Propagation of Uncertainty in Range Dependent Bellhop

*Real Bathymetry, Partial Correlation, Dominant Terms, n2-linear Comparison and Gaussian Width Choices*

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Abstract

The Bellhop Gaussian Beam solution for acoustic propagation in an underwater environment is a complicated function of the environmental parameters of sound speed, boundary reflectivity and water depth. One of the purposes of this paper is to extend the previous derivations for the derivative with respect to sound speed uncertainty of the Gaussian Beam solution contained in McCammon, DRDC Atlantic CR 2012-100, to treat range dependence in bathymetry and bottom losses. In addition, the statistical solution for the variance of the intensity is augmented by providing for partial correlation of the sound speed uncertainties in near surface layers. In all cases tested, it is shown that the Gaussian function is the dominant term in the TL variance, and the largest variances occur whenever the ray has its turning point very near a sound speed gradient change. The derivations for the derivatives were made assuming a c-linear sound speed profile segments, so this report also examines an n²-linear sound speed profile and finds that the variances from both profile forms are qualitatively the same. Finally, the effect of Gaussian width choice on TL variance is studied.

Résumé

La solution à faisceaux gaussiens de la propagation acoustique dans un milieu sous-marin du modèle Bellhop est une fonction compliquée des paramètres environnementaux que sont la vitesse du son, la réflexivité des limites et la profondeur de l’eau. Un des buts du présent article est d’étendre les calculs de la dérivée par rapport à l’incertitude de la vitesse du son de la solution à rayons gaussiens déjà faits dans le rapport CR 2012-100 de RDDC Atlantique (McCammon) afin de tenir compte de la variation de la profondeur et de la perte au fond en fonction de la distance. De plus, la solution statistique pour la variance de l’intensité est améliorée afin de tenir compte de la corrélation partielle des incertitudes de la vitesse du son dans les couches près de la surface. Dans tous les cas soumis à l’essai, on démontre que la fonction gaussienne est le terme dominant de la variance de la perte de transmission et que les plus grandes variances se produisent chaque fois que le rayon change de direction très près d’un changement de gradient de vitesse du son. Dans le précédent rapport, les dérivées ont été calculées en supposant un profil de vitesse du son dont les segments sont linéaires par rapport à c. Le présent rapport examine donc un profil de vitesses du son dont les segments sont linéaires par rapport à n², ce qui permet de constater que les variances des deux formes de profil sont qualitativement identiques. Finalement, l’effet de la largeur du faisceau gaussien sur la variance de la perte de transmission est étudié.
Executive summary

Propagation of Uncertainty in Range Dependent Bellhop
McCammon, D.F.; DRDC Atlantic CR 2013-082; Defence R&D Canada – Atlantic; September 2013.

Introduction

Bellhop is a widely-used underwater acoustic propagation model that DRDC has adapted for use in the Environmental Model Manager, a component of the System Test Bed, for both passive and active acoustic transmission loss and reverberation calculations. The model has also been incorporated into a tool for estimating the sensitivity of acoustic propagation to environmental uncertainty and variability. This was extended in some simplified cases to study the theoretical impact of environmental variability on active sonar, using the mathematics of the Gaussian Beam model that has been incorporated in Bellhop. Here, this work is extended to study the effects of a range-dependent environment and to examine the impact of partially correlated sound speed uncertainties.

Results

Several results were obtained in this study. The Gaussian Beam intensity derivative was derived including range dependent bathymetry and bottom losses, and it was found that obtaining an accurate estimate for the speeds surrounding the source depth will be of most benefit in reducing the uncertainty in Bellhop’s output. The differences in uncertainty transfer between correlated and uncorrelated sound speed profile variances were computed, and the dominant term in the contribution to acoustic intensity was determined, revealing a fundamental limitation to ray theory.

In Section 3, the mathematics used to transfer uncertainty in sound speed into uncertainty in transmission loss are amended to include partial correlation among sound speed uncertainties, such as might occur in near-surface points that are influenced by the atmospheric conditions. It is shown that the fully uncorrelated SSP variances, in which the derivatives sum as squares, provide the greatest variance in transmission loss, while the fully correlated SSP variances, in which the intensity derivatives sum coherently, provides almost zero transmission loss variance. This implies that transmission loss variance in the surface duct will be reduced by the partial correlation of the duct’s defining points. Finally, the effects of choices in sound speed profile interpolation techniques and in Gaussian beam widths were examined.

Significance

The Bellhop model is the model used in the System Test Bed and Pleiades for passive and active sonar performance predictions. This work extends the previous work done in allowing changes in Bellhop transmission loss to be directly calculated given knowledge of the underlying uncertainty in the environment, for range dependent environments. The sensitivity of the Gaussian Beam
solution to environmental inputs can now be more easily and accurately evaluated. The tactical applications of this result include the ability to define environmentally-driven smoothing of detection ranges and levels for signal excess and figure of merit calculations. This research can also be used to define the required accuracy of the environmental inputs given an acceptable level of error in transmission loss. This work also highlights some of the limitations of the propagation model.

**Future plans**

Further work may take place in several areas. Current limitations of the Bellhop model should be addressed, by investigating alternate forms for the jump condition that is the dominant term in the variance of the acoustic intensity, and by improving the treatment of the low frequency bottom interactions found in Bellhop. The application of this work to the development of a user tool computing sonar equation uncertainty is also ongoing. Computation of transmission loss variance provides a significantly more realistic transmission loss estimate. This variance, in conjunction with a noise variance, could be used in forming the statistical terms of the sonar equation, such as the probability of detection. Implementing this closed-form computation in the active and passive versions of BellhopDRDC would provide DRDC Atlantic with a unique modeling capability.
Sommaire

Propagation of Uncertainty in Range Dependent Bellhop

McCammon, D.F.; DRDC Atlantic CR 2013-082; R & D pour la défense Canada – Atlantic; septembre 2013.

Introduction

Le modèle Bellhop est un modèle de propagation acoustique sous-marine largement utilisé que RDDC a adapté en vue de l’employer dans le gestionnaire de modèles environnementaux, un élément du Banc d’essai de système, pour les calculs de perte de transmission et de réverbération en acoustique passive et active. Le modèle a également été intégré dans un outil servant à évaluer la sensibilité de la propagation acoustique à l’incertitude et à la variabilité du milieu. Cet outil a été étendu à certains cas simplifiés pour étudier l’effet théorique de la variabilité du milieu sur le sonar actif à l’aide de la théorie mathématique du modèle à faisceaux gaussiens qui a été intégré au modèle Bellhop. Ces travaux sont ici approfondis afin d’évaluer les effets d’un milieu qui varie en fonction de la distance et des incertitudes de la vitesse du son partiellement correlées.

Résultats

L’étude a permis d’obtenir plusieurs résultats. La dérivée de l’intensité du faisceau gaussien a été calculée en incluant une profondeur et une perte au fond qui varient en fonction de la distance. L’obtention d’une estimation précise des vitesses autour de la profondeur de la source s’est avérée être la meilleure façon de réduire l’incertitude des résultats du modèle Bellhop. On a calculé les différences de transfert d’incertitude entre les variances du profil de vitesse du son corrélées et non corrélées et déterminé le terme dominant de la contribution à l’intensité acoustique, ce qui révèle une limite fondamentale de la théorie des rayons.

Dans la section 3, les calculs mathématiques utilisés pour déterminer le transfert d’incertitude de la vitesse du son vers la perte de transmission sont modifiés pour tenir compte de la corrélation partielle des incertitudes de la vitesse du son, comme la corrélation qui peut se produire dans les points près de la surface qui sont influencés par les conditions atmosphériques. Il est montré que les variances complètement non corrélées, pour lesquelles on additionne le carré des dérivées, donnent la plus grande variance de la perte de transmission, tandis que les variances complètement corrélées, pour lesquelles on additionne les dérivées de l’intensité de façon cohérente, donnent une variance de la perte de transmission quasi nulle. Ce résultat indique que la variance de la perte de transmission dans le chenal de surface est réduite par la corrélation partielle des points de définition du chenal. Finalement, les effets du choix des techniques d’interpolation du profil des vitesses du son et de la largeur des faisceaux gaussiens ont été étudiés.
Portée

Le modèle Bellhop est le modèle qui sert à prédire les performances de sonar actif et passif dans le Banc d’essai de système et Pleiades. Ces travaux complètent les travaux précédents en permettant de calculer directement les changements de la perte de transmission du modèle à partir de l’incertitude sous-jacente du milieu pour les milieux qui varient en fonction de la distance. La sensibilité de la solution à faisceau gaussien aux données d’entrée environnementales peut maintenant être évaluée plus facilement et plus précisément. Les applications tactiques de ce résultat sont notamment de permettre de définir un lissage des distances de détection et des niveaux à partir des données du milieu pour les calculs d’excès de signal et de facteur de mérite. Ces travaux peuvent également servir à définir la précision nécessaire pour les données environnementales pour obtenir un certain niveau acceptable d’erreur de perte de transmission. Ces travaux font également ressortir certaines limites du modèle de propagation.

Recherches futures

Des travaux subséquents pourraient toucher plusieurs domaines. Les limites actuelles du modèle Bellhop devraient être étudiées en examinant d’autres formes pour la condition de saut qui constitue le terme dominant de la variance de l’intensité acoustique et en améliorant le traitement des interactions avec le fond en basses fréquences du modèle Bellhop. L’application de ces travaux pour mettre au point un outil utilisateur servant à calculer l’incertitude de l’équation du sonar est également en cours. Le calcul de la variance de la perte de transmission donne une estimation beaucoup plus réaliste de la perte de transmission. Utilisée avec la variance du bruit, cette variance pourrait servir à établir les termes statistiques de l’équation du sonar, comme la probabilité de détection. La mise en oeuvre de ce calcul analytique dans les versions active et passive de BellhopDRDC fournirait à RDDC Atlantique une capacité de modélisation unique.
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1. Introduction

1.1 Research Described in This Report

The Bellhop Gaussian Beam solution formulated by Porter and Bucker [1] for acoustic propagation in an underwater environment is a complicated function of the environmental parameters of sound speed, boundary reflectivity and water depth. The purpose of this paper is to extend the previous derivations for the derivative with respect to sound speed uncertainty of the Gaussian Beam solution contained in McCammon [2] to treat range dependence in bathymetry and bottom losses. The derivative of the reflection loss as a function of angle is included to provide the intensity derivative with angle dependent losses. Then various topics listed below are explored using this range dependent code.

In Section 2, the derivative of the intensity is expanded to accommodate range dependence in the bathymetry and bottom loss. The code is tested against 4 different bathymetries.

In Section 3, the statistical solution for the variance of the intensity is augmented by providing for partial correlation of the sound speed uncertainties in near surface layers.

In Section 4, the various terms in the derivative solution are compared and the dominant terms are identified. It is found that the Gaussian function is the dominant term, and the Gaussian width, $\sigma$, dominates whenever the ray has its turning point very near a sound speed gradient change.

The derivations in reference 2 were made assuming linear sound speed profile segments. In Section 5 of this report, an $n^2$ linear sound speed profile is also tested, and the intensity derivatives using the two different functional forms of the SSP are compared.

In Section 6, the effect of Gaussian width choice on output sensitivity is studied. And Section 7 provides a summary of the findings of these various research topics and a recommendation for additional topics of interest.

1.2 Review of Concept of Propagation of Uncertainty

The uncertainty in sound speed will be assumed to be random so that a relationship given by Papoulis [3] on the standard errors of functions of random variables can be used to obtain the variance of the acoustic intensity given the variances of the underlying uncertainties. The variance of the intensity, $\sigma_i^2$, is derived from the Bellhop solution for the acoustic intensity as a sum of intensities from propagated rays with Gaussian weights. Using the standard error relationship between the randomness x and each ray’s intensity $I_i$, $\sigma_i^2$ will be given by:

$$\sigma_i^2 = \left( \frac{dI_i}{dx} \right)^2 \bigg|_{x=0} \sigma_x^2 = \left( \frac{d \sum I_i}{dx} \right)^2 \bigg|_{x=0} \sigma_x^2 = \left( \sum \frac{dI_i}{dx} \right)^2 \bigg|_{x=0} \sigma_x^2$$ (1)
For the case of more than one underlying random variable, the variance of the intensity would be the sum of the contributions from each variable, and in Section 3, partial correlation between the different uncertainties is addressed. The transmission loss, (TL), in dB with this uncertainty will be

\[ TL_{\text{uncertain}} = -10 \times \log_{10}(I + \sigma_I) \]  

(2)

1.2.1 Bellhop Mathematics for Intensity

The mathematics governing the Bellhop Gaussian Beam solution are presented by Porter and Bucker [1] and also Jensen, Kuperman, Porter and Schmidt [4]. Quoting from [1], the Gaussian Beam tracing “begins with the integration of the usual ray equations to obtain the central ray of the beam. Beams are then constructed about the rays by integrating a pair of auxiliary equations, which govern the evolution of the beam in terms of the beam radius and curvature as a function of arc length. The resulting pressure field describes a beam, in that the field falls off in a Gaussian fashion as a function of normal distance from the central ray of the beam”. One of the results from reference 2 was the derivation of the derivative of the intensity with respect to the sound speed uncertainty in closed form.

The intensity at any point \((r,z)\) in the Bellhop implementation can be written as

\[ I(r, z) = \sum A \frac{c}{c_0} \exp\left(\frac{n \pi^2}{2 \sigma^2}\right) \frac{2^{nb} s^{ns}}{r_{bs} r_{s}^{2ns}} = \sum I_i \]  

(3)

constant \(A = \frac{\delta \cos \theta_0 e^{-2\alpha r}}{3.1425} \)

\[ \sigma = \begin{cases} \frac{q}{q_0} \\ \pi \lambda \\ 0.2s / \lambda \end{cases} \text{ depending on the size of } q/q_0 \]

The criterion for choosing \(\sigma\) is

\[ \sigma = \text{MAX}\left(\frac{q}{q_0}, \text{MIN}\left(\frac{\pi \lambda}{0.2s / \lambda}\right)\right). \]

In these intensity equations, \(A\) is a constant, independent of the ray number, which carries the cylindrical spreading and attenuation. The \(\theta\) subscript refers to the value at the source point at the start of the propagation. The variable \(r\) is the horizontal range in meters while \(\alpha\) is the water column attenuation. The letter \(c\) is the speed at the ray’s \((r,z)\) position. In the argument of the Gaussian term, \(n\) is the normal distance from the ray’s position to the receiver. The Gaussian width variable \(\sigma\) can take one of three values depending on the size of \(q/q_0\). Note that \(\sigma\) appears in both the denominator of the intensity and the denominator of the Gaussian argument, making it a critically important variable.
In the three choices for $\sigma$, the first choice is the ratio of $q/q_0$ where $q_0 = c_0/\delta$, $c_0$ the sound speed at the source and $\delta$ the angle increment between successive rays being traced. According to Dr. Porter, $q$ is the factor associated with the beam radius and curvature of the ray as it propagates. It is a ray-traced quantity that depends on the sound speed and gradients in each layer. Because there are large changes in $q$ from step to step, the variable is interpolated according to the distance the ray has traveled along the arc. The closed form solution for $q$ for a linear SSP is derived in reference 2 and can be expressed for the $j^{th}$ step as

$$q(j) = q(j - 1) + q_L \left\{ 1 + \sum_{k=1}^{j} q(k - 1) N_k \right\}$$

and $q_L = \frac{\sin \theta_j - \sin \theta_0}{\delta^2}$, $q(0) = 0$

Here, Snell’s constant, $D = \cos \theta/c$, and the linear constant gradient sections of the sound speed profile have the gradient $g$. The symbol $N$ is referred to by Porter and Bucker [1] as a jump condition. It provides a correction to the beam curvature whenever the ray has traversed a SSP layer or reflected from a boundary. The form of this jump condition will be discussed in detail in Section 2 when range dependence is added.

In the third choice for $\sigma$, $s$ represents the total arc length of travel of the ray, and in the second and third choices for $\sigma$, $\lambda$ is the positional wavelength $c/f$ with $f$ the frequency in Hertz. The terms in the reflection coefficients $\mathcal{R}$ contain the product of the boundary losses suffered on each reflection from surface or bottom, with $n_S$ and $n_B$ being the number of bounces.

### 1.2.2 Bellhop Intensity Derivative

An individual ray in the Bellhop intensity in Eq. 3 is differentiated with respect to the random sound speed perturbation $x$ as:

$$\frac{dl_i}{dx} = \left[ \frac{dc}{dx} - \frac{dc_0}{c_0} + \left( -n^2 \right) \left( \frac{2}{n} \frac{dn}{dx} - \frac{2}{\sigma} \frac{d\sigma}{dx} \right) - \frac{2}{\sigma} \frac{d\sigma}{dx} + \sum_{n_S} \frac{2}{\mathcal{R}_S} \frac{d\mathcal{R}_S}{dx} + \sum_{n_B} \frac{2}{\mathcal{R}_B} \frac{d\mathcal{R}_B}{dx} \right] l_i$$

where

$$\frac{d\sigma}{dx} = \left\{ \begin{array}{l} \frac{dq}{q} - \frac{dc_0}{c_0} \\ \frac{dc_0}{c_0} \\ \frac{ds}{s} - \frac{dc}{c} \end{array} \right\}$$

depending on $\sigma = \left\{ \begin{array}{l} q/q_0 \\ \pi \lambda \\ 0.2s/\lambda \end{array} \right\}$ and $\mathcal{R}$

The reflection coefficients are shown as a sum of terms over the number of bounces because each term will in general be different for each bounce given the change in angle associated with a non-flat bathymetry reflection.
2. Changes in Intensity Derivatives due to Range Dependence

2.1 Bathymetry variations

2.1.1 Bottom jump condition

The bathymetry variations affect the intensity by changing the ray propagation angle on reflection. In the Bellhop mathematics, this occurs within the subroutine REFLECT where the ray tangent is incremented by the subtraction of twice the bottom slope angle \( \alpha_B \).

The jump condition is obtained by complex phase matching at the interface and is evaluated on the bottom interface. The jump condition \( N_{\text{refl}} \) for the bottom is defined in Bellhop as

\[
N_{\text{refl}} = \frac{2 \rho}{\xi} \left\{ 2 \left( \vec{g} \cdot \vec{N} \right) - \frac{\rho}{\xi} \left( \vec{g} \cdot \vec{T} \right) \right\} \quad (6)
\]

The symbol \( g \) is the vector of range and depth gradients of the SSP at the bottom. The terms \( \rho \) and \( \xi \) represent the components tangential and normal to the boundary and they are given by the dot product of the incident ray tangent with the tangent and normal of the input bathymetry.

\[
\rho = \left( \vec{T} \cdot \vec{T}_B \right), \quad \xi = \left( \vec{T} \cdot \vec{N}_B \right)
\]

Let \( r = \) range of a bathymetry point and \( h = \) depth of the bottom at that range. Then the tangent and normal vectors of the bathymetry are

\[
\vec{T}_B = \frac{[r_2 - r_1, \ h_2 - h_1]}{\sqrt{(r_2 - r_1)^2 + (h_2 - h_1)^2}}, \quad \vec{N}_B = \frac{[h_1 - h_2, \ r_2 - r_1]}{\sqrt{(r_2 - r_1)^2 + (h_2 - h_1)^2}}
\]

or in terms of the bottom slope angle \( \alpha_B \),

\[
\vec{T}_B = [\cos \alpha_B, \sin \alpha_B] \quad \text{and} \quad \vec{N}_B = [-\sin \alpha_B, \cos \alpha_B].
\]

In Bellhop, the ray tangent and normal are defined in terms of the ray angle \( \theta \) and sound speed as,

\[
\vec{T} = \frac{[\cos \theta, \ \sin \theta]}{c} \quad \text{and} \quad \vec{N} = \frac{[-\sin \theta, \ \cos \theta]}{c}.
\]

Thus the terms \( \rho \) and \( \xi \) become the trigonometric functions of the difference in angles, \( \theta - \alpha_B \).
\[ \rho = \frac{\cos(\theta - \alpha_B)}{\sin(\theta - \alpha_B)} = \text{ctn}(\theta - \alpha_B) \]

and \( N_{\text{refl}} \) can be written as

\[ N_{\text{refl}} = 2 \frac{\text{ctn}(\theta - \alpha_B)}{c} \left\{ 2 \left( \mathbf{g} \cdot \mathbf{N} \right) - \text{ctn}(\theta - \alpha_B) \left( \mathbf{g} \cdot \mathbf{T} \right) \right\} \tag{7} \]

In this form, it is clear that the term is inversely proportional to the sine and sine\(^2\) of the angles involved, and therefore it will be infinite when \( \theta = \alpha_B \). The derivative of the bathymetry jump condition with respect to the uncertainty is inversely proportional to sine\(^3\) of the angles involved. It is

\[ \frac{dN_{\text{refl}}}{dx} = N_{\text{refl}} \left( -\frac{dc}{dx} \frac{d\theta}{dx} + \frac{\frac{d\theta}{dx}}{\sin^2(\theta - \alpha_B)} \right) \]

\[ + 2 \frac{\text{ctn}(\theta - \alpha_B)}{c} \left\{ 2 \left( \frac{d\mathbf{g}}{dx} \cdot \mathbf{N} \right) + 2 \left( \mathbf{g} \cdot \frac{d\mathbf{N}}{dx} \mathbf{T} \right) \right\} \frac{d\theta}{dx} \]

\[- \text{ctn}(\theta - \alpha_B) \left[ \left( \frac{d\mathbf{g}}{dx} \cdot \mathbf{T} \right) + \left( \mathbf{g} \cdot \frac{d\mathbf{T}}{dx} \right) \right] \] \tag{8}

The derivatives of the ray tangent and normal are given by

\[ \frac{d\mathbf{T}}{dx} = -\frac{dc}{dx} \mathbf{\tilde{T}} + \mathbf{\tilde{N}} \frac{d\theta}{dx} \quad \text{and} \quad \frac{d\mathbf{N}}{dx} = -\frac{dc}{dx} \mathbf{\tilde{N}} - \mathbf{\tilde{T}} \frac{d\theta}{dx} \]

After these jump conditions are computed, the ray tangent is incremented to include the bottom slope angle as \( \mathbf{\tilde{T}}_{j+1} = \mathbf{T}_j - 2\zeta \mathbf{\tilde{N}}_B \), which translates in terms of ray propagation angles as \( \theta_{j+1} = \theta_j - 2\alpha_B \). This is equivalent to a shift in ray incident angle over a flat bottom. That is, the non-flat bottom reflection causes one ray to be transformed into another. The range region of the ray between bottom reflections will be governed by the properties at the start of that region, therefore Snell’s Constant will change at the boundary interface, from

\[ D = \cos\theta/c \quad \text{to} \quad D = \cos(\theta - 2\alpha_B)/c \] \tag{9}

The derivative of \( D \) with respect to the uncertainty in any SSP layer will remain unchanged because the slope angle of the bottom is not a function of SSP uncertainty and the derivative of \( D \) does not depend on the starting angle of the ray.
In the following sub-sections, comparisons are shown between the computed intensity derivative and a numerically generated derivative. Also, the full field distribution of the intensity derivative is plotted.

### 2.1.2 Flat bathymetry intensity derivative comparisons

#### 2.1.2.1 Source at 30m

These examples show the log of the intensity derivative as a function of range for a flat bottom at 200m depth. The bottom is a hard layered sediment consisting of a 4 m layer with a speed of 1630 m/s, density ratio 2.0 and attenuation 0.157 dB/mkHz overlying a half space of speed 1880 m/s, density ratio 2.175 and attenuation 0.085 dB/mkHz. The single sound speed profile is generally upward refracting with a surface duct and a small subsurface channel about 125m down. In this example, the source is in the surface duct at 30m and the receiver is below the short negative gradient in the lower layer at 150m. The profile is shown in Figure 1. Each of the four defined SSP points is denoted X1-X4. This profile was chosen to provide structure to the acoustic field with both ducted and deep refracted rays. The number of rays being calculated are those internally assigned by Bellhop by default and range between 300 and 400 for all these examples.

![Figure 1. SSP with ducts defined by 4 points.](image)

In the level comparison figures that follow, the intensity derivative will be taken with respect to an uncertainty in one of these four points. The numerical derivatives are obtained by running Bellhop with the basic SSP and again with one of the points perturbed by 0.0001 m/s, then taking the difference in intensity divided by 0.0001.

Figure 2 displays comparisons between the analytically computed intensity derivatives and the numerically generated derivatives at a receiver depth of 150m for an uncertainty in each of the four points of the SSP respectively.
Figure 2. Flat bottom: Analytical and numerical comparison of intensity derivative with the perturbation in X1-X4 respectively.

The comparison in levels is reasonably good, and particularly good at the high intensity derivative points, which lends confidence to the derivations.

It is interesting to compare the positions in the water column where the highest intensity derivatives occur. In Figure 3, the frames show a full field calculation of the transmission loss (top left) and a TL slice (top, right), followed by four frames showing the intensity derivative for each of the four SSP points. The TL slice is computed for a receiver depth of 150m. The full field TL vs range frame (top left) shows rays caught in the surface duct and several oscillating pairs that break through the duct and reflect from the bottom.

The TL slice (top, right) includes the uncertainty in the TL associated with one standard deviation of the intensity as computed in Eq 1. The blue line represents the sum of errors associated with just the top two points in the SSP, \( \sigma_{x1} = \sigma_{x2} = 1 \) and \( \sigma_{x3} = \sigma_{x4} = 0 \), with simple unity weighting. The red line shows the sum of variances associated with just the lower two points in the SSP, \( \sigma_{x1} = \sigma_{x2} = 0 \) and \( \sigma_{x3} = \sigma_{x4} = 1 \). It can be seen that the uncertainties associated with the top two SSP points that surround the source are the largest contributor to the TL variance at this depth.
Figure 3. Flat bottom: Full field TL, TL slice, and dIdx for each SSP point X1-X4, S=30m.

Each of the four frames below the TL show the full field intensity derivative vs range for an uncertainty in each of the four SSP points, named in the plot title line. Frames 3 and 4 (middle left and right) show perturbations in the first two points which enclose the source, and these are seen to be strongly associated with the strongest propagating rays, that is, the greatest TL variance due to the SSP uncertainty is associated with the strongest rays. The lowest two frames, 5 and 6, show
the derivative with respect to uncertainty in the third and fourth SSP points respectively. Here, the perturbation does not affect the rays in the surface duct at all, as might be expected since they do not penetrate past the duct floor and into the perturbed region. In these two frames, the strongest variance is associated with the rays that reflected from the bottom.

2.1.2.2 Source at 150m.

For contrast, in Figure 4, the same computation is made with the source positioned on the lower side of the surface duct at 150m. In the TL frame (top, left), the strongest rays are shown trapped below the surface duct, except for one set that reflects from the surface at about 3km. The TL slice (top, right) is taken at a receiver depth of 30m. In this slice, as in the last case, the blue line represents the sum of variances associated with just the top two points in the SSP, \( \sigma_{x1} = \sigma_{x2} = 1 \) and \( \sigma_{x3} = \sigma_{x4} = 0 \). The red line shows the sum of variances associated with just the lower two points in the SSP, \( \sigma_{x1} = \sigma_{x2} = 0 \) and \( \sigma_{x3} = \sigma_{x4} = 1 \). It is found that in this case, the uncertainties associated with the red line, the lower two SSP points that surround the source, are almost identical to the contribution to the TL variance from the upper two SSP points that surround the receiver, that is, the red and blue lines nearly overly each other at this receiver depth.

Frame 3, the first intensity derivative frame (middle, left) shows that the uncertainty at the ocean surface point only affects the rays that propagated in the first layer because only that layer’s gradient was affected by the uncertainty. The uncertainty in the surface layer has very little effect on the deep oscillating rays as expected. Intensity derivative frames 4 (middle, right) and 5 (bottom, left) are for uncertainty in the second and third points in the SSP which form the top and axis of the channel. These positions of uncertainty do affect all the strong rays, almost equally. The final frame (bottom, right) for uncertainty in X4, the last SSP point below the bottom, shows that all the rays are affected because the source is located in the layer defined by this point, but not as strongly as the higher points at X3 or X2.
2.1.3 Peaked bathymetry intensity derivative comparisons

In this example, the SSP is the same as Figure 1, with the source at 30m. The 20km bathymetry features a symmetric up and down slope ranging from 350m to 250m and forming a peak at 10km, as shown in Figure 5. The two layer bottom is lossy with a 10m layer having a density
ratio of 1.41, speed of 1453 m/s and attenuation of 0.038 dB/mkHz overlying a halfspace with speed 1557 m/s, density ratio 1.73 and attenuation 0.156 dB/mkHz.

![Bathymetry](image)

**Figure 5.** Peak bathymetry with source and receiver depths marked for numerical comparisons.

For this test case, a comparison of 5 different numerical perturbations is provided to show that the numerical estimate can vary widely at different ranges. Figure 6 demonstrates this variation for perturbations at SSP positions X1 (left) and X4 (right). Using one sided differences versus central differences for the numerical estimate did not change this variation.

![Comparison of numerical estimates](image)

**Figure 6.** Comparison of numerical estimates using different numerical perturbation increments in SSP points X1 (left) and X4 (right) over the peak bathymetry.
Figure 7 compares the analytic result with the numerical estimate using 0.0001 for each SSP point X1-X4. It seems that deviations in the comparisons might be a function of the choice of perturbation for the numerically generated curve.

![Graphs showing comparison of intensity derivatives](image)

*Figure 7. Peak bottom: Analytical and numerical comparison of intensity derivative with the perturbation in X1-X4 respectively and the receiver at 150m.*

Figure 8 displays full field plot of TL, the TL slice for a receiver at 150m, and the intensity derivatives for each of the SSP points X1-X4 over the peak bathymetry. As in the flat bottom case, the intensity derivatives are strongest along the rays that oscillate in the layer enclosing the perturbations. Positions X1 and X2 produce uncertainty in the ducted rays, while positions X3 and X4 primarily affect the deep reflected and refracted rays.
Figure 8. Peak bottom: Full field TL, TL slice, and ddx for each SSP point X1-X4, S=30m.

The variance displayed on the TL slice plot, (top, right) is greatest from the sum of the intensity derivatives for the upper two SSP points, shown in blue. The sum of variances associated with the perturbations in the lower two SSP points, shown in red, is much less.
2.1.4 Rough bathymetry intensity derivative comparisons

In this example, the SSP is the same as Figure 1. The 50km bathymetry features a random collection of 12 points that describe a rough hump as shown in Figure 9. The bottom is a hard layered sediment consisting of a 4 m layer with a speed of 1630m/s, density ratio 2.0 and attenuation 0.157 dB/mkHz overlying a half space of speed 1880m/s, density ratio 2.175 and attenuation 0.085dB/mkHz. For the comparisons with numerically generated derivatives, a perturbation of 0.0001m/s was used at each sound speed point.

![Figure 9. Rough bathymetry.](image)

2.1.4.1 Source at 30m

Figure 10 displays the analytical results compared to the numerical estimates for a 30m source and a 150m receiver. As in the previous cases, there are discrepancies between these curves that might be caused by precision problems in the numerical estimates, but the curves seem to agree in strong intensity areas, therefore, for the purposes of finding physical insight in the behavior of the derivatives, these discrepancies are not significant.
Figure 10. Rough bottom: Analytical and numerical comparison of intensity derivative with the perturbation in X1-X4 respectively with the source at 30m and the receiver at 150m.

Figure 11 displays the full field distribution of the TL, the TL slice for a receiver at 150m, and the full field distribution of the intensity derivatives for the rough bottom case with a 30m source. In this longer range case, the TL frame (top, left) shows there are many cycles of trapped rays that gradually dissipate in strength. The TL slice (top, right), evaluated at the receiver depth of 150m, shows many repeated ray cycles and generally a high loss. In the first 20 km, the variances for the red and blue lines are about the same. Beyond the peak in the rough bathymetry at about 28 km, the uncertainty in the two upper layers sum to provide a much larger variance (blue) than those from the two lower SSP points (red).

The distributions of the intensity derivatives for each of the SSP points (frames 3-6) show the same characteristics as in the two previous cases. Rays propagating within the constant gradient depth region bounded by an uncertainty are most strongly affected by that uncertainty.

It is interesting to note that there is no dissipation in the strength of the surface ducted intensity derivatives with range (frames 3 and 4 for example), unlike the actual transmission loss.
Figure 11. Rough bottom: Full field TL, TL slice, and $d\ln{\text{dx}}$ for each SSP point X1-X4, $S=30m$. 
2.1.4.2 Source at 150m

Lowering the source to 150m, below the surface duct, provides a different case. The comparisons with numerically generated derivatives at a receiver depth of 30m are shown in Figure 12, and again reasonably good agreement is obtained, particularly at the strong intensity derivative ranges.

Figure 12. Rough bottom: Analytical and numerical comparison of intensity derivative with the perturbation in X1-X4 respectively with the source at 150m and the receiver at 30m.

Figure 13 displays the full field distribution of the TL, a TL slice at 30m, and the full field intensity derivatives for the rough bottom case with a 150m source. Again, it is noted that there is little dissipation in the strength of the channeled rays’ intensity derivatives with range (frame 5 for example), unlike the actual transmission loss. The TL slice shows that the sum of intensity derivatives from the lower two SSP points (red line) is generally providing a slightly larger variance than the sum of intensity derivatives from the upper two SSP points (blue line).
Figure 13. Rough bottom: Full field TL, TL slice, and dldx for each SSP point X1-X4, S=150m.
2.1.5 Slope Bathymetry Intensity Derivative Comparisons

2.1.5.1 Uniformly negative SSP gradient

This case contains an upward sloping bottom out to 20km with a simple downward refracting profile. The bottom loss is a 3-point table describing a hard bottom: 0° grazing with 0.95 pressure reflection coefficient, 20° grazing with a 0.70 pressure reflection coefficient and 90° grazing with 0.50 pressure reflection coefficient. Surface losses are given by a table using Eckart’s model for a 10kt windspeed. Losses are linearly interpolated between these points by Bellhop. The bathymetry rises from 250m at the start to 120m at 20km. Source is located at 30m. Figure 14 shows the bathymetry and the linear negative gradient SSP for this case.

![Bathymetry and SSP plots](image)

*Figure 14. Bathymetry (left) and SSP (right) for negative gradient, sloping bottom case.*

Choosing the receiver depth to be 10m, Figure 15 compares the analytic intensity derivatives with the numerically generated values. This case shows some of the poorest comparisons, for some unknown reason.
Figure 15. Sloping bottom, negative gradient: Analytical and numerical comparison of intensity derivative with the perturbation in $X_1$-$X_4$ respectively with the source at 30m and the receiver at 10m.

The full field distribution of the acoustic transmission loss, the TL slice at 10m, and the full field intensity derivatives are shown in Figure 16 for the negative gradient over the upward sloping bottom case. In this case, all sound is refracted downward toward the highly reflective bottom. The intensity derivatives for each of the points in the SSP seem to provide about the same amount of strength and are associated with the same group of rays. The TL slice taken at a receiver depth of 10m shows the uncertainty caused by the two upper SSP points (blue) is approximately the same as the uncertainty from the two lower SSP points (red).
Figure 16. Sloping bottom, negative gradient: Full field TL, TL slice, and \( d\text{ddl}x \) for each SSP point X1-X4, \( S=30 \text{m} \).
2.1.5.2 Uniformly Positive SSP Gradient

In these comparisons, the SSP is given as a uniformly positive gradient, shown in Figure 17, over the same upward sloping bottom shown in the left in Figure 14.

![Figure 17. Positive gradient for sloping bottom case.](image)

The analytical derivatives are compared to the numerically generated derivatives in Figure 18. As in the last sub-section, these comparisons are among the poorest and the reason is unknown.
Figure 18. Sloping bottom, positive gradient: Analytical and numerical comparison of intensity derivative with the perturbation in X1-X4 respectively with the source at 30m and the receiver at 10m.

Next Figure 19 displays the full field TL, a TL slice at 10m, and the full field intensity derivatives for this upslope, positive gradient case. The full field TL (top, left) shows that all the sound is refracted upward, but the addition of angle dependent boundary losses serves to selectively diminish many of the rays, leaving just the two main groups of shallowly refracted rays and a bottom bounce path. In this case, clearly the top two SSP points that enclose both the source and the receiver are contributing the largest majority of the TL variance, as shown in frame 2 (top, right) by the blue line, while uncertainties in the bottom two points play a much smaller role (red line). The full field frames of intensity derivatives show that the derivatives of the shallowly refracted rays do not diminish with range.
Figure 19. Sloping bottom, positive gradient: Full field TL, TL slice, and $\partial dL/\partial x$ for each SSP point X1-X4, $S=30m$. 
2.2 Bottom loss range variations

Presently in Bellhop, there is provision for changes in the bottom properties and losses with range. The algorithm simply changes loss quantities as required, with no interpolation between defined points. To accommodate this range dependence within the uncertainty calculation, it is only necessary to use computer bookkeeping in the same way as the Bellhop code. Thus, as the ray steps out in range, the bottom loss is changed whenever specified and this will likely change the value of the derivative of the loss coefficient with respect to angle at that range. It will be shown in Section 4 that the boundary loss contribution to the SSP uncertainty is far smaller than the contribution from the Gaussian, therefore this bottom loss range variation will probably not affect the total uncertainty predictions in any significant way.

2.3 Sound speed range variations

Bellhop provides an option for using a bilinear quadrilateral interpolation of SSP data if range dependent SSP are input. This technique computes the speed and gradients by finding the values in the same layer in each range segment and interpolating them in range within that segment. This technique will produce a horizontal and vertical sound speed gradient that will change with each ray step. In the closed form solution in reference 2, a constant gradient sound speed profile was utilized between input SSP points to derive all the relevant terms for the ray as it steps through the layer. To accommodate a non-constant gradient that changes with each step in an SSP layer, the derivations will have to be re-examined and extensive changes to the equations will be necessary, which are beyond the scope of this current tasking.
3. Partial Correlation of Sound Speed Layers

The formation of the variance of the intensity, $\sigma^2$, was expressed in Eq. 1 in the form that assumes the randomness in each SSP sound speed point is independent of the others. If instead, there is partial correlation between adjacent points in an SSP, then the expression must be extended. Partial correlation will usually occur in near surface layers because the atmosphere’s heating or cooling that determines the surface sound speed will also, to an increasingly lesser extent, affect the lower layers until the thermocline is reached. Kendall and Stuart [5] provide an equation for the standard errors of functions of random variables. The derivations in this report have assumed that each of the $m$ SSP layers may have some uncertainty associated with it and the acoustic intensity from the rays will be a function of all of the SSP points, and thus of all of the randomness, $I(x) = I(x_1, x_2, \ldots, x_m)$. Then from [5]

$$\text{var}(I(x)) = \sum_{i=1}^{m} \left(\frac{dI}{dx_i}\right)^2 \text{var}(x_i) + 2 \sum_{i=1}^{m} \sum_{j=i+1}^{m} \frac{dI}{dx_i} \frac{dI}{dx_j} \text{cov}(x_i, x_j)$$

(10)

where, by our notation, $\text{var}(I(x)) = \sigma_I^2$, $\text{var}(x_i) = \sigma_{x_i}^2$, and $\text{cov}(x_i, x_j) = \rho_{ij}\sigma_{x_i}\sigma_{x_j}$. The parameter in the covariance, $\rho_{ij}$ is the correlation coefficient between points $i$ and $j$. To give an explicit example of this equation, in the cases studied in this paper there were four SSP points defined ($m=4$). Therefore,

$$\sigma_I^2 = \left(\frac{dI}{dx_2}\right)^2 \sigma_{x_2}^2 + \left(\frac{dI}{dx_3}\right)^2 \sigma_{x_3}^2 + \left(\frac{dI}{dx_4}\right)^2 \sigma_{x_4}^2 + 2\frac{dI}{dx_2} \frac{dI}{dx_3} \rho_{12}\sigma_{x_1}\sigma_{x_2} + 2\frac{dI}{dx_2} \frac{dI}{dx_4} \rho_{13}\sigma_{x_1}\sigma_{x_3} + 2\frac{dI}{dx_3} \frac{dI}{dx_4} \rho_{23}\sigma_{x_2}\sigma_{x_3} + 2\frac{dI}{dx_2} \frac{dI}{dx_3} \rho_{24}\sigma_{x_2}\sigma_{x_4} + 2\frac{dI}{dx_3} \frac{dI}{dx_4} \rho_{34}\sigma_{x_3}\sigma_{x_4}$$

Figure 20 shows some evaluations of the error quantity $(1 + \frac{\sigma_I^2}{I})$ from the sloping bottom case in Section 2.1.5, changing the correlation coefficient (same value for all four points) while keeping the individual SSP point standard deviations $\sigma_{x_i}$ constant at 1.0 m/sec. The value of the correlation coefficient has the same large impact on the level of the error in either the upward or downward refraction cases over the sloping bottom.

In the case of fully correlated points, (black line on Figure 20), $\rho_{ij} = 1$, and Eq. 10 reduces to

$$\sigma_{I,\text{correlated}}^2 = \left(\sum \frac{dI}{dx_i} \sigma_{x_i}\right)^2$$

Within this sum, the intensity derivatives nearly cancel each other in sign and the result is an extremely low variance. This makes sense because with $\rho_{ij} = 1$ and $\sigma_{x_i} = 1$, the uncertainty is added to each sound speed point in the same direction and by the same amount, therefore the
entire SSP is merely shifted uniformly. The propagation within the SSP is governed by the gradients which will remain the same. Thus the field will remain the same and the derivative should be zero. The reason it is not quite zero in the examples shown is because the boundary losses are dependent on the actual speed at the boundary, so when the speed is shifted, the losses are slightly different.

In the case of fully uncorrelated points, (red line on Figure 20), \( \rho_{ij} = 0 \), and the derivatives sum as squares without regard to sign and therefore are the largest.

\[
\sigma_{I \text{uncorrelated}}^2 = \sum \left( \frac{dI}{dx_i} \sigma_{x_i} \right)^2
\]

Clearly partial correlations among the upper points in a surface duct will cause the total variance to fall between these two extremes. This behavior with correlation coefficient is found in all the other test cases examined in this report, and therefore will not be shown.

**Figure 20.** Example variances for different correlation coefficients among all points, sloping bottom downward refraction (left) and upward refraction (right).
4. Dominant Terms and Physical Insight

Using the test cases shown in Section 2.1, the seven separate terms in the intensity derivative in Eq. 5 are computed and compared to each other. It is found that in all cases tested here, at every range, for all combinations of source and receiver position, SSP, and for all uncertainty layers, the two terms that comprise the derivative of the Gaussian term $W = \exp\left(-\frac{n^2}{2\sigma^2}\right)/\sigma$ are dominant, as illustrated in Figure 21.

![Figure 21. Relative strength of variance terms for four representative cases. Blue squares represent the Gaussian derivative terms. Red stars represent the sound speed derivative terms and light blue stars are derivatives of the boundary losses.](image-url)
The two relevant terms in the derivative of the intensity that come from the Gaussian factor are

\[
\left(-\frac{n^2}{\sigma^2}\right) \frac{dn}{dx} + \left(\frac{n^2}{\sigma^2} - 1\right) \frac{d\sigma}{\sigma}
\]  

(11)

### 4.1 Normal distance to the receiver \( n \)

The symbol \( n \) in the Gaussian is the normal distance from the ray position to the receiver coordinates. Let \( \phi \) be a vector of differences between the positions of the ray at step \( j-1 \) and step \( j \):

\[
\phi = [(r_j - r_{j-1}), (z_j - z_{j-1})]
\]

and let \( \beta \) be a vector of differences between the receiver position and the ray at step \( j-1 \):

\[
\beta = [(r_{r_{curv}} - r_{j-1}), (z_{r_{curv}} - z_{j-1})]
\]

Then the ray unit normal is defined as

\[
n_{\text{ray}} = [-\phi(2), \phi(1)]/|\phi|
\]

and the normal distance \( n \) between the receiver and the ray is defined as

\[
n = \beta \cdot n_{\text{ray}} = -\beta(1)\phi(2) + \beta(2)\phi(1)
\]

\[
\sqrt{\phi(1)^2 + \phi(2)^2}
\]

(12)

Note that the ray unit normal can be written more simply as

\[
n_{\text{ray}} = [-\sin \theta, \cos \theta]
\]

and therefore \( n \) can be expressed as

\[
n = -(r_{r_{curv}} - r_{j-1}) \sin \theta + (z_{r_{curv}} - z_{j-1}) \cos \theta.
\]

The derivative of this normal distance \( n \) with respect to the uncertainty in SSP is given by

\[
\frac{dn}{dx} = \frac{dr}{dx_{j-1}} \phi(2) - \beta(1) \left(\frac{dz}{dx_j} - \frac{dz}{dx_{j-1}}\right) - \frac{dz}{dx_{j-1}} \phi(1) + \beta(2) \left(\frac{dr}{dx_j} - \frac{dr}{dx_{j-1}}\right)
\]

\[
- n \left(\frac{\phi(1) \left(\frac{dr}{dx_j} - \frac{dr}{dx_{j-1}}\right)}{\phi(1)^2 + \phi(2)^2} + \frac{\phi(2) \left(\frac{dz}{dx_j} - \frac{dz}{dx_{j-1}}\right)}{\phi(1)^2 + \phi(2)^2}\right)
\]

(13)

Or, using the simpler form for \( n \),

\[
\frac{dn}{dx} = \frac{dr}{dx_{j-1}} \sin \theta - (r_{r_{curv}} - r_{j-1}) \cos \theta \frac{dz}{dx_{j-1}} \cos \theta - (z_{r_{curv}} - z_{j-1}) \sin \theta \frac{dz}{dx}
\]

(14)
The relation for the change in ray propagation angle with respect to uncertainty is given by a relation derived from the arc length traveled for each range step. The derivative is incremented by the values from the previous step.

\[
\frac{d\theta}{dx} = \frac{d\theta_{j-1}}{dx} - (\theta_{j-1} - \theta) \left( \frac{dg}{dx} + \frac{dD}{dx} \right)
\]  

(15)

In equations 13 and 14, the derivatives of the ray range and depth with respect to the uncertainty \( x \) are computed from the linear constant gradient equations for range and depth of a ray. Using the symbol \( D \) for Snell’s constant, \( D = \cos \theta/c \), and \( g \) for the constant gradient of the SSP in the layer being traversed, \( g = \Delta c/\Delta z \), the horizontal range is \( r = r_{j-1} + \frac{\sin \theta_{j-1} - \sin \theta}{gD} \) and the depth traversed is \( z = z_{j-1} + \frac{\cos \theta - \cos \theta_{j-1}}{gD} \). The derivatives of these quantities with respect to the uncertainty \( x \) are

\[
\frac{dr}{dx} = \frac{dr}{dx_{j-1}} + \frac{\cos \theta_{j-1} \frac{d\theta_{j-1}}{dx}}{gD} - \left( \frac{\sin \theta_{j-1} - \sin \theta}{gD} \right) \left( \frac{dg}{dx} + \frac{dD}{dx} \right)
\]

\[
\frac{dz}{dx} = \frac{dz}{dx_{j-1}} + \frac{-\sin \theta \frac{d\theta}{dx} + \sin \theta_{j-1} \frac{d\theta_{j-1}}{dx}}{gD} - \left( \frac{\cos \theta - \cos \theta_{j-1}}{gD} \right) \left( \frac{dg}{dx} + \frac{dD}{dx} \right)
\]  

(16)

4.2 Gaussian width \( \sigma \)

The equations for the Gaussian width \( \sigma \) are given by Eq. 3 and the derivatives of \( \sigma \) are given in Eq. 5. The most important term in this set of choices is the ratio \( \sigma = q/q_0 \), with the quantity \( q_0 \) being a constant (the speed at the source divided by the ray angle increment). Suppressing the summation indices, the derivative of Eq. 4 for \( q \) at the \( j \)th step with respect to the SSP uncertainty \( x \) is

\[
\frac{dq(j)}{dx} = \frac{dq(j-1)}{dx} + \frac{dq_L}{dx} (1 + \sum q(j - 1) \, N) + q_L \sum \left( \frac{dq(j-1)}{dx} \, N + q(j - 1) \frac{dN}{dx} \right)
\]  

(17)

and the derivative of \( q_L \) is

\[
\frac{dq_L}{dx} = \frac{\cos \theta_{j-1} \frac{d\theta_{j-1}}{dx} - \cos \theta \frac{d\theta}{dx}}{gD^2} - q_L \left( \frac{dg}{dx} + \frac{2dD}{dx} \right)
\]
4.3 Jump conditions

The expressions for the jump conditions from the bathymetry are given in Eq. 7 and 8. Although the jump conditions for the surface reflection and SSP layer transitions are not affected by a range dependent bathymetry, it is useful to list them here as they show many similarities with the bottom jump condition.

For flat bottoms where \( \alpha_B = 0 \) the reflection jump condition and its derivative can be reduced to

\[
N_{\text{refl}} = -\frac{2D^2g}{\sin \theta} \tag{18}
\]

\[
\frac{dN_{\text{refl}}}{dx} = N_{\text{refl}} \left( \frac{\gamma D}{D} + \frac{dg}{g} - \frac{\cos \theta \frac{d\theta}{dx}}{\sin \theta} \right)
\]

and for surface reflection, this is also the form employed. It can easily be seen that these terms will be infinite at grazing incidence because of their denominator’s dependence on \( \sin \theta \).

For layer transitions, the jump condition changes slightly with \( \Delta g = g_2 - g_1 \) (the difference in the gradients of each layer) replacing \( 2g \) from the reflection case. For transitions, the jump condition can be reduced to

\[
N_{\text{Tran}} = -\frac{D^2 \Delta g}{\sin \theta} \tag{19}
\]

\[
\frac{dN_{\text{Tran}}}{dx} = N_{\text{Tran}} \left( \frac{\gamma D}{D} + \frac{\Delta g}{\Delta g} - \frac{\cos \theta \frac{d\theta}{dx}}{\sin \theta} \right)
\]

It can be seen that all these expressions for jump conditions have singularities where the angles become zero which in turn will introduce singularities in the equation for \( q \) and \( \frac{dg}{dx} \).

4.4 Relative contributions of Gaussian derivative terms

For these examples, a value of unity will be used for the underlying variance of the sound speed (see Eqs. 1 and 10) so that the standard deviation of the intensity will be equal to the intensity derivative. Only one point will be uncertain in order to simplify the comparisons and remove the complication of correlated points.
4.4.1 Flat bottom, S=30m, R=150m

To examine the relative contributions from the two Gaussian derivative terms in Eq. 11, we examine the flat bottom case with the source at 30m and the receiver at 150m using the 4-point profile shown in Figure 1. The derivatives are taken with respect to the surface sound speed, $X_1$. Figure 22-left, displays a ray trace of the limiting rays, $\pm 4.093^\circ$ and $\pm 4.094^\circ$, in the surface duct between 0 and 120m depth. Figure 22-right plots the log of $q$ (Eq. 4) for these four rays as a function of the step along the ray path. It can be seen that the value for $q$ increases by several orders of magnitude on the rays that break through the surface duct (black) as compared to the value for $q$ on the two trapped rays (red). This occurs because the jump condition for transition between two SSP layers (Eq 19) injects a large value into $q$ since the angle is nearly grazing at the transition point.

Figure 22. Left: Ray trace of surface duct limiting rays. Right: Value of $q$ at each step along these 4 rays.

Figure 23 shows the contributions vs launch angle from the source at a fixed range of 5km. Here, the angles that define the surface duct are shown by the dot-dash line and there are no ray angles within these lines that will reach the 150m deep receiver. On this figure, the total intensity derivative normalized by the intensity for each ray angle is plotted as the black stars, the $\frac{dx}{dx}$ term is shown in green and the $\frac{dr}{dx}$ term is shown in red. The total normalized derivative will be the sum of all the black points. (i.e. the intensity is the sum of all launch angles at this range and receiver depth).
This figure shows that the largest contribution to the intensity derivative occurs on either side of the bounding rays of the surface duct where the grazing angle is very small. The figure also shows that this contribution is coming from the $\frac{d\sigma}{dx}$ term, which is caused by the presence of $\sin^2 \theta$ in the denominator of $\frac{dN_{ran}}{dx}$. When summing all these rays, the majority of high angle terms will offset each other, however the contributions from the edges of the surface duct floor will produce a sum total positive value of nearly 2dB.

**4.4.2 Flat bottom S=150m, R=30m**

In this case, with the deep source, the lower duct traps rays with launch angles inside ±2.84° as shown in the ray trace in Figure 24-left. The normalized derivative is shown on the right for uncertainty in the 130m SSP point, X3 which is the minimum of the SSP and the axis of the channel formed. The large term that will dominate the sum of angle contributions occurs at +2.9° on the edge of the sound channel (in this case negative). It is composed of the sum of the $\frac{dn}{dx}$ (green) and the $\frac{d\sigma}{dx}$ (red) terms, which in this case are both negative. The $\frac{d\sigma}{dx}$ term is the larger and
it comes from the large increase in the transition jump condition derivative \( \frac{dN_{\text{trans}}}{dx} \) at near grazing on the top of the channel. Note that on the graph, only the +2.9° ray is contributing to the intensity at the 30m receiver. This occurs because the -2.9° ray that also escapes the sound channel is sufficiently displaced horizontally from the chosen range of 5km that the Bellhop logic for ray selection (based on the normal distance to the receiver point) rejects this ray. The TL standard deviation this set of conditions will produce is 0.48dB.

![Bellhop Ray Plot](image)

*Figure 24. Left: Ray trace of four adjacent rays. Right: Angular distribution of the normalized intensity derivative and its contributing terms at 5km, flat bottom, X3, S=150, R=30.*

### 4.4.3 Flat bottom S=R=30m

For this comparison, a collocated source and receiver is chosen, with both in the surface duct of the 4-point profile shown in Figure 1. Figure 25-left displays a ray tracing plot between -4.1 and 4.1 degrees for the flat bottom case. On the right, the angle distribution of intensity at 5km for ray launch angles is shown, windowed to ±10°. The dot-dash line shows the angular limits of the surface duct in which the source and receiver are located. In this case, all the rays in the surface duct will contribute to the field at the receiver. Because these rays are forming caustics as they propagate, the value of \( q/q_0 \) becomes small and eventually the limits for the choice of the Gaussian width, \( \sigma \) expressed in Eq. 3 are reached. The dotted line labeled sig=2 shows the angular region where \( \sigma \) becomes limited by the 2nd choice shown in Eq. 3, that is, the factor \( \pi \lambda \). Outside this dotted region, the Gaussian width is given by \( q/q_0 \).
In Figure 25-right we see that for the most part, the $\frac{dn}{dx}$ term is dominant, particularly in the surface duct. The notable exception is at the edge of the duct at $-4.1^\circ$, where the $\frac{dz}{dx}$ term is very large, as was the case in Figures 22-24. This large factor is only partially offset by the values of the other ray angles, and the net result is a change in TL of about 2.1 dB due to the uncertainty in the first SSP point.

### 4.4.4 Slope bottom, negative SSP gradient, S=30m, R=10m

In this case, the bottom has a uniform upward slope and the SSP has a uniform negative gradient, therefore all rays are refracted downward to interact with the bottom which will steepen the ray propagation angle on contact. In Figure 26–left, the rays are traced between $\pm 4^\circ$. In the right plot, the normalized intensity derivatives are plotted at 10km by launch angle from the source. Here, there are no channels or ducts and so there are no large derivatives. The bottom jump condition will still be large at grazing incidence, however in this case, the rays will never graze the bottom, therefore the red symbols for $\frac{dz}{dx}$ are very small, and the intensity derivative is dominated by the green symbols for $\frac{dn}{dx}$. The total sum of these derivatives over angle produces a TL change of 0.17 dB due to the uncertainty in the first SSP point.
4.4.5 Slope bottom, positive SSP gradient, S=30m, R=10m

In this case, the positive gradient will refract sound upward, so that as the bottom rises, there will be a near grazing incidence at some point for all rays shallower than about 10°. The ray trace in Figure 27-left shows two such rays (red) that nearly graze the bottom and then rise to arrive close to the receiver depth of 10m at 10km. In Figure 27-right, the intensity derivative is shown as a function of launch angle. For angles below 10°, these grazing rays generate a large increase in the bottom jump condition which makes $\frac{d\sigma}{dx}$ (red) large.

Notice that for the launch angles between -1.25° and +0.6° shown by the dotted lines on the plot, the value of $\sigma$ becomes limited by the factor $\pi\lambda$. (See the 2nd choice for $\sigma$ in Eq. 3). A careful examination of the black rays in the ray trace shows that there is a caustic generated (where the two rays cross) very near the 10m receiver depth at 10km. This caustic has caused the value of $\sigma$ to be limited, and in this region, the error is entirely given by the $\frac{d\sigma}{dx}$ term. The structure looks very similar to that shown in Figure 25 where the value of $\sigma$ was limited for the same reasons. The sum of derivatives for all angles yields a TL change of 4.5dB.
4.5 Mitigation of sensitivity

The results of the tests shown above have proven that the jump condition is providing the largest increase in intensity derivative (and variance) whenever the choice of Gaussian width is $q/q_0$ and the grazing angle is very small going into the layer change or boundary. Reference 1 suggests an alternative form for the jump condition at small grazing angles, however this form is not implemented in the current version of Bellhop. This ought to be examined, as perhaps a more careful treatment of the phase-matching jump condition would reduce the intensity derivative and therefore reduce the Bellhop sensitivity to sound speed.

The insight gained by this dominant term analysis is that, in Bellhop, there may be one ray, the one with the very small angle as it passes through the layer that dominates all the other ray’s contributions to the error. The large variance comes from the sound speed uncertainty that would make a ray just barely break through a layer instead of refracting just before the layer. It seems clear that the inability of ray theory to trace split rays means that the sound speed uncertainty pushes the ray one way or the other, all or nothing. This is a fundamental limitation of ray theory, and redefining the jump condition may not repair this problem.

4.6 Relationship of intensity derivatives between adjoining uncertainty points

In Section 3, it was demonstrated that if the uncertainties in the all the SSP points are fully correlated with equal standard deviations, then the errors will sum to nearly zero because of opposing signs of the intensity derivatives. Let us examine the reasons for the change of sign in the intensity derivatives as the uncertainty is moved to different points in the SSP.

Examine the dominant term in the intensity derivative which is the Gaussian derivative (Eq. 11), as shown in Section 4.1. The two terms in the Gaussian derivative are the derivative of the normal
distance to the receiver from the ray, \( dn/dx \), and the derivative of the Gaussian width factor, \( dσ/dx \), which are listed in Eqs. 13–15. These equations and their underlying functions (the change in range, depth and angle of the ray, \( dr/dx \), \( dz/dx \) and \( dθ/dx \)), are all found to be functions of the sum of the derivatives of the gradient \( (dg/dx)/g \) and the Snell’s constant \( (dD/dx)/D \) in each layer.

The definition of the gradient, for points \([c_1, z_1]\) and \([c_2, z_2]\) is \( g = \frac{c_2 - c_1}{z_2 - z_1} \).Injecting a random value of \( x_1 \) into the first speed and taking the derivative of the gradient with respect to this random value, yields

\[
\frac{dg}{dx_1} = \frac{d}{dx_1} \left( \frac{c_2 - (c_1 + x_1)}{z_2 - z_1} \right) = \frac{-1}{z_2 - z_1}
\]

while injecting the random value of \( x_2 \) into the second speed and taking the derivative with respect to this value yields

\[
\frac{dg}{dx_2} = \frac{d}{dx_2} \left( \frac{(c_2 + x_2) - c_1}{z_2 - z_1} \right) = \frac{+1}{z_2 - z_1}
\]

Thus the quantity \( (dg/dx)/g \) will be \( ±1/Δc \), the sign depending on whether the uncertainty is in the upper speed or the lower speed of the adjacent points.

The Snell’s Law constant \( D \) is defined at the source position as \( D = \frac{c_0}{\cosθ_0} \) where \( θ_0 \) is the launch angle of the ray and \( c_0 \) is the sound speed at the source depth. For generality, we will assume that the SSP does not have a defined point at the source depth \( z_0 \). If the source layer is \( L \), then \( c_0 = c_L + (z_0 - z_L)g \), and \( D \) will vary with the uncertainty at either end point of the gradient of the layer containing the source. If the source layer is \( L \), then

\[
\frac{dD}{dx_L} = -\frac{D}{c_0} \left( \frac{dc_L}{dx_L} + (z_0 - z_L)\frac{dg}{dx_L} \right)
\]

\[
\frac{dD}{dx_{L+1}} = -\frac{D}{c_0} \left( \frac{dc_L}{dx_{L+1}} + (z_0 - z_L)\frac{dg}{dx_{L+1}} \right)
\]

Now, the product of \( (z_0 - z_L)\frac{dg}{dx} = ± \frac{z_0 - z_L}{z_{L+1} - z_L} \) will always be a fraction less than 1 since \( z_0 \) is in the layer between \( L+1 \) and \( L \). Examining the upper equation, if the uncertainty is in layer \( L \), \( \frac{dc_L}{dx_L} = 1 \), and the quantity in brackets in the upper equation is a positive number even though the gradients derivative is negative. If the uncertainty is in layer \( L+1 \), \( \frac{dc_L}{dx_L} = 0 \) but the gradients derivative is positive, therefore the upper equation is still a positive number. The lower equation also retains the same sign whether the uncertainty is in layer \( L \) or \( L+1 \). Even with the bathymetry
reflections discussed in Section 2, where the propagation angle is changed by the bottom slope, this derivative ratio will remain valid because the bathymetry angle is not a function of SSP uncertainty. It can be concluded that this quantity will not change sign with a changing position of the uncertainty x, but it is several orders of magnitude smaller than the gradient term, therefore the changes of sign in the Gaussian derivative are caused by the prominent presence of the (linear) gradient in the propagation equations.

The conclusion is that if the two sound speed points defining a layer are correlated with each other in terms of a random change and possess the same standard deviation, then there will be almost no intensity variance introduced by propagation in that layer because of the simple change in sign of the linear gradients derivative.
5. Comparison of derivatives using \( n^2\)-linear and \( c\)-linear profile functions

5.1 \( n^2\)-linear profile functional form

The functional form called \( n^2\)-linear describes the sound speed profile which causes rays to follow parabolic ray trajectories. This form is particularly useful for those Normal Mode solutions that use Airy functions for their eigenfunctions, (such as KRAKEN and AP6) and it has been suggested that this form would not generate false caustics, thus a comparison of the variances in Bellhop generated by this form and the \( c\)-linear form is useful.

Let the layer be defined by the points \([z_0, c_0]\) and \([z_1, c_1]\) and the incident ray angle \( \alpha \). The index of refraction, \( n \), is the ratio of the sound speed at the start of a layer divided by the sound speed in that layer, \( n(z) = c_0/c(z) \). The assumption of \( n^2\)-linear form requires

\[
n^2(z) = a + bz
\]  

(20)

At \( z = z_0 \), \( n = 1 \), therefore \( a = 1 - b z_0 \) and the equation for \( n \) becomes \( n^2(z) = 1 + b(z - z_0) \)

At \( z = z_1 \), \( n = c_0/c_1 \), therefore \( b = c_0^2 \frac{c_1^2 - c_0^2}{z_1 - z_0} \) and the sound speed is given by

\[
c(z) = \frac{c_0}{\sqrt{1 + b(z - z_0)}}
\]

(21)

The gradient of the \( n^2\)-linear profile, \( g \), is

\[
g = \frac{dc}{dz} = -\frac{1}{2} c_0 b (1 + b(z - z_0))^{-3/2} = -\frac{1}{2} \frac{b c(z)^3}{c_0^2}.
\]

Notice that the gradient of the \( n^2\)-linear profile is a function of \( z \) through the sound speed \( c(z) \) in the numerator. Therefore in this \( n^2\)-linear profile case, the second depth derivative of \( c \), denoted \( c_{zz} \) in reference 1 is

\[
c_{zz} = \frac{3g^2}{c(z)}
\]
Contrast this with the c-linear profile where the gradient is constant, \( g_{lin} = \frac{c_1 - c_0}{x_1 - x_0} \), and the second depth derivative of the sound speed, \( c_{zz} \), is zero. Figure 28 displays a simple two point profile and its gradient using the two functional forms, red for c-linear and black for \( n^2 \)-linear.

![Figure 28. Examples of sound speed profile (left) and gradient (right) for the two profile forms.](image)

The functional form of an \( n^2 \)-linear profile arises naturally in elastic media from the definition of the compressional wave speed in terms of Lamé constants in which the reciprocal of the non-vanishing Lamé constant \( \lambda \) is allowed to vary linearly with depth [6].

### 5.2 Closed form equation for \( q \) using \( n^2 \)-linear profile functional form

Reference 1 provides a derivation of the closed form solutions for the p-q equations using the \( n^2 \)-linear sound speed profile. These solutions are

\[
\begin{align*}
p(s) &= Ap_1 + Bp_2 \\
q(s) &= Aq_1 + Bq_2
\end{align*}
\]

where

\[
\begin{align*}
p_1 &= \left( b \frac{dz}{dr} \right) \frac{\cos^3 \alpha}{n^3 \sin \theta} \\
p_2 &= \frac{dz}{dr} \cos \alpha \left( \frac{1}{n} + \frac{2 \cos^2 \alpha}{n^3} \right) \\
q_1 &= \sin \theta \\
q_2 &= \left( \frac{2c_0}{b} \right) \left( n - 2 \frac{\cos^2 \alpha}{n} \right)
\end{align*}
\]
Using the initial conditions, \( p(0) = 1 \) and \( q(0) = 0 \), \( \theta = \alpha, r = r_0, z = z_0, c = c_0, \) \( \frac{dz}{dr} = \tan \alpha \) and \( n = 1 \), then

\[
p_1(0) = \frac{b \cos^2 \alpha}{2c_0} \]
\[
p_2(0) = \sin \alpha (1 + 2 \cos^2 \alpha) \]
\[
q_1(0) = \sin \alpha \]
\[
q_2(0) = \frac{2c_0}{b} (1 - 2\cos^2 \alpha) \]

The solution for the constants \( A \) and \( B \) are

\[ A = \frac{-q_2(0)}{p_2(0)q_1(0) - p_1(0)q_2(0)} \quad \text{and} \quad B = \frac{q_1(0)}{p_2(0)q_1(0) - p_1(0)q_2(0)} \]

The denominator of these constants, \( p_2(0)q_1(0) - p_1(0)q_2(0) = 1 \). Therefore,

\[ q(s) = -q_2(0) q_1(s) + q_1(0) q_2(s) \]

The dependence on arc length position \( s \) is contained in the index of refraction and the ray angle, \( n = c_0/c(s) \) and \( \theta = \theta(s) \), while \( \alpha \) and \( c_0 \) are the values at the start of the layer and \( b \) is the constant in the \( n^2 \)-linear profile definition in Eq 20.

Let us denote this equation by the symbol \( q_{Ln} \) to indicate it is the \( q(s) \) in the layer using an \( n^2 \)-linear profile, because we have previously used the symbol \( q_L \) for \( q(s) \) in the layer using a \( c \)-linear profile (in Eq. 4). In keeping with the previous notation for a step from \( j-1 \) to \( j \), the angle at the start of the step \( \alpha = \theta_{j-1} \), the speed at the start of the step \( c_0 = c_{j-1} \), and \( n = c_{j-1}/c \). These two relations for the two different profile forms are shown in Eq. 22, where the functional forms of the first ratio are the same, but the definition of the gradient \( g \) is different.

\[
q_{Ln} = \frac{\sin \theta_{j-1} - \sin \theta}{g D^2} (2\cos^2 \theta - 1) \cos^2 \theta \quad \text{n}^2\text{-linear} \quad (22)
\]
\[
q_L = \frac{\sin \theta_{j-1} - \sin \theta}{g D^2} \quad \text{c-linear}
\]

Now the jump conditions at each interface must be added to this solution. This requires an integration of the first of the p-q equations, \( dq/ds = cp = cq_N \). where \( q_N \) is the extra term supplied by the interface phase matching. On the interface \( q_N \) is constant, therefore we have
\[ q - q_{j-1} = q_j N \int c \, ds. \] The integral of the sound speed along the ray trajectory is
\[
\int c \, ds = \frac{\sin \theta_{j-1} - \sin \theta}{\partial D^2}
\] which has the same functional form as that found for the c-linear profile, except that the two gradients are different. The jump conditions \( q_j N \) also have the same functional form for either sound speed profile type except the gradients are different. Finally, we can express the beamwidth factor \( q \) at step \( j \), including the jump conditions, as

\[
q(j) = q(j - 1) + \left\{ q_{\ln} + q_L \sum_{k=1}^{j} q(k - 1) N_k \right\} \quad \text{n}^2\text{-linear}
\]

\[
q(j) = q(j - 1) + \left\{ q_L + q_{\ln} \sum_{k=1}^{j} q(k - 1) N_k \right\} \quad \text{c-linear}
\]

The differences between these two equations are the factor \((2 \cos^2 \theta - 1) \cos^2 \theta\) that appears in \( q_{\ln} \) and the different definitions for the gradients which are found in \( q_L, q_{\ln} \) and \( N \). These two expressions have the same form when \( \theta = 0 \), in which case, only the gradients are different.

### 5.3 Comparisons of \( q \) using \( n^2 \)-linear and c-linear functional forms

#### 5.3.1 Discrepancy between ray-traced values and closed form values

To test the \( n^2 \)-linear equation for \( q \) shown in Eq. 23, the closed form solution for \( q \) is compared to the ray-traced solution from Bellhop. In Figure 29, the fractional differences are plotted for a source depth of 99m. These are obtained by subtracting the ray-traced value from the closed form value and dividing the result by the closed form value. The figure shows the logarithm of this fractional difference for four different rays, the \(-1^\circ, 2^\circ, 8^\circ, \text{and } 12^\circ\), for the flat bottom case that was used earlier in this report. The most striking aspect of this figure is that the fractional differences in the \( n^2 \)-linear cases are orders of magnitude greater than those in the c-linear cases. The largest fractional difference in the \( n^2 \)-linear case is 1.39 (139%) while the largest fractional error in the c-linear case is 0.02 (2%).

We speculate that this difference occurs because of the technique used for the ray-tracing in Bellhop. The code contains the following description of the ray-tracing technique:

“The numerical integrator used here is a version of the polygon (a.k.a midpoint, leapfrog, or Box method), and similar to the Heun (second-order Runge-Kutta method). However it’s modified to allow for a dynamic step change, while preserving the second order accuracy.”

The technique consists of taking two half-steps sequentially and linearly summing the contributions from each half step. The quantities being traced are the ray position and tangent, the
values for $p$ and $q$, and the travel time $\tau$, and these quantities come from the speeds, gradients, and second derivatives of the profile. In the c-linear case, the speeds are linearly related and gradients are constant over the full step, therefore the closed form solution is very close to the traced solution when the two half steps are added together. In the $n^2$-linear case, the speeds are not linearly related and the gradients change at each half step, therefore, the simple addition of the two half-steps will not match the full-step closed form solution as closely. The largest differences in Figure 29 occur for the shallowly refracted ray at 2°. Figure 30-left shows a linear ray trace for the 2° ray, which transitions through the middle layer of the SSP and refracts in the lower layer. The c-linear gradients are listed on the figure. Figure 30-right shows the dB loss from $1/q$ for the 2° ray with range for both profile types. The ray traced values are shown in black while the closed form solutions are shown in green (c-linear) and red ($n^2$-linear) The green line representing the c-linear closed form solution matches the ray traced value so closely that the green line overlays that black line. In the $n^2$-linear case, the red line begins to depart from its black line at about 1.3km and shows a wide difference beyond 3km. Note that the $n^2$-linear ray actually refracts at a slightly farther range than the c-linear ray.

![Diagram](image.png)

**Figure 29.** Fractional differences in $q$ between closed form and ray-traced values. The black lines are the $n^2$-linear results and the green lines are the c-linear results for 4 different rays.
In Figure 31, an even more striking example of the traced/closed form difference is shown. In this figure, the traced value of \( \frac{1}{q} \) is compared to the closed form value for the 4.1° ray using source depth of 30m. Note that they agree reasonably well until about 4.5km when the traced value (red dashed) departs significantly from the closed form (black), although it retains the same shape as the closed form solution.

Because the \( n^2 \)-linear closed form values for \( q \) do not always closely match the ray-traced values, the ray’s intensity and its derivative will both be computed using the closed form values in the remainder of this study. The comparison between the derivatives of the closed forms of the two SSP types will still be a valid metric to judge and contrast the sensitivity of the transmission loss to these two functional forms of the sound speed profile.

Figure 31. Comparisons of the dB loss from \( 1/q \) using the closed form and traced solutions for the 4.1° ray.
5.4 Derivative of $q$ in an $n^2$-linear profile

The derivative of Eq. 23 is similar to Eq. 17 in form with the exceptions being the derivative of $q_{Ln}$ and the gradient in the $n^2$-linear case.

From Eq. 22, the definition of $q_{Ln}$ is $q_{Ln} = q_L (2\cos^2\theta - 1)\cos^2\theta$. Comparing the equations for the derivatives of $q_L$ and $q_{Ln}$ for the c-linear and $n^2$-linear cases respectively,

$$\frac{dq_L}{dx} = \frac{\cos\theta_{j-1} \frac{d\theta_j}{dx} - \cos\theta_j \frac{d\theta_{j-1}}{dx}}{gD^2} - q_L \left( \frac{\frac{d\theta}{dx}}{g} + \frac{\frac{d\theta}{dx}}{D} \right)$$

$$\frac{dq_{Ln}}{dx} = \frac{dq_L}{dx} (2\cos^2\theta - 1)\cos^2\theta - q_L \left( 4\cos^3\theta \sin\theta \frac{d\theta}{dx} \right) - q_L (2\cos^2\theta - 1)(2\cos \theta \sin \theta \frac{d\theta}{dx})$$

And comparing the equations for the gradients,

$$g = (c_1 - c_0)/(z_1 - z_0) \quad \text{c-linear}$$

$$g(z) = -\frac{1}{2}c_0 b \left( 1 + b(z - z_0) \right)^{-3/2} \quad \text{n}^2\text{-linear}$$

where $b = c_0^2 \left( \frac{c_1}{z_1} - \frac{c_0}{z_0} \right)$.

The derivatives of the gradients with respect to an uncertainty in $c_0$ are

$$\frac{dg}{dx_0} = \frac{-1}{z_1 - z_0} \quad \text{c-linear}$$

$$\frac{dg}{dx_0} = g(z) \left\{ \frac{1}{c_0} + \frac{db}{dx_0} b - \frac{3}{2} \left( 1 + b(z - z_0) \right)^{-1} \frac{db}{dx_0} (z - z_0) \right\} \quad \text{n}^2\text{-linear}$$

where, $\frac{db}{dx_0} = \frac{c_0}{z_1 - z_0}$.

And, the derivatives of the gradients with respect to an uncertainty in $c_1$ are

$$\frac{dg}{dx_1} = \frac{1}{z_1 - z_0} \quad \text{c-linear}$$

$$\frac{dg}{dx_1} = g(z) \left\{ \frac{db}{dx_1} b - \frac{3}{2} \left( 1 + b(z - z_0) \right)^{-1} \frac{db}{dx_1} (z - z_0) \right\} \quad \text{n}^2\text{-linear}$$
where, \( \frac{dP}{dx_1} = \frac{\frac{dP}{dz}}{\frac{dz}{dx_1}} \).

### 5.5 Comparisons of TL and \( dI/dx \)

The figures that follow demonstrate the differences between the transmission loss and the derivative of the intensity computed using the \( c \)-linear and \( n^2 \)-linear profile forms.

The first set of examples are shown in Figure 32 which uses a simple two point profile with a positive gradient in 300m water depth. The bottom is a hard sediment halfspace and the surface has zero loss. The surface sound speed is 1450 m/s. The speed at the bottom in these figures is 1451, 1460, 1550 and 1750 m/s, respectively, to increase the steepness of the gradient.

**Figure 32.** TL and \( dI/dx \) comparisons in an upward refracting profile with different gradients.

The constant \( c \)-linear gradient is shown on each figure and the results from this profile function are shown in black. The \( n^2 \)-linear intensity derivative is shown in green (uncertainty in surface...
speed) and red (uncertainty in bottom speed) on the lower curve in each graph. The upper curve on each graph shows the transmission loss in dB with black being the c-linear calculation and red being the n²-linear value. Note that the intensity derivatives for this two point profile are nearly identical whether the uncertainty is placed in the upper or lower speed (i.e. the green lines very nearly overly the red lines). As the gradient steepens, the transmission loss becomes increasingly oscillatory and the two profile functions begin to depart in period of oscillation. In the two smaller gradient cases, the two derivatives are almost exactly equal, however as the gradient steepens, they begin to diverge, with the c-linear result being slightly lower than the n²-linear result at the longer ranges.

The next set of frames in Figure 33 come from the simple 2-point profile constructed as a downward refracting profile with the bottom speed being 1450 m/s and the surface speed being 1451, 1460, 1550 and 1750 m/s, respectively, to increase the steepness of the gradient.

**Figure 33.** TL and dl/dx comparisons in a downward refracting profile with different gradients.

In the last frame, the derivative of the intensity is normalized by the intensity to show that the values are still quite similar, although the intensity itself is quite a bit higher in the n²-linear (red).
case. In the higher gradient cases in both these figures, the $n^2$-linear form seems to show slightly more error, although the peak error lines up in both cases pretty closely.

The next examples are computed using the 4-point profile shown in Figure 1. Figure 34 contains four graphs computed with the source and receiver set at 125m, near the axis of the channel. The graphs show the transmission loss and intensity derivatives (normalized by the intensity) for the uncertainty in each of the four profile points respectively. In this case, black displays the c-linear results and red displays the $n^2$-linear results. These figures show that the intensity derivatives of the c-linear and $n^2$-linear SSP functional forms are very nearly alike. They differ in range because of the differences in the ray paths, but their magnitudes are very similar.

Figure 34. TL and normalized intensity derivative for uncertainty in each point of the 4-point profile respectively, source and receiver at 125m in the channel.

Figure 35 shows cross-channel propagation with the source at 30m and the receiver on the channel axis.
a) uncertainty in the top point at the surface.  

b) uncertainty in the 2nd point at 120 m  

c) uncertainty in the 3rd point at the channel axis  

d) uncertainty in the bottom point  

Figure 35. TL and normalized intensity derivative for uncertainty in each point of the 4-point profile respectively, source above the channel at 30m, receiver on the axis at 130m in the channel.

Again, there is little difference between these intensity derivatives.

5.6 Conclusion of comparison of variances using c-linear and n²-linear SSP profile forms

The conclusion from these limited tests is that either functional form of the sound speed profile will produce very closely the same magnitude of intensity derivative and therefore the same transmission loss variances.

However, it is interesting to note that in the case of the steep upwardly refracting profile, shown in Figure 33d, the transmission losses generated with these two functional forms of the SSP differ rather significantly, the n²-linear form giving much less loss than the c-linear form, while in all the other figures, the transmission losses from the two functional forms are either the same or the n²-linear form gives slightly more loss.
6. Effect of Gaussian width choice on TL variances

6.1 Gaussian width choices in Bellhop

All ray-theoretic models face the difficulty of producing a finite amplitude when the rays reach caustics \((q \to 0)\) and the Gaussian Beam solution is no different. The technique employed by Bellhop is to limit the value of the Gaussian width \(\sigma\) using the logic expressed in Eq. 3, namely:

\[
\sigma = \begin{cases} 
  q/q_0 & \text{depending on the size of } q/q_0 \\
  \pi \lambda & \\
  0.2s/\lambda
\end{cases}
\]

The first choice is the ratio of \(q/q_0\) where \(q_0 = c_0/\delta\), \(c_0\) the sound speed at the source and \(\delta\) the angle increment between successive rays being traced. In the third choice for \(\sigma\), \(s\) represents the total arc length of travel of the ray, and in the second and third choices, \(\lambda\) is the wavelength \(c/f\) with \(f\) the frequency in Hertz. The criterion for choosing is:

\[
\sigma = \text{MAX}\left(\frac{q}{q_0}, \text{MIN}\left(\pi \lambda, \frac{0.2s}{\lambda}\right)\right).
\]

The derivative of \(\sigma\) is

\[
\frac{d\sigma}{dx} = \begin{cases} 
  \frac{dq}{dx} - \frac{dc_0}{q} & \\
  \frac{dc}{\sigma} & \\
  \frac{ds}{\sigma} - \frac{dc}{c}
\end{cases}
\]

The third choice, \(0.2s/\lambda\), is a nearfield limitation. For example, at 1000Hz, the arc length \(s\) that must be traveled to make \(0.2s/\lambda < \pi \lambda\) must be less than 35m, at 500 Hz the arc length traveled must be less than 141m. In all the examples presented here, the frequency was high enough and the range far enough that the third choice was never selected. However, in two of the examples, shown in Figures 25 and 27, the second choice was selected because the rays were close to a caustic. Figure 25 shows surface duct propagation where caustics occur all along the duct. Therefore the Gaussian width changes will be examined using the first and second choices for \(\sigma\).

6.2 Comparison between no choice and the first two choices for \(\sigma\)

Using the flat bottom environment with source and receiver in the duct of the 4-point profile shown in Figure 1, the transmission loss and normalized derivative of the intensity are plotted in Figure 36. The black lines show the TL and normalized intensity derivative computed using the
choice logic explained above, that is, the black lines show the intended Bellhop result. The red lines show the same two quantities with the logical choice for $\sigma$ disabled, so that $\sigma = q/q_0$ always. It is clear that disabling the limitation on $\sigma$ increases the intensity and its derivative dramatically.

![Transmission loss and $dl/dx$](image)

**Figure 36.** Comparison of TL and normalized error for different Gaussian width choices. Red shows the width $q/q_0$, while black shows the choice $\text{MAX}(q/q_0, \pi\lambda)$.

Next, the size of the limiting factor is tested. Figure 37 displays the TL and normalized intensity derivative for different factors multiplying the second choice, ranging from $16\pi\lambda$ to $\pi\lambda/16$. In this figure, the heavy black line is the Bellhop programed choice of $\pi\lambda$. When this limiting value is decreased, the TL and its variance rise in the caustic regions, while when this limiting value is increased, the TL and its variance fall in the caustic regions. It is interesting to note that the limiting factor or ‘minimum aperture’ in CASS/Grab is $2\pi\lambda$, the light blue color whose TL falls just slightly below the dark black line of Bellhop. The CASS/Grab developers say their choice was empirical. The Bellhop limiting factor was probably determined by a best match to the normal mode result from KRAKEN [7].
Figure 37. Comparison of TL and normalized error for different sigma limiting factors from 16 to $1/16^{th}$. 
7. Summary and Recommendations

7.1 Summary

The purpose of this report is several-fold. In it, there are topics dealing with diverse aspects of the intensity derivative which is used to compute the transmission loss variance due to sound speed uncertainty.

In Section 2, the intensity derivative is enhanced to include range dependent bathymetry and bottom losses. To demonstrate the effect of range dependence, four different bathymetries were examined. In every case, the analytic intensity derivative is compared to a numerically computed derivative to demonstrate the validity of the analytic closed-form equations. Next, full field plots of the distribution of the intensity and its derivatives are shown for uncertainties in each of the four sound speed points defined in the input profile. These full field plots show that the regions of high derivative coincide with the regions of high intensity and they are largest whenever the uncertainty is found in the two SSP points bracketing the source. This immediately implies that obtaining an accurate estimate for the speeds surrounding the source depth will be of most benefit in reducing the uncertainty in Bellhop’s output.

In Section 3, the mathematics used to transfer uncertainty in sound speed into uncertainty in transmission loss are amended to include partial correlation among sound speed uncertainties, such as might occur in near-surface points that are influenced by the atmospheric conditions. It is shown that the fully uncorrelated SSP variances, in which the derivatives sum as squares, provide the greatest variance in transmission loss, while the fully correlated SSP variances, in which the intensity derivatives sum coherently, provides almost zero transmission loss variance. This implies that transmission loss variance in the surface duct will be reduced by the partial correlation of the duct’s defining points.

In Section 4, the various terms in the intensity derivatives are compared to determine the dominant terms. It is found (for sound speed uncertainty) that the derivative of the Gaussian term provides the strongest contribution. The Gaussian term contains the normal distance to the receiver \( n \) and the Gaussian width \( \sigma \) in the form of \( \exp \left( -\frac{n^2}{2\sigma^2} \right) / \sigma \). The normal distance to the receiver depends on the ray path and it doesn’t exhibit any instabilities. The Gaussian width however, is another matter. It is a very important quantity that is related to the beamwidth and curvature of the ray and it is directly proportional to the ray-traced quantity \( q \). In this paper, the analytic equations for \( dq/dx \) are used to determine its behavior. It is found that the largest contributions occur whenever a ray passes through a SSP layer or strikes a boundary at a very small grazing angle. The quantity injecting the large increase in \( dq/dx \) is the derivative of the jump condition which is proportional to \( 1/\sin^3 \theta \).

The results of the dominant term comparisons have proven that the jump condition is providing the largest increase in intensity derivative (and TL variance) whenever the choice of Gaussian width is \( q/q_0 \) and the grazing angle is very small going into the layer change or boundary. Reference 1 suggests an alternative form for the jump condition at small grazing angles; however this form is not implemented in the current version of Bellhop. This ought to be examined, as
perhaps a more careful treatment of the phase-matching jump condition would reduce the intensity derivative and therefore reduce the Bellhop sensitivity to sound speed.

The insight gained by this dominant term analysis is that, in Bellhop, there may be one ray, the one with the very small angle as it passes through the layer, that dominates all the other ray’s contributions to the intensity derivative. The large value comes from the sound speed uncertainty that would make the ray just barely break through a layer instead of refracting immediately before the layer. It seems clear that the inability of ray theory to trace split-rays means that the sound speed uncertainty pushes the ray one way or the other, all or nothing. This is a fundamental limitation of ray theory and redefining the jump condition may not repair this problem.

Also in Section 4, the reason for the highly correlated SSP points at either end of a layer producing very little TL variance (the result of Section 3) is examined. The study has determined that this occurs because the derivatives being summed from the two points defining the layer are off-setting in sign. The influencing term is the simple change in sign of the derivative of the linear gradient between the two points. The conclusion is that if the two sound speed points defining the layer are correlated with each other in terms of a random change and possess the same standard deviation, then there will be almost no intensity variance in that layer.

Section 5 examines the intensity derivatives using two different functional forms for the interpolation of the sound speed profile: c-linear and \(n^2\)-linear. The equations for the intensity derivative using the \(n^2\)-linear profile are derived and compared to the c-linear form. It is found that while the predictions of the transmission loss may be different, the normalized intensity derivatives are very similar. It is not possible to declare that one form of SSP interpolation is more stable or less sensitive than the other in the mathematics of Bellhop.

Section 6 compares the effect of the limiting value for the Gaussian width \(\sigma\) on the intensity derivatives. The limiting value is invoked whenever the size of the quantity \(q/q_0 < \pi \lambda\) which occurs in the vicinity of a caustic. It is demonstrated that the limiting value \((\pi \lambda)\) serves to dampen the intensity derivative as compared to the unrestricted case. Different multiples of this limiting value show that as the limit decreases, \((\pi \lambda/4, \pi \lambda/16)\) both the intensity and the intensity derivative increase dramatically in the caustic region showing sharp peaks. On the other hand, as the limiting value increases \((2\pi \lambda, 4\pi \lambda, 16\pi \lambda)\) both the intensity and the intensity derivative decrease in the caustic region, smoothing out the transmission loss curve. This factor \(\pi \lambda\) was probably chosen by comparison with a benchmark model.

### 7.2 Recommendations

Recommendations arising from this research are:

1. This report’s research and the proof-of-concept shown in reference 2 have demonstrated the feasibility of computing the variance of the transmission loss due to sound speed uncertainty in range dependent environments using Bellhop. Given the inherent uncertainty of a changing sound speed profile and the strong sensitivity of the Bellhop transmission loss to the sound speed, the computation of the TL variance provides a significantly more realistic transmission loss estimate. The TL variance, in conjunction with the noise variance, could be used in forming the statistical terms of the sonar equation, such as the probability of detection. Implementing this closed-form...
computation in the active and passive versions of BellhopDRDC would provide DRDC Atlantic with a unique modeling capability.

2. From Section 4, it was found that the dominant term in the variance of the intensity is the jump condition that comes from complex phase matching at a sound speed layer boundary or the surface or bottom. Rays that reach these boundaries at very low grazing angles cause large values of the jump condition because it is proportional to $1/sin^3\theta$. There is an alternate form for the jump condition said to be valid for shallow grazing angles, given in reference 1. It is an obvious area for study to determine the decrease in sensitivity that it would produce.

3. It was noted in Section 2.3 that sound speed range dependence employs a bilinear quadrilateral interpolation technique and the intensity derivative equations from reference 2 were not able to properly handle the non-constant gradient that this interpolation produces. Computing the transmission loss variance in this case will require re-derivation of the underlying equations in the intensity derivative; those of the depth, range, and angle derivatives. This line of research may not be very important because if the assumption is that the uncertainty remains fixed by the speed’s depth rather than by the speed’s value, then one might expect the range dependent speed changes would have little impact on the size of the transmission loss variance.

4. The bottom reflection treatment currently implemented in Bellhop uses Rayleigh reflection coefficients. There is code within Bellhop to include the effects of beam displacement but this has not been made available to the user. There is a discussion of beam and time displacement in Chapman and Ellis [8] which would provide a starting point for examining the Bellhop code. There are also publications by Chris Tindle (for example [9]) on the topic. If the beam and time displacement were activated in BellhopDRDC it would extend its frequency range because it would enable a more realistic treatment of low frequency bottom interactions.

5. Dr. Porter has indicated to the author that he has finished a 3D version of Bellhop that will be released in 2013. It would be interesting to experiment with this model, to determine the changes in transmission loss it would predict in Nova Scotia’s littoral waters that feature a lot of azimuthally changing topography as compared to using the 2D model.
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The Bellhop Gaussian Beam solution for acoustic propagation in an underwater environment is a complicated function of the environmental parameters of sound speed, boundary reflectivity and water depth. One of the purposes of this paper is to extend the previous derivations for the derivative with respect to sound speed uncertainty of the Gaussian Beam solution contained in McCammon, DRDC Atlantic CR 2012-100, to treat range dependence in bathymetry and bottom losses. In addition, the statistical solution for the variance of the intensity is augmented by providing for partial correlation of the sound speed uncertainties in near surface layers. In all cases tested, it is shown that the Gaussian function is the dominant term in the TL variance, and the largest variances occur whenever the ray has its turning point very near a sound speed gradient change. The derivations for the derivatives were made assuming a c-linear sound speed profile segments, so this report also examines an n²-linear sound speed profile and finds that the variances from both profile forms are qualitatively the same. Finally, the effect of Gaussian width choice on TL variance is studied.

La solution à faisceaux gaussiens de la propagation acoustique dans un milieu sous-marin du modèle Bellhop est une fonction compliquée des paramètres environnementaux que sont la vitesse du son, la réflexivité des limites et la profondeur de l’eau. Un des buts du présent article est d’étendre les calculs de la dérivée par rapport à l’incertitude de la vitesse du son de la solution à rayons gaussiens déjà faits dans le rapport CR 2012-100 de RDDC Atlantique (McCammon) afin de tenir compte de la variation de la profondeur et de la perte au fond en fonction de la distance. De plus, la solution statistique pour la variance de l’intensité est améliorée afin de tenir compte de la corrélation partielle des incertitudes de la vitesse du son dans les couches près de la surface. Dans tous les cas soumis à l’essai, on démontre que la fonction gaussienne est le terme dominant de la variance de la perte de transmission et que les plus grandes variances se produisent chaque fois que le rayon change de direction très près d’un changement de gradient de vitesse du son. Dans le précédent rapport, les dérivées ont été calculées en supposant un profil de vitesse du son dont les segments sont linéaires par rapport à c. Le présent rapport examine donc un profil de vitesses du son dont les segments sont linéaires par rapport à n², ce qui permet de constater que les variances des deux formes de profil sont qualitativement identiques. Finalement, l’effet de la largeur du faisceau gaussien sur la variance de la perte de transmission est étudié.