments could be questioned.

3. Experiment two

3.1. An altered configuration of shapes

The speed and accuracy of perceptual tasks may depend on the symmetries of the shapes involved. 'Symmetry', like 'invariant', is a relative term: both terms are relative to groups of transformations (cf. the description given in Wagemans et al., 1994). There is one feature of the ellipse pairs that may be considered an important feature or symmetry. The ellipses have a common centre (the midpoint of their foci). The ellipses can be translated (that is, 'slid') against one another so that their centres no longer coincide. That is a geometric change which alters the values of projective invariants of the pair. Will accurate reproduction of the invariants depend upon the preservation of the feature, or will the invariants be reproduced accurately despite such an alteration?

3.2. Method

3.2.1. Observers

Six students and employees of DCIEM were tested, of whom two were women and four were men. All had normal uncorrected acuity; their mean age was 30 years. None were aware of the specific purpose of the study, apart from the experimental instructions given. Observers were given a few dollars per diem allowance for their participation.

3.2.2. Display

The basic stimuli for Experiment two consist of 36 pairs of ellipses. Half of these are the 18 pairs that were presented in Experiment one. The ellipses of those pairs have a common centre, that is, they have a common midpoint between their foci. The
18 new pairs of ellipses are formed from the old pairs by changing the relative positions of the ellipses in each pair, so that the ellipses no longer have a common centre, but are offset. The effect of this translation is depicted in Fig. 4. The relative translation changes the values of projective invariants that can be calculated on the pairs of ellipses. The dependent measures were the same as in Experiment one.

Fig. 4. Another 18 pairs of ellipses are formed for Experiment two, in which all 36 pairs are presented. These pairs differ from the previous pairs: now the ellipses are offset, not mutually centred. The midpoints of the foci of the two ellipses no longer coincide, that is to say, one of the ellipses has been translated or transposed. In other respects each one of these pairs corresponds to one of the 18 pairs of ellipses presented earlier.
3.2.3. Procedure

The instructions to subjects were the same as in Experiment one. Most of the display parameters were retained unaltered (as listed in Appendix A). Tilt limit and Update interval were not varied systematically in Experiment two; instead these parameters were set to their default values. One other display parameter was altered: Viewpoint distance was specified to be 35.0, from 30.0 before (this parameter is relative to the other dimensions of the depiction). Each of the 36 ellipse pairs was presented a number of times: twice each day, for ten Days of testing. Two Blocks of 36 trials were presented each day; order of presentation was randomized within blocks. Then each ellipse pair was viewed 20 times by each observer.

3.3. Results

Two analyses of variance were performed on the results: ANOVAs of similar form were executed for the log ratio of invariant one (as before, this is the log of the estimated value of the invariant, minus the log of the actual value of the invariant) and the log ratio of invariant two. These analyses were two-way factorial ANOVAs with all measures repeated: one factor was Centred versus Offset, which contrasts the ellipse pairs used in Experiment one with the offset versions of the same pairs. Another factor is Day, which represents the ten days of practise on the task. No significant differences were found among the conditions of these analyses, for either of the dependent measures. The overall degree of association between estimated values and actual values of the invariants is high. The overall regression coefficient for all observations on invariant one was 0.85; the regression equation was \( Y = 0.85X + 0.03, \ n = 4320 \). Similarly, the overall regression coefficient for all observations on invariant two was 0.68; the regression equation was \( Y = 0.76X + 0.21, \ n = 4320 \). Performance on invariant two was not as accurate as performance on invariant one, a finding which will be repeated later in these studies.

Regression analysis was applied to the results for each observer. Separate analyses will be reported for invariant one (as the log values of invariant one for the standard shapes: \( x \) values, versus the log values of invariant one for the comparison shapes: \( y \) values) and for invariant two. Regression coefficients are reported in Table 3, together with the slope and intercept of the fitted regression line. There were 720 observations for each regression equation. The degree of association between estimated values and actual values of the invariant is often very high, though this varies by observer. Observer one has poorer performance than the other observers. Those five observers reproduce both invariants with accuracy.

The difference in shape between centred pairs of ellipses and offset pairs does have another effect on estimates. This may be described informally as an effect of variability rather than as an effect of mean tendency: offset pairs give rise to a greater variability in estimate than do centred pairs. It is the first business of these analyses to reveal changes in central tendency where they exist. Concurrent changes in variability may be present, but specification of the magnitude of such changes requires a more massive effort in experiment than has been attempted here.
Table 3
Regression coefficients and equations are tabled for six observers in Experiment two

| Observer one | Invariant one | r = 0.60 | y = 0.5 I x + 0.33 |
|             | Invariant two | r = 0.21 | y = 0.17 x + 0.74 |

| Observer two | Invariant one | r = 0.90 | y = 1.01 x - 0.02 |
|             | Invariant two | r = 0.86 | y = 0.96 x + 0.02 |

| Observer three | Invariant one | r = 0.91 | y = 0.94 x + 0.08 |
|               | Invariant two | r = 0.90 | y = 0.99 x + 0.01 |

| Observer four | Invariant one | r = 0.94 | y = 1.04 x + 0.08 |
|              | Invariant two | r = 0.90 | y = 1.01 x + 0.04 |

| Observer five | Invariant one | r = 0.89 | y = 0.80 x + 0.18 |
|              | Invariant two | r = 0.78 | y = 0.63 x + 0.35 |

| Observer six  | Invariant one | r = 0.85 | y = 0.78 x + 0.10 |
|              | Invariant two | r = 0.77 | y = 0.82 x + 0.11 |

*The equations relate the log values of invariants for the standard figures (x values), to the log values of invariants in the comparison figures which are adjusted by the observers in experiment (y values). Results for invariant one and invariant two are expressed separately. Each equation is based on 720 observations.

3.4. Discussion

The observers in the experiment demonstrate a fair degree of shape constancy in projective terms, both with and without the preservation of a certain degree of symmetry. These observers achieve some degree of shape constancy through vision: that much is uncontroversial. What may be questioned is the description, or the best construal of shape constancy. The central tendency of estimate for centred pairs of ellipses was not found to differ significantly from the central tendency of the actual values of those invariants. This may bolster hope that a description by invariants can take precedence in the description of shape constancy, over more colloquial descriptions. What I am urging is a change in criteria, to say that constancy in vision is better quantified in these terms. Sameness of shape often connotes Euclidean properties such as distance, which is apt insofar as solid bodies retain their shape if their Euclidean properties are unchanged. Shape constancy is the ability of observers to discern sameness of shape under changing conditions of observation. The list of changing conditions usually includes direction of illumination, and perspective effects associated with changes in viewpoint. A sufficient condition for shape constancy is that observers discern such sameness in shape. Yet this condition seems too strong, given perennial anxieties about the perception of distance.
A better strategy may be to adopt a necessary condition: there are grounds for supposing shape constancy when sameness of shape is achieved in projective terms. Perhaps vision apprehends the projective invariants best of all, among properties of shape. The projective invariants touch the eyes, in a manner of speaking. These invariants – unlike some others – are available at the eyes because of an optical constraint in homogeneous media. Then some worries or concerns about scaling and psychophysics may be alleviated by the judicious choice of a measure (or family of measures) in the study of visual form.

4. Experiment three

4.1. Two sets of instructions with static shapes

At least two tasks can be assigned to observers of a flat-screen display that depicts figures tilted in depth. Observers may be asked to attend to properties of the picture, that is, to geometrical properties measurable on the display's surface. Else the observers may be asked to attend to the properties represented by the display, that is, to the three-dimensional geometry of figures that are represented in perspective projection. It is not clear which task ought to be easier or more natural. We may ask observers to perform both tasks. Under either of these tasks, observers may be able to reproduce the geometric properties common to a flat picture in perspective and the properties in solid geometry it may represent. Various measures of constancy assume that observers attain only partial performance on either of these tasks: individual differences in estimates of shape are thought to range between geometric properties measurable on the picture, and the geometric properties represented by the picture. One example of such a measure of constancy is the Thouless index (Thouless, 1931a,b; Brunswik, 1934, p. 61, footnote 1, claims precedence). One claim associated with those measures is that the measures are neutral with regard to the metric of shape constancy: in other words, that any measure of shape constancy may be subject to a 'regression to the phenomenal object'. Yet by their nature, projective invariants may be said to cut through changes in perspective, and consequently estimates of those invariants may not be affected by shifts in perspective, or by the change in instructions.

4.2. Method

4.2.1. (Observers

Five students and employees of DCIEM were tested; three were women and two were men. All had taken part in one of the previous studies.

4.2.2. Display

The same pairs of ellipses were used as in Experiment one (see Fig. 1). No motion was applied to the pair of ellipses, the checkered background, or the viewpoint of the observer, and the pair of ellipses cast no shadow on the plane of the background.
Three Blocks of 18 pairs were shown to each observer each day of testing. The constant depicted Tilt of the pairs (see the definition of Viewpoint tilt in Appendix A) varied from one day to another, but not within days. The Tilt of the plane of the ellipse pairs to the picture plane could be 0°, 12.5°, 25°, 37.5°, 50°, or 62.5°. A tilt of 0° implies that the standard ellipse pair was seen directly from above, that is, at a direction normal to the textured ground plane depicted on the left-hand side of the display.

4.2.3. Procedure

Observers were given two sets of instructions, and the order of instructions was varied across observers. Call the first set of instructions the 'Copy' condition. In this condition observers were asked to copy the shape of the ellipse pair exactly as it lay before them. They were told that their copies should coincide exactly in size and shape with the image on the screen:

Trace the image presented on the left-hand side of the computer screen, as accurately as possible. When tracing or copying the screen image, you will be copying the image of the shape exactly as it appears on the left-hand side of the computer screen, regardless of what it depicts. That is, you will be presented with two shapes on the left-hand screen. Your task is to reproduce, as accurately as possible, these shapes within the workspace on the right. Imagine that you are tracing out the screen image on the left within the workspace on the right, and the final image you create could overlay the screen image exactly. You may use the mouse and slider switches to adjust the size, rotation, and 'stretch' of the copy figures. You may also 'click and drag' the objects on the right-hand side of the screen in order to adjust their placement within the workspace so that they match the placement of the shapes on the left to which they correspond.

The other set of instructions, called the 'Recovery' condition, required observers to recover the original shape of the ellipse pair. That is, they were asked to reconstruct the shape in the original drawing as seen face-on; its actual shape despite the distortion induced by the angle of the line of sight to the flat surface. Those instructions have been cited in full already, as the instructions to observers for Experiment one. The Copy and Recovery conditions can be considered as a change in Task.

4.3. Results

Two analyses of variance (ANOVAs) were performed on the results: they had the same form, but were executed on different dependent measures (namely on the log ratio of invariant one, and the log ratio of invariant two). These analyses were two-way factorial ANOVAs, with all measures repeated. In these analyses the effects of Tilt, of Task, and of the interaction Tilt X Task were evaluated. The outcome of two of these analyses can be stated shortly: no significant differences were found for the
dependent measure of log ratio of invariant one, nor were there any for the log ratio of invariant two. The degree of association between estimated values and actual values of the invariant was high overall. The observers have reproduced the actual values of the invariants, in the main. The overall regression coefficient for all observations on invariant one (actual: $x$ values; estimated: $y$ values) was 0.86; the regression equation was $Y = 0.83X + 0.15$, $n = 3240$. Similarly, the overall regression coefficient for all observations on invariant two was 0.70; the regression equation was $Y = 0.78X + 0.13$, $n = 3240$.

Although there does seem to be some difference between the Copy task and the Recovery task, that difference may be explained in terms of the variability of estimates under the tasks. Observers' estimates in the Recovery task are appreciably more variable than estimates in the Copy task, and an increase in the variability of estimate will deflate the value of a regression coefficient. The variance of the log ratio of invariant one in the Recovery condition can be compared to the variance of the log ratio of invariant one in the Copy condition. This ratio is taken for each invariant and each observer separately. The ratios for invariant one are: (Observer one: 2.93; Obs. two: 0.97; Obs. three: 2.02; Obs. four: 19.45; Obs. five: 12.05; each ratio is based on 324 pairs of estimates.) The same comparison can be made for invariant two in the Recovery condition and invariant two in the Copy condition. (Observer one: 1.81; Obs. two: 0.60; Obs. three: 1.29; Obs. four: 13.06; Obs. five: 17.86; each ratio is based on 324 pairs of estimates.) The log ratios of the invariants are more variable for the Recovery condition than for the Copy condition, except where as for Observer two, the results evince near-perfect performance for both conditions. A similar change in variability seems to obtain between the conditions of tilt; Fig. 5 contrasts the small variability of error in the condition of 0° tilt ($n = 540$, includes data from all observers) with the appreciable variability of error in the condition of 62.5° tilt ($n = 540$).

The log ratio of invariant one ($x$ values) was plotted against the log ratio of invariant two ($y$ values), for all observations in the two conditions of tilt. In the condition of 0° tilt, a circle of 0.27 units in radius and centred on the origin will encompass 50% of the observations, while a circle of 1.08 units radius will encompass 95% of the observations. By contrast in the condition of 62.5° tilt, a circle of 0.77 units in radius and centred on the origin is needed to encompass 50% of the observations, while a circle of 3.99 units is needed to encompass 95% of the observations.

4.4. Discussion

James Gibson stressed the primacy of motion in his studies of visual form perception. Observers' accuracy of estimation is not always as great for static shapes as it is for shapes undergoing dynamic transformation in perspective, as in a casual comparison of the results of this experiment to those of Experiment one. Gibson may well have been right in his surprising prediction, that "The classical puzzles that arise with this kind of vision [surfaces seen with the head fixed and the array frozen] are resolved by recognizing that the invariants are weaker and the ambiguities stronger
Fig. 5. Two scatterplots serve to indicate that variability in error of estimate may change between conditions of Experiment three. Errors in observer's estimates of invariant one (log ratio of invariant one, x-axis of both graphs) are plotted against errors in observer's estimates of invariant two (log ratio of invariant two, y-axis of both graphs), for two different conditions. In the condition for which the variable of tilt was 0°, errors are generally small, and cluster about zero. In the condition for which the variable of tilt was 62.5°, there is a greater variability in these errors, though their central tendency is not significantly different. Small dots represent one observation by one observer; all 540 observations from each condition are represented.
when the point of observation is motionless" (Gibson, 1979, p. 303). His phrase "the invariants are weaker" may be glossed as "are estimated with greater variability". What could he have meant, to say that static form is not basic to the study of visual form perception? His allusion may mean that invariants are better perceived under dynamic changes in perspective than they are from a static viewpoint, in fixed perspective. Where measures related to distance and area will vary with the slant and tilt of a static form in perspective, the measures of the associated projective properties do not vary. Neither do the central tendencies of observers' estimates of those projective properties vary significantly, though one may well go on to make claims about changes in the variability of such estimates. In this experiment, estimates of shape that are made of objective shape despite perspective are more variable in projective terms than estimates that are made of perspective shape at the surface of a display.

5. Experiment four

5.1. Sustained practice by one observer

In the previous experiments, pairs of shapes were rotated about an observer's line of sight, which is to say that the slant and tilt of these pairs were varied in random steps. It could be argued that accurate performance in those experiments may not reflect the invariants themselves. Instead, observers' productions might be thought to represent a central tendency of shapes: observers might "take an average" of shapes over the span of rotations. The observers' line of sight to the background plane can be varied, in order to change the central tendency of the shape pairs without altering the value of the associated invariants. When the line of sight is varied, changes in the limit applied to the degree of slant and tilt a pair may have to the picture plane will change the central tendency of the pair of shapes, as well. In this way a new test can be made of the robust estimates observers make of the projective invariants of conics. The new test involves varying Viewpoint tilt (see Appendix A) at the same time as the Tilt limit of the standard ellipse pairs is varied.

A large number of shape pairs will be presented to one observer under varying degrees of tilt. In any extended test of performance, the effects of practice on performance may become evident. Performance comes to mirror ability given extended learning and practice, but how much is enough? One observer was asked to perform the experimental task once a day over 32 days, to examine changes in his performance due to practice without feedback.

5.2. Method

5.2.1. Observer

A graduate student in psychology (CS) participated in this experiment. Observer CS had normal uncorrected acuity; he did not take part in other experiments that are reported in this article.
5.2.2. Procedure

The experiment was conducted using the same software as before, upgraded to run on a Silicon Graphics Crimson graphics engine (IRIS 5.3 operating system). Some changes were made to the assignments of colours in the display; otherwise the program was unchanged. The standard figures appeared in continuous motion, wobbling in rotation about the common centre of the ellipses. The magnitude of their rotation changed in steps of five degrees every 100 ms independently for tilt and slant; the direction of this rotation was decided at each step for tilt and for slant. This rotation was constrained by an absolute limit on the tilt or slant that the plane of the pair could make to the picture plane. The limits imposed on tilt were either 20°, 40°, 60°, or 80°, while the viewpoint angle could be either 0°, 25°, 50°, or 75°. These conditions were randomized over 16 of the 32 days of testing. The standard figures appeared on the left side of the monitor, with a grey and green checkerboard pattern background that could recede in perspective, depending on the implicit viewpoint of the observer. The comparison figures appeared on the right side of the monitor, on a neutral grey background. The observer was asked to indicate with a key press when his adjustments to the comparison ellipses constituted a satisfactory match.

5.3. Results

Observer CS reproduced the invariants well; his performance improved over the first few days of testing, and it was stable across time. A regression analysis was performed to compare the estimated and the actual values of the log invariant. The overall regression coefficient for Observer CS on invariant one was 0.98; the regression equation was \( Y = 0.94X + 0.08, \ n = 1728 \). This represents a relation close to a one-to-one ratio for the two quantities. The regression coefficient for Observer CS on invariant two was 0.95; the regression equation was \( Y = 0.88X + 0.18 \).

Observer CS's performance improved over the first four days of testing, and was maintained at a near-ideal level thereafter (Fig. 6). The log ratios of the invariants might not differ across the conditions of an ANOVA; still the log ratios could be consistently different from zero. The mean values of the log ratio invariants have been tabled for the conditions of Tilt limit and Viewpoint angle for this observer (Table 4). Observer CS reproduced the precise value of invariant one on average, in all but two conditions. A two-tailed t-test was applied to single groups of observations from each condition, to determine if the mean of the observations differed from zero. This mean value differed from zero only for two conditions: a viewpoint angle of 75° combined with a tilt limit of 60°, and a viewpoint angle of 75° combined with a tilt limit of 80° (for Observer CS on invariant one). All of the mean values for Observer CS on invariant two differed from zero, by the same tests. (Asterisks mark differences at \( p < 0.001 \) in Table 4). Again, estimates of invariant one were found to be reliable and unbiased, to a closer tolerance than were estimates of invariant two.
Fig. 6. Mean error in estimate of invariant one is plotted for a series of trials in Experiment four. Each observation is plotted as a point. The line of moving average (n = 109; Kendall, 1976) is plotted together with these individual observations. Each day of testing represents 54 trials; Observer CS attained accurate performance early, and maintained this accurate performance. The analogous scatterplot for invariant two is presented separately; it shows much the same pattern.
Table 4
Mean error in the reproduction of projective properties is tabulated by levels of tilt limit and viewpoint angle for Experiment four \((n=108, \text{Observer CS})\).  

<table>
<thead>
<tr>
<th>View</th>
<th>Tilt limit</th>
<th>20°</th>
<th>40°</th>
<th>60°</th>
<th>80°</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>0°</td>
<td>25°</td>
<td>50°</td>
<td>75°</td>
</tr>
<tr>
<td></td>
<td>EstimatedLn invariant one, minus Ln invariant one</td>
<td>0.00</td>
<td>0.01</td>
<td>0.03</td>
<td>0.06</td>
</tr>
<tr>
<td></td>
<td>Estimated Ln invariant two, minus Ln invariant two</td>
<td>-0.15°</td>
<td>-0.10°</td>
<td>-0.17°</td>
<td>-0.19°</td>
</tr>
</tbody>
</table>

"Tilt limit" expresses the maximum pictured slant or tilt with respect to the background plane that a moving ellipse pair may be subjected to in the display. 'View' or Viewpoint tilt expresses the angle that the observer's line of sight makes to a direction perpendicular (i.e., normal) to that background plane.

5.4. Discussion

There is a definite pattern to the observer's responses, when those responses are quantified in terms of projective invariants. Though precise estimates of invariant one do take a little time to stabilize, accurate performance can be maintained over long periods. Observer CS's performance does not depart from ideal performance as a matter of a simple average of views; that observer failed to reproduce the value of invariant one only in extreme combinations of conditions of tilt limit and viewpoint angle. Though estimates of invariant one may be accurate for a variety of conditions, it may be helpful to know what the limits of this shape constancy may be. The next experiment pursues the investigation of these extreme conditions of observation, using a group rather than an individual. There may be some question how such departures from shape constancy can be explained in more familiar terms of geometry. Yet perhaps the assessment of visual performance is perspicuous and intelligible already in projective terms; what may be needed is some effort of education, as in Van Gool et al. (1994) and Weiss (1993), to make projective geometry more familiar.

6. Experiment five

6.1. Range of motion and an effect of viewpoint

A restricted range of tilt and a raking viewpoint were found to affect Observer CS's estimates of invariants in Experiment four. It is important to know whether those departures from constancy are idiosyncratic effects, or robust effects that can be demonstrated for a collection of observers. Here the performance of seven ob-
servers is examined under the same range of conditions for which departures from constancy have already been indicated.

6.2. Method

6.2.1. Observers

Seven employees of Armstrong Laboratory’s Warfighter Training Research Division (Arizona State University East Campus, Mesa, AZ) participated; of these five were men and two were women. Their ages ranged from 23 to 40; all had normal or corrected-to-normal vision. None were aware of the specific purpose of the study, and each was given a few dollars per diem allowance for participation.

6.2.2. Procedure

The experiment was conducted using the same software as before, run on a Silicon Graphics Crimson graphics engine. The orientation of the standard figure changed in steps of five degrees every 100 ms independently for tilt and slant. A new factor was introduced. The presence of a textured background, though detached from the shape pairs, may have allowed the dimensions of the pair to be estimated more readily. An alternate version of the software was prepared that eliminated the background plane in favour of a uniform grey colour. The observers' judgments could then be made with the background, or without a background. The dependent measures were the log ratio of invariant one, and the log ratio of invariant two.

The observers' task was to reconstruct pairs of ellipses on the model of a standard pair of ellipses that appeared on the computer monitor. The same set of 18 shapes were used, and the instructions given were those administered in Experiment one. In this experiment, two conditions of Viewpoint tilt were given: 0° and 75°, to determine whether the effect found in the previous study can be found for a number of different observers. The conditions of Tilt limit were varied, to 20° and 80°. All combinations of these factors of Viewpoint tilt, Tilt limit, and Background were presented in eight blocks of an experiment. The order of administration of the eight blocks was randomly assigned. Each of the 18 Shapes were presented under every one of these combinations of conditions, for a total of 144 trials per observer. The blocks of trials were divided across two sessions of testing, on two different days.

6.3. Results

The correlations between actual (ln) values of the invariant and estimated (ln) values of the invariant for each observer are high, but not so high as for the results of Experiment one (the values of regression coefficients range between 0.59 and 0.95 for different observers and the two invariants). Performance is not as accurate because a variety of extreme conditions have been presented.

Two analyses of variance were performed on the results: first, ANOVAs of similar form were executed for the log ratio of invariant one and the log ratio of invariant two. These analyses were four-way factorial ANOVAs with all measures repeated;
the effects of Background (present or absent), Tilt limit (20° or 80°), Viewpoint tilt (0° or 75°), Shape, and their interactions were evaluated. The first analysis concerns the log ratio of invariant one, where there was only one significant effect: that of Shape (F(17,102) = 8.07, p < 0.05). The second analysis concerns the log ratio of invariant two, where there were three significant effects: that of Shape, (F(17,102) = 8.96, p < 0.05), that of Viewpoint tilt, (F(1,6) = 6.68, p < 0.05), and that of an interaction of Background, Tilt limit, and Viewpoint tilt (F(1,6) = 7.12, p < 0.05). It could be argued that this interaction represents the effect of a stimulus artifact: though a background may have been present for some conditions of Viewpoint tilt, the background did not frame the ellipse pair when it was pictured at the larger (75°) viewpoint angle and the smaller Tilt limit.

6.4. Discussion

Though estimates of the projective invariants of conics may be precise and robust, they are not infallible. There is an effect of large variations in Viewpoint tilt on invariant two: an effect that is not idiosyncratic, but which persists in a small group of observers. This is a reminder of limits to shape constancy. Still, the effect of Viewpoint tilt that was found in Experiment four did not produce a significant difference in estimates of invariant one among this group of observers, under the same conditions as before. Instead a significant difference in their estimates of invariant one was found to depend on the factor of Shape. Departures from shape constancy can also be described in terms of shape itself; where departures from shape constancy occur, they may eventually be predicted by the measure of invariants associated with the stimulus shapes, rather than by the measure of the transformations due to perspective that may be applied to those shapes. Though estimates of a projective invariant of conics may be robust, facts about the perception of projective invariants are not necessary facts, nor are they the analytic consequence of common sense about vision. Rather they are contingent facts for investigation by experiment. "We seem again to be led to a point where the analogy between the invariants of perception and those of geometry disappears ... It is but empirical observation that can tell us in which domains of sense-perception there exist phenomena of constancy and how far their influence extends." (Cassirer, 1944, p. 26).

7. General discussion

Mais la vue, l’odorat, et le gout sont capables des memes progres scientifiques. Nos sens, distribues en autant d’etes pensans, pourroient done s’elever tous aux speculations les plus sublimes de l’arithmetique et de l’algebre; sonder les profondeurs de l’analyse; se proposer entre eux les problemes les plus compliques sur la nature des equations, et les resoudre comme s’ils estoient des Diophantes: c’est peut-etre ce que fait l’huître dans sa coquille (Condillac, 175411984, p. 279)
Visually guided estimation of the projective invariants of conics can be both precise and robust, though it is not infallible in either respect. Estimates of these invariants were found to be precise in Experiment one, where the rate of a stepwise rotation was varied, and the limiting magnitude of that rotation was also varied. Those estimates were precise both for the group and for individual observers on average; a small bias was found to be associated with one version of the invariant. In a second experiment, estimates of the invariants were found to be robust under a simple geometric operation, that consists of a relative translation of the ellipses in a pair. Though this operation increased the variability of estimates, estimates remained accurate on average. Observers in the third experiment were asked to perform two different tasks on static pairs of ellipses in perspective. When the observers were given instructions to reconstruct the shape of the ellipse pairs despite the effects of perspective, they produced estimates that were appreciably more variable than when the same observers were given instructions to copy the exact outline or profile of the ellipse pairs. An extreme raking viewpoint was found to affect estimates reliably for a single observer, in Experiment four, and variations in shape were found to affect estimates for a group of observers, in Experiment five. The effect of viewpoint was not replicated for invariant one with the group of observers.

The precision of the estimates of invariant one raises something of a problem. Niall (1992, p. 157) states that the main finding of his experiments is that "observers' estimates of projective properties are often biased, to a large fraction of cross ratio". Naturally projective measures — including the cross ratio of points on a line, the analogue of the cross ratio for points in a plane, and the projective invariants of conics — stand in more or less direct geometric relation. Two facts may help to reconcile these disparate findings. The first is that the magnitude of error in estimate is similar on average for the two sets of studies. And in the present studies, projective invariants are varied over a much greater range. The studies concur on a second point: the importance of shape for predicting estimates of invariants. Estimates of invariants are robust despite factors such as rate of rotation, extent of rotation, changes in orientation in one direction, and so on (as invariants one and two in the first experiment, and invariant one but not invariant two in the fifth experiment). That is, such estimates are largely robust under transformation of shapes in perspective. But proportionally large changes in estimates of invariants can occur across the values of those invariants, that is, with changes in shape. Biases in estimate and changes in variability of estimate are often functions of the value of the invariant, which emphasizes the place and importance that measurement of those invariants should have in the conduct of experiments on visual form perception.

The 'problems' of visual perception take on a different cast, given serious consideration of the proposition that projective invariants provide an appropriate family of measures for the study of visual form perception. I propose a projective stance in the study of visual form perception: stimulus materials should be considered in terms of, and measured for their projective properties (read: absolute projective invariants). Changes in distance, angle, size, or parallelism are then parcelled as transformations extraneous to projective properties; the projective properties themselves are regarded as basic. (It may be the case that these factors affect the estimation of
projective properties by observers: that is another matter.) Colloquially, this shifts the dividing line between what is to count as a transformation and what is to count as an invariant, for the purposes of standard descriptions of visual form perception: strictly Euclidean properties are then deemed to count as transformations, and strictly projective properties are invariants. This falls short of making projective geometry a competence theory for human vision, a research strategy I have criticized as "the projective thesis". Under the projective thesis, projective geometry is instrumental in explaining how humans see; under the projective stance, projective geometry is instrumental in describing how well individuals see. Projective geometry is not the necessary form of visual imagination, or the form of sensible intuition (as Kant once made Euclidean geometry the form of sensible intuition). But accustomed as we are to thinking of shape in terms of distances and angles, it may require an effort of imagination to adopt the projective stance, as one strategy for vision research.

An objection to adoption of the projective stance might run: "but that's not how things look!". How would things look if we considered that their projective properties were their only constant properties? In other words, how would things look, if we considered that at most their projective properties counted as invariants? Things would look no different than they ever have, when seen by daylight through air. One might be tempted to say: if solid things were subjected to all the transformations which count as projective transformations, then things should look non-rigid (or somehow wobbly). That conceptual picture narrows in on the transformations of a single object; it neglects or excludes the projective relations which hold among parts of the environment. Projective invariants may span separate solid objects, as distances can be measured between solid objects just as well as along the sides of one object. A projective transformation applied to one object in isolation may alter the invariants it shares among the other objects in the scene. Things do not look non-rigid or wobbly without changes in some of those shared invariants. As for global transformations of the scene in projective terms, this is what light achieves anyway. The global action of light defines the ambient geometric conditions of vision: the action of light in air provides a model for projective invariants and transformations. Those are the ordinary conditions under which we see. They are the conditions under which things 'look normal', and no process of reconstruction or restitution need be invoked to explain how things should look under normal conditions. We see by the projective medium of light: this is how we see if anything counts as explanation of "how we see", and it is a fact that does not stand in need of further explanation (that is, unless one's interests lie in empirical knowledge of geometry rather than in vision). The projective stance is not an hypothesis about the way things look; it does not alter the way things look. In vision research there are many ways to measure form; we are free to choose our conventions for better or for worse. The measurement of projective properties is especially well-suited to description of constancies under the action of light, i.e., to the medium of sight. The projective stance is simply an invitation to this better way of measurement.

Under the projective thesis, the visual perception of form is governed by geometrical rules that are psychological in nature, or otherwise hidden in the mind. By
contrast, under the *projective stance* study of the visual perception of form is regulated by geometrical rules of the propagation of light, specifically by criteria of projective geometry. Projective invariants can be estimated well by eye, under some circumstances. That is an empirical fact. We may learn to apply a better system of measurement in the study of vision, consonant with this fact. But a demonstration of shape constancy does not constitute an understanding of sight as an achievement. Talk of a 'natural geometry' in the visual system, or innate principles of kinematics only serves to stall such an understanding. Perhaps better application of geometry – and better understanding of the place of projective geometry in the study of vision – will prevent the theoretical misunderstanding of vision as a distance sense, and may encourage the better understanding of vision as a light sense, consonant with the development of an ecological optics.

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**Appendix A. Stimulus variables for the graphics display**

These variables describe the general conditions of the display: when a variable is not mentioned in the text of an experiment and a default value is listed, the variable is assumed to take on the default value.

*Background interval:* The frequency at which the background texture is redrawn, in milliseconds. (Default: once every 350 ms.)

*Slant excursion:* Magnitude of the change in the target's slant at each interval. The target slant refers to the rotation of the shape about the x-axis. (Default: 5.0 degrees.)

*Slant limit:* The target's maximum deviation from the horizontal, where slant rotation is about the x-axis. This must be greater than or equal to the slant increment. (Default: 20.0 degrees.)

*Tilt excursion:* The magnitude of the change in the target's tilt at each interval. The target tilt is the rotation of the shape about the z-axis. The tilt excursion is specified in degrees. This value must be non-negative (even though, for each interval, the change in tilt may be either positive, negative, or zero). (Default: 5.0 degrees.)

*Tilt limit:* The target's maximum deviation from the horizontal (tilt rotation is about the z-axis). Value must be higher than that of the tilt excursion. (Default: 20.0 degrees.)

*Update interval:* The interval at which the target changes its tilt and slant, in milliseconds. They change simultaneously, in tandem. (Default: every 1000 ms.)
**Viewpoint distance:** The fixed distance between the centre of the target display and the centre of the plane section containing the viewpoint. (The viewpoint drifts, but it is always bound within a plane defined by the variables of viewpoint tilt and viewpoint slant.)

**Viewpoint excursion:** The random drift of the observer's viewpoint within the viewing plane is also called the viewpoint excursion. The excursion value specifies how far the viewpoint travels from its present position at the expiration of a view interval. The angular direction in which this step is taken is random, between 0.0 and 360.0 for each interval. (Default: 2.0.)

**Viewpoint interval:** The period of time within which the observer's viewpoint changes, in milliseconds. (Default: 1900 ms.)

**Viewpoint limit:** The upper bound on the distance between the viewpoint and the viewpoint's initial position (at the onset of the trial). The limit is a radius defining a circular boundary around the initial position (at the onset of the trial). In the course of its drift, the viewpoint may not go beyond this boundary. (Default: 10.0.)

**Viewpoint slant:** The rotation, in degrees, of the viewing plane about the y-axis. A positive value rotates the viewing plane forward in depth, into the picture.

**Viewpoint tilt:** The rotation, in degrees, of the viewing plane about the y-axis. The viewing plane is a plane about which the viewpoint wanders. This plane is defined by the viewpoint distance, viewpoint tilt, and viewpoint slant. To understand the effect of tilt, imagine taking a plane parallel to the $xy$ plane (right-hand coordinates, $z$ is equal to the viewpoint distance), and swivelling the plane clockwise or counterclockwise about the $x$-axis. A positive value is a counterclockwise rotation, again in a right-hand coordinate system.

References

Thouless, R.H., 1931b. Phenomenal regression to the 'real' object. II. British Journal of Psychology 22 (1), 1-30.