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Rational Numbers and Three Pulse PRI Estimation

Dimitrios Prodanos

Defence R&D Canada

TECHNICAL MEMORANDUM
DREO TM 2000-051
September 2000

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Rational Numbers and Three Pulse PRI Estimation

Dimitrios Prodanos
Electronic Support Measures Section

TECHNICAL MEMORANDUM
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Abstract

The pulse repetition interval (PRI) of a radar is an important parameter that can be used to classify the radar, identify the emitter type, and ascertain the current operating mode. There are many situations where only a limited number of radar pulses are detected, resulting in a sparse train of radar pulses. The theory of rational approximation provides one means of accurately determining the PRI of a sparse pulse train.

Two useful representations of the rational numbers are the Stern-Brocot tree and the Farey series The Stern-Brocot tree is a binary tree containing all of the rational numbers in lowest terms. The Farey series is an ordering of all fractions with denominators less than a certain size. These two representations are useful when three pulses are used to provide an initial estimate of the PRI. Using a Gaussian time of arrival error model, this work shows how the Stern-Brocot tree and the Farey series can be used to quickly determine the most likely PRI estimates.

Résumé

L'intervalle de répétition des impulsions (IRI) d'un radar est un paramètre important qui peut être exploité pour classifier le radar, identifier le type d'émetteur, et déterminer le mode d'opération courant. Dans bien des cas, on ne détecte qu'un nombre limité d'impulsions radar, ce qui ne donne qu'un train clairsemé d'impulsions La théorie de l'approximation par des nombres rationnels fournit une méthode pour déterminer correctement l'IRI dans un train clairsemé d'impulsions.

L'arbre de Stern-Brocot et la série de Farey sont deux représentations utiles des nombres rationnels. L'arbre de Stern-Brocot est un arbre binaire qui contient tous les nombres rationnels sous leur forme réduite. La série de Farey est une mise en ordre de toutes les fractions qui ont un dénominateur inférieur à une valeur déterminée. Ces deux représentations sont utiles lorsque trois impulsions sont utilisées pour faire une estimation de l'IRI. Avec l'aide d'un modèle gaussien de l'erreur sur le temps d'arrivée, nous montrons comment l'arbre de Stern-Brocot et la série de Farey peuvent être employés pour en arriver rapidement à une estimation de l'IRI le plus probable.

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Executive summary

The pulse repetition interval (PRI) of a radar is an important parameter that can be used by an electronic support measures (ESM) system to classify the radar, identify the emitter type, and ascertain the current operating mode. The PRI is also used to deinterleave overlapping pulse trains from multiple radars.

There are many situations where only a limited number of radar pulses are detected, resulting in a sparse train of radar pulses. Constraints in ESM system design often require a scanning architecture that must search through a large volume of space, such as when a spinning antenna is used, or when a single receiver must multiplex between many antennas. An ESM system must also scan through a wide range of frequency in order to detect radar emitters. Scanning in space and/or frequency reduces the probability that any particular radar pulse will be detected. Many radar systems must also scan in space, creating a scan-on-scan effect which further reduces the probability of pulse detection.

The capability to accurately determine the PRI of sparse and noisy pulse trains increases the range of situations under which the PRI can be used to reliably classify and identify a threat radar. This capability can also improve deinterleaving performance, since it is robust against missing pulses.

Three pulses are sufficient to provide an initial estimate of the PRI of a sparse and noisy pulse train. Using only three pulses increases the number of ambiguities (i.e., other estimates that are just as probable). As more pulses are considered, the ambiguities are resolved and the accuracy of the PRI measurement is improved. In ESM systems that use an initial three pulse PRI estimate, it is important that all ambiguous estimates are retained so that the correct value is not prematurely removed from further consideration. In this way, the final PRI measurement that is presented to the ESM operator is more likely to be the correct one.

Two useful approaches to three pulse PRI estimation are the Stern-Brocot tree and the Farey series. This work introduces these representations of the rational numbers and presents some of their main properties. It also shows how they can be used to quickly determine all of the ambiguous PRI estimates. Future work will explore techniques that use additional pulses to select the correct PRI estimate.

Dimitrios Prodanos, "Rational numbers and three pulse PRI estimation," Defence Research Establishment Ottawa, Technical Memorandum 2000-051, September 2000.

Sommaire

L'intervalle de répétition des impulsions (IRI) d'un radar est un paramètre qui peut être exploité par un système de Mesures de Soutien Électronique (MSE) pour classifier le radar, identifier le type d'émetteur et déterminer le mode d'opération courant. L'IRI est employé aussi pour désentrelacer les trains d'impulsions de plusieurs radars, trains qui peuvent se chevaucher.

Dans bien des cas, on ne détecte qu'un nombre limité d'impulsions radar, ce qui ne donne qu'un train clairsemé d'impulsions. Les contraintes auxquelles est soumise la conception d'un système MSE exigent souvent une architecture exploitant une technique de balayage qui doit chercher dans un grand volume, par exemple, lorsqu'on utilise une antenne tournante, ou lorsqu'un seul récepteur doit être multiplexé pour servir plusieurs antennes. Un système MSE doit aussi balayer un large spectre de fréquences pour détecter les émetteurs radar. Le balayage en espace et/ou en fréquence diminue aussi la probabilité de détecter l'impulsion d'un radar spécifique. Plusieurs systèmes radar doivent aussi balayer en espace, produisant un effet de balayage-sur-balayage, ce qui réduit d'autant plus la probabilité de détecter une impulsion.

La capacité de déterminer avec précision l'IRI de trains d'impulsions clairsemés et entachés de bruit augmente le nombre de situations où l'IRI peut être employé pour classifier et identifier de façon fiable un radar ennemi. Cette capacité peut aussi améliorer la qualité du désentrelacement, puisqu'elle offre une bonne résistance à l'erreur lorsqu'il manque des impulsions.

Trois impulsions sont suffisantes pour faire une estimation initiale de l'IRI d'un train d'impulsions clairsemé et entaché de bruit. N'utiliser que trois impulsions augmente le nombre d'ambiguïtés (i.e. d'autres estimations qui sont aussi probables) À mesure que plus d'impulsions entrent en considération, les ambiguïtés sont résolues et la précision des mesures de l'IRI s'améliore. Pour les systèmes MSE qui utilisent une estimation initiale de l'IRI à l'aide de trois impulsions, il est important que toutes les estimations ambiguës soient conservées pour ne pas rejeter prématurément la valeur correcte. De cette façon, il est plus probable que la mesure finale de l'IRI, qui est présentée à l'opérateur MSE, soit celle qui est la bonne

L'arbre de Stern-Brocot et la série de Farey sont deux approches utiles pour l'estimation de l'IRI à l'aide de trois impulsions. Les travaux rapportés ici présentent ces représentations des nombres rationnels et quelques unes de leurs principales propriétés. On montre aussi comment ces approches peuvent être utilisées pour déterminer rapidement toutes les estimations ambiguës de l'IRI. Les travaux futurs exploreront des techniques qui emploieront des impulsions supplémentaires pour en arriver à une estimation correcte de l'IRI.

Dimitrios Prodanos, "Nombres rationnels et estimation de l'intervalle de répétition des impulsions à l'aide de trois impulsions," Centre de recherches pour la défense Ottawa, Technical Memorandum 2000-051, Septembre 2000. (en anglais)

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1 Introduction

1.1 Radar pulse trains

Electronic Support Measures (ESM) is the theory and practice of monitoring the electromagnetic spectrum in order to provide information on the location and nature of both friendly and hostile emitters. Radars are one important class of emitters that must be detected and analysed. They typically emit a periodic sequence of radio frequency (RF) pulses. Estimating the periodicity of these sequences is an important discriminator between different radars.

The simplest periodic pulse train occurs when the time between two successive pulses, known as the pulse repetition interval (PRI), is a constant value. This results in a highly regular series of time of arrival (TOA) measurements within an ESM system. Depending upon how noisy the TOA measurements are, a highly accurate estimate of the radar PRI can be made. This estimate can often be used to classify the radar, identify the emitter type, and ascertain its current operating mode.

More complex sequences of radar pulses are also possible, making it more difficult to estimate the PRI. In fact, some radars generate random intervals between successive pulses. However, it is possible to derive the PRI of a radar in certain cases. One such case is when the sequence of TOA measurements results from incomplete detection of a constant PRI pulse train. If a large proportion of radar pulses are missing, the now sparse pulse train can appear to have a very complex PRI pattern, confusing conventional ESM systems. This report explores advanced techniques to determine the underlying PRI of such a sparse pulse train.

1.2 Sparse pulse trains

There are many situations where only a limited number of radar pulses are detected, resulting in an incomplete set of TOA measurements. Constraints in ESM system design often require a scanning architecture that must search through a large volume of space, such as when a spinning antenna is used, or when a single receiver must multiplex between many antennas. An ESM system must also scan through a wide range of RF in order to detect a radar emitter. Scanning in space and/or frequency reduces the probability that any particular radar pulse will be detected and its TOA measured. Many radar systems must also scan in space, creating a scan-on-scan effect which further reduces the probability of pulse detection.

An incomplete or sparse set of TOA measurements of a constant PRI pulse train t_i can be represented by,

$$t_i = t_0 + k_i T + \varepsilon_i,$$

where period T is the PRI, and t_0 is a constant offset. The ε_i represent possible measurement errors. The integers k_i represent the index of the detected pulse. For example, if only the first, third, and eighth pulses are detected by the ESM system, then $k_1 = 1$, $k_2 = 3$, and $k_3 = 8$. The k_i are usually unknown and must be estimated along with the underlying period T.

It is possible to guide the estimation of the period T and the integers k_i with prior information. Knowledge of what radars are likely to be detected will influence the estimation of the period T. Even with no prior information of what radars will be encountered, basic radar theory will somewhat constrain the possible values for T. The integers k_i will depend upon the various scans being performed by the radar and/or ESM receiver and the resulting probabilities of pulse detection. With no prior information, it is easiest to assume that all integer values are equally likely, although this can lead to unrealistic results. Some limits are necessary to ensure reasonable estimates of the period that are not vanishingly small.

Given a set of noisy measurements t_i, the problem is to simultaneously estimate the underlying PRI or period T, the integers k_i, and the offset t_0. This is not a well defined problem, since it is invariant with respect to any estimates which leave the terms $t_0 + k_i T$ fixed. For example, halving the period T and doubling the integers k_i results in the same measurements t_i. Even with some reasonable limits based on limited prior information, there may exist many such ambiguities, or parameter estimates that are equally likely. Furthermore, as measurement errors increase, the number of ambiguities increase.

A number of approaches to this estimation problem have been explored [1, 2]. All of these fundamentally rely on the theory of approximation by lower order (smaller denominator) rational numbers. Most previous work is well suited to finding best estimates, but is less effective at finding all of the ambiguous estimates.

While searching for all ambiguous estimates is more computationally intensive than finding a single best estimate, it is a requirement when there is a good chance that the best estimate is not the correct one. This is often the case when the ambiguous estimates are almost as reasonable as the best estimate.

1.3 Deinterleaving of sparse pulse trains

When many radars are present, it becomes less likely that successive TOA measurements belong to the same pulse train. These simultaneous pulse trains must be deinterleaved in order to obtain an estimate of the PRI. When only the PRI can be used to deinterleave (because there are no other discriminating parameters), a chicken and egg problem results. In this case, deinterleaving and PRI estimation occur simultaneously. Optimal approaches that identify all likely possibilities are computationally intensive. Less optimal approaches are typically used since they can be implemented in real time.

An optimal approach to deinterleaving must guarantee to find the most likely grouping of pulses. This is complicated by the fact that there may be ambiguities, which are groupings that are equally likely. In addition, other pulse groupings may be almost as likely, and need to be considered. Searching through all likely candidate pulse groupings is a combinatorial problem, with a computational cost that increases exponentially with the total number of pulses being deinterleaved.

Most suboptimal approaches involve selecting a very limited number of candidate pulses from a buffer of detected pulses. It is then assumed that these candidates all belong to the same pulse train, and their PRI is estimated. The remaining TOA measurements are compared with this hypothesis, and if enough other pulses are consistent, a pulse train is declared. This is a trial and error approach that is relatively fast, but its performance is difficult to evaluate.

This suboptimal approach achieves most of its speed by removing each declared pulse train from the search process. As soon as a train is declared, the pulses belonging to that train are removed and not considered in the subsequent search of the pulses remaining in the buffer. With fewer pulses, the next pulse train takes less time to declare, and so on until only the residual pulses that could not be assigned remain However, once pulses are first assigned to a pulse train, they are not available for other, potentially more likely pulse groupings. Furthermore, only one set of pulse trains is generated using this approach, even when there are many different pulse groupings that are almost as likely.

Two pulses are sufficient to estimate the PRI of a constant PRI pulse train. In the case of a sparse pulse train where many pulses are missing, three or more pulses are necessary. More pulses improve performance, but also add to the computational cost. This report will focus upon initial PRI estimation based upon three pulses.

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2 Three pulse PRI estimation

2.1 Gaussian error model

We desire to extract the period T and associated integer counts k_i from the time of arrival measurements t_i,

$$\begin{aligned} t_1 &= t_0 + k_1 T + \varepsilon_1, \\ t_2 &= t_0 + k_2 T + \varepsilon_2, \\ t_3 &= t_0 + k_3 T + \varepsilon_3, \end{aligned}$$

where t_0 is a constant offset and the ε_i are independent, identically distributed measurement errors.

In order to eliminate the constant offset, we can calculate the time difference of arrival (TDOA),

$$\begin{aligned} d_1 &= t_2 - t_1 = (k_2 - k_1)T + \varepsilon_2 - \varepsilon_1 = mT + e_1, \\ d_2 &= t_3 - t_2 = (k_3 - k_2)T + \varepsilon_3 - \varepsilon_2 = nT + e_2, \end{aligned}$$

where m and n are integer counts of the respective differences. The errors e_1 and e_2 are no longer independent, since they both depend upon ε_2. If we assume the original errors ε_i are independent and have Gaussian distributions with mean $E(\varepsilon_i) = 0$ and variance $E(\varepsilon_i^2) = \sigma^2$, then the errors e_1 and e_2 also have Gaussian distributions with,

$$E(e_1) = E(e_2) = 0, \quad E(e_1^2) = E(e_2^2) = 2\sigma^2, \quad E(e_1 e_2) = -\sigma^2.$$

Using this, we can determine the probability p of making an observation (d_1, d_2) given parameters (m, n, T),

$$\begin{aligned} p &= \frac{\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}^{-1/2}}{2\pi\sigma^2} \exp\left(\frac{\begin{pmatrix} d_1 - mT \\ d_2 - nT \end{pmatrix}^T \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} d_1 - mT \\ d_2 - nT \end{pmatrix}}{-2\sigma^2} \right) \\ &= \frac{1}{2\sqrt{3}\pi\sigma^2} \exp\left(\frac{(d_1 - mT)^2 + (d_2 - nT)^2 + (d_1 - mT)(d_2 - nT)}{-3\sigma^2} \right) \end{aligned}$$

2.2 Most likely estimate

Given the TDOA measurements d_1 and d_2, we are interested in finding the most likely estimates of the period T and the integers m and n. Assuming that all values of m, n, and T are equally likely (perhaps within a region), then the most likely estimates are those that maximize the probability p for the given measurements. Unfortunately, this is not a well defined problem, since many choices of the parameters m, n, and T will give the same values of mT and nT, and hence leave the probability p unchanged. However, some conclusions can still be drawn.

The value of the period T which maximizes the probability p is given by,

$$T = \frac{2d_1 m + 2d_2 n + d_1 n + d_2 m}{2(m^2 + n^2 + mn)},$$

which gives a probability of,

$$p = \frac{1}{2\sqrt{3}\pi\sigma^2} \exp\left(\frac{(d_1 n - d_2 m)^2}{-4\sigma^2(m^2 + n^2 + mn)} \right).$$

It is more difficult to maximize the probability p with respect to the integers m and n, since they are not continuous. However, it is possible to consider the probability p as a function of the ratio m/n,

$$p = \frac{1}{2\sqrt{3}\pi\sigma^2} \exp\left(\frac{(d_1 - d_2(m/n))^2}{-4\sigma^2(1 + (m/n) + (m/n)^2)} \right). \quad (1)$$

This expression makes clear the fact that many different values of the integers m and n can maximize the probability p. However, as a function of m/n, the probability p is maximized only when $m/n = d_1/d_2$, as one would expect.

In a previous example, we chose $k_1 = 1$, $k_2 = 3$, and $k_3 = 8$. This should lead to $m = 2$, $n = 5$, and, assuming a period of $T = 100$, $d_1 = 200$ and $d_2 = 500$. If, assuming an error variance of 1, our actual measurements lead to TDOA values of,

$$d_1 = 198.4063, \quad d_2 = 498.559,$$

then the probability p tells us how likely other values of m and n are. This is shown in Figure 1 as a function of m/n, with the probability p normalized such that the maximum at d_1/d_2 is equal to 1. Notice that the fraction $19/48 = 0.3958$ is just as likely as $2/5 = 0.4$, which is the value we would hope to find. In fact, limiting ourselves to denominators of less than 50, the fractions,

$$\frac{16}{41}, \frac{9}{23}, \frac{11}{28}, \frac{13}{33}, \frac{15}{38}, \frac{17}{43}, \frac{19}{48}, \frac{2}{5}, \frac{19}{47}, \frac{17}{42},$$

all lie between $m/n = 0.39$ and $m/n = 0.405$, and are hence reasonable estimates of the integers m and n.

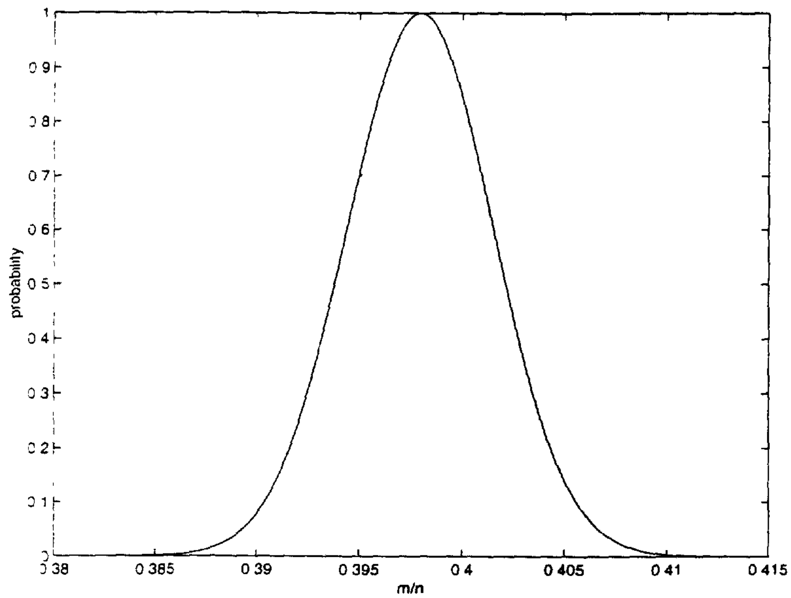

Figure 1: Probability of m/n with $d_1/d_2 = 0.398$

Figure 1 shows that the best estimate of the period T is obtained by choosing integers m and n such that $m/n = d_1/d_2$. This suggests that the overall estimation problem can be reduced to finding good rational approximations to the ratio d_1/d_2 How quickly the probability p falls off as m/n moves away from d_1/d_2 gives a measure of how close these approximations need to be in order to generate likely estimates

The probability p provides a measure of the goodness of an estimate, but does not provide a method of determining good estimates. The following sections describe some representations of the rational numbers, and how these representations can be used to obtain a set of likely candidate integers m and n. The basic idea is to use the probability p to determine the smallest and largest values for m/n, and then use representations of the rational numbers to quickly find fractions within that range

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3 Rational approximation

3.1 Stern-Brocot tree

The Stern-Brocot tree is one way of constructing the set of all nonnegative fractions m/n in lowest terms (i.e., the greatest common divisor of m and n is one, or $\gcd(m,n) = 1$). It was discovered independently by Moriz Stern, a German mathematician, and Achille Brocot, a French clockmaker. Many of its properties are given in [3]; only those that are relevant will be discussed here.

While the entire Stern-Brocot tree is infinite in size, it can be built up from smaller, finite trees. This is done by inserting the *mediant* fraction $(m+m'/n+n')$ between the adjacent fractions m/n and m'/n'. Starting from the two fractions $(\frac{0}{1}, \frac{1}{0})$, although the latter is technically undefined, their mediant is added to obtain,

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{0}.$$

By continuing to construct mediant fractions, two more are added,

$$\frac{0}{1}, \frac{1}{2}, \frac{1}{1}, \frac{2}{1}, \frac{1}{0},$$

and then four,

$$\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}, \frac{3}{2}, \frac{2}{1}, \frac{3}{1}, \frac{1}{0}.$$

and then 8, 16, and so on At every stage the number elements in the tree is approximately doubled. If the initial two fraction tree is denoted by zero, then there are $2^k + 1$ elements at stage k

It is easy to show that if $m/n < m'/n'$ and if all values are nonnegative, then the new mediant fraction satisfies

$$m/n < (m+m')/(n+n') < m'/n'.$$

This proves that the Stern-Brocot tree preserves order. It also guarantees that a fraction cannot appear in more than one place.

To show that every fraction is in lowest terms, use is made of the fact that at every stage of the construction,

$$m'n - mn' = 1, \tag{2}$$

whenever m/n and m'/n' are consecutive fractions. Since the greatest common divisor of m and n must also divide the left hand side of (2), it must divide 1. This is only possible if the greatest common divisor is equal to 1, making m and n relatively prime. Similarly for m' and n'. To prove (2), it must be shown that (2) is true initially ($1 \cdot 1 - 0 \cdot 0 = 1$), and is also true whenever a new mediant is added:

$$(m+m')n - m(n+n') = m'n - mn' = 1.$$
$$m'(n+n') - (m+m')n' = m'n - mn' = 1.$$

It has almost been verified that the Stern-Brocot tree is a number system for representing the (positive) rational numbers. All that remains to be shown is that every nonnegative fraction appears once. A proof of this appears in [3]. Thus, the Stern-Brocot tree can be considered as a binary tree representation of the rationals. The initial levels of this structure are shown in Figure 2.

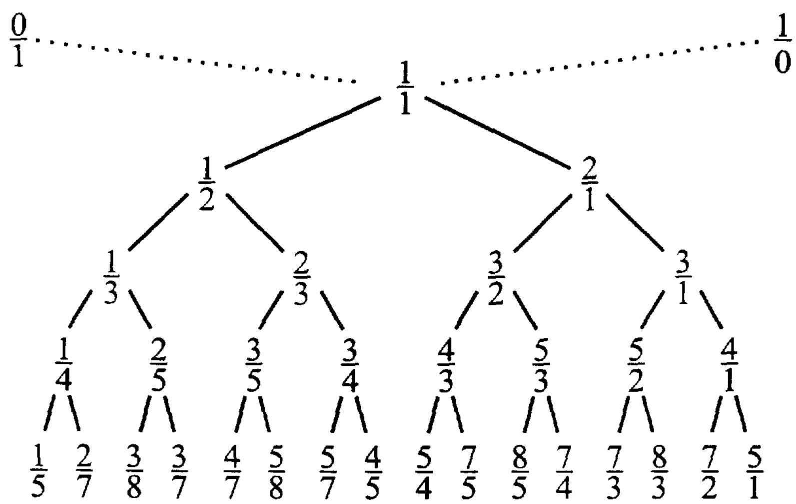

Figure 2: Stern-Brocot Tree

The Stern-Brocot tree allows for the fast approximation of both rational and real numbers The number to be approximated is compared with the initial node, and the left or right fork is chosen. This fork determines the subsequent node for comparison. This process of moving down the tree can be continued, each subsequent node providing a more accurate approximation

To formalize this observation, a 2×2 matrix can be used to represent each node in the Stern-Brocot tree. Each node is itself the mediant of two higher nodes in the tree. These two fractions m/n and m'/n' form the left and right columns of the matrix representation:

$$\begin{pmatrix} m & m' \\ n & n' \end{pmatrix}$$

A downward step to the right involves replacing the left column with the mediant fraction $(m+m')/(n+n')$. Similarly, a downward step to the left involves the replacement of the right column. This behaviour can be captured using matrix multiplication. To do this, define the following three matrices:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

The matrix M defines the top level node of the Stern-Brocot tree. Right multiplication (recall that matrices can be multiplied from the left or the right with different results) by the matrix L moves one node down and to the left, while R moves down and to the right. Thus any node in the tree, and hence any rational number, can be represented by an initial M right multiplied by a number of L and R matrices. For example,

$$MLRRL = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix},$$

represents the fraction $(2+3)/(3+4) = 5/7$, which is exactly what one obtains after moving left, right, right, and then left from the top level node.

Real numbers can be approximated to any accuracy desired by moving far enough down the tree. Approximations of rational numbers necessarily halt when the actual number is itself reached.

3.2 Navigating the Stern-Brocot tree

Using this representation, the Stern-Brocot tree can be easily navigated. Any real number can be approximated to whatever accuracy is desired by moving far enough downwards through the tree. Given any node in the tree, neighbouring fractions can easily be determined. One can also move back up the tree, determining lower order approximations.

As has been previously discussed, any node in the Stern-Brocot tree, and hence any rational number, can be represented by a 2×2 matrix. This matrix is itself the product of an initial matrix M (which represents the top level node of the Stern-Brocot tree), followed by a number of matrices L and R, which represent moving left and right down the tree. Thus, any rational number can be represented as a character string consisting of symbols "M", "L", and "R".

To actually calculate the fraction that corresponds to any given string representation, the matrices M, L, and R must be used. They are then multiplied in the same order as the string representation. For example, the fraction corresponding to the string "MLRRL" can be obtained by calculating the matrix $MLRRL$. This matrix was previously shown in Section 3.1 to represent the fraction $5/7$.

The inverse procedure of calculating the string representation given a particular fraction m/n requires a binary search through the Stern-Brocot tree. This is easiest if the matrix representation S is maintained while generating the string representation. Starting with $S = M$, the fraction m/n is compared to the fraction corresponding to S. If m/n is smaller, append "L" to the string representation and set $S = SL$. Otherwise, append "R" and set $S = SR$. The process continues until the fraction corresponding to S is equal to m/n.

For example, to find the correct representation of the fraction $5/7$, start with the top level node $1/1$ which is represented by the matrix $S = M$. Since $5/7$ is less than $1/1$, move down to the left, appending "L" to the initial "M" and setting

$$S = ML = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},$$

which represents the fraction $(0+1)/(1+1) = 1/2$. Since $5/7$ is greater than $1/2$, the next step down the tree is to the right, so append "R" to the string representation to give "MLR" so far. It is also necessary to set

$$S = SR = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix},$$

which represents the fraction $(1+1)/(2+1) = 2/3$. This is still less than $5/7$, so append another "R" and set

$$S = SR = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix},$$

to obtain "MLRR" and the fraction $3/4$. Since this is greater than $5/7$, the next step down the tree is to the left, appending "L" and setting

$$S = SL = \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}.$$

This represents the initial fraction $5/7$, so the process is halted, obtaining the string representation "MLRRL" as expected.

Unlike the matrix representation of a rational number, the character string representation contains information that correctly places that rational number within the Stern-Brocot tree. One can move down the tree by appending symbols "L" and "R" as well as move up the tree by taking away those symbols. The current matrix representation can be updated with right matrix multiplication, using either the matrices L and R to move down the tree, or their inverses if to move upwards.

For example, to find lower order approximations to $5/7$, one can simply truncate its string representation "MLRRL". To ensure that the approximation is either smaller (larger) than $5/7$, truncation is repeated until the first "R" ("L") symbol is removed. This means that "MLRR"$= 3/4$ is the next larger approximation to $5/7$, and that "MLR"$= 2/3$ is the next smaller approximation. To look for higher order fractions by moving down the Stern-Brocot tree, the string representation must be extended. A larger fraction is found by appending "RLLL..." to whatever resolution is desired. The string "MLRRLRLLL"$= 23/32$ is a fraction larger than $5/7$, no other fraction with a denominator less than 32 lies between these two fractions. These two techniques are useful in generating two or more consecutive fractions

3.3 Farey series

The Farey series is closely related to the Stern-Brocot tree and provides another means of representing the rational numbers. It is not as useful for generating successively closer approximations, but a recursive formula exists to generate all fractions of a certain resolution (which loosely corresponds to depth in the Stern-Brocot tree) from just two consecutive elements in the series.

The Farey series of order N, denoted by F_N, is the set of all reduced fractions between 0 and 1 with denominators less than or equal to N, arranged in increasing order. For example, if $N = 7$,

$$F_7 = \frac{0}{1}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{1}{1}.$$

The Farey series defines a subtree of the Stern-Brocot tree, obtained by pruning off all branches with denominators greater than N, as well as all fractions greater than 1 This means that any two consecutive elements of F_N also satisfy (2).

It turns out that given two consecutive elements of F_N, the entire series can be generated. Let m/n and m'/n' be consecutive elements of F_N. Then the subsequent element m''/n'' is given by

$$\begin{aligned} m'' &= \left\lfloor \frac{n+N}{n'} \right\rfloor m' - m, \\ n'' &= \left\lfloor \frac{n+N}{n'} \right\rfloor n' - n. \end{aligned}$$

Returning to the example of F_7 and starting with $1/5$ and $1/4$,

$$\begin{aligned} m'' &= \left\lfloor \frac{5+7}{4} \right\rfloor 1 - 1 = 3 - 1, \\ n'' &= \left\lfloor \frac{5+7}{4} \right\rfloor 4 - 5 = 12 - 5, \end{aligned}$$

which gives $m''/n'' = 2/7$ as expected. This formula is now proven in general.

First, notice that $m''n' - m'n'' = m'n - mn' = 1$, since consecutive elements of F_N must satisfy (2). This implies that $m''/n'' > m'/n'$. It also implies that the greatest common divisor of m'' and n'' is 1 (the gcd divides m'' and n'' and hence divides $m''n' - m'n'' = 1$), which implies that m''/n'' is in lowest terms. To show $m''/n'' \in F_N$ it must be shown that $n'' \leq N$. This is clear from

$$N = \left(\frac{n+N}{n'} \right) n' - n \geq \left\lfloor \frac{n+N}{n'} \right\rfloor n' - n = n''$$

Also notice that m''/n'' is always defined, since

$$n'' > \left(\frac{n+N}{n'} - 1 \right) n' - n = N - n' \geq 0.$$

All that remains to be shown is that m''/n'' is the next consecutive element of F_N. If not, then it must be greater than the next consecutive element $\bar{m}/\bar{n} \in F_N$, since it was already greater than m'/n'. This implies that $m''\bar{n} - \bar{m}n'' > 0$. Using this fact, it can be shown that

$$\bar{n} = \left(m''n' - m'n''\right)\bar{n} = n'\left(m''\bar{n} - \bar{m}n''\right) + n''\left(\bar{m}n' - m'\bar{n}\right) \geq n' + n''.$$

This leads to the contradiction that $\bar{n} > N$, since it has been previously shown that $n'' > N - n'$. Hence, m''/n'' must itself be the next consecutive element in F_N.

There is a similar formula for obtaining the previous element. Let m'/n' and m''/n'' be consecutive elements of F_N. Then the previous element m/n is given by

$$\begin{aligned} m &= \left\lfloor \frac{n''+N}{n'} \right\rfloor m' - m'', \\ n &= \left\lfloor \frac{n''+N}{n'} \right\rfloor n' - n''. \end{aligned}$$

The proof closely follows that already given for the next consecutive element, and will not be repeated here.

These results can be extended to fractions greater than 1. Indeed these formulas can be directly applied, since nowhere in their proofs is it required that numerators be less than denominators. However, the proofs do require positive numerators and denominators, so these formulas cannot be directly extended to negative fractions.

3.4 Application to three pulse PRI estimation

It has been previously remarked that finding good PRI estimates relies on finding good rational approximations. This section will briefly describe how the Stern-Brocot tree and the Farey series can be used to more rapidly estimate the PRI of three radar pulses.

Recall that two time difference of arrival measurements can be calculated from three time of arrival measurements,

$$\begin{aligned} d_1 &= t_2 - t_1 = mT + e_1, \\ d_2 &= t_3 - t_2 = nT + e_2, \end{aligned}$$

where the period T is the PRI we would like to estimate, m and n are integers, and the e_i are error terms. Given particular measurements d_1 and d_2, and for every choice of integers m and n, there exists a most probable estimate of the period T. However, this maximum probability is higher for some choices of m and n than it is for others. We earlier derived Equation 1, which is the probability that particular values of m and n lead to measurements d_1 and d_2. Figure 1 is a plot of this probability for some particular values of d_1 and d_2.

Choosing m and n such that m/n is closer to d_1/d_2 leads to better estimates of the period T. It is in this way that estimating the PRI relies on finding good rational approximations. We seek to approximate the ratio d_1/d_2 by rational numbers m/n. The better the approximation, the better the eventual PRI estimate.

Equation 1 can be used to define what range of m/n values need to be considered. For example, it is clear from Figure 1, that choices of m and n outside the range $.39 < m/n < .405$ will lead to very unlikely estimates of the PRI. Limits can be tightened further at the risk of eliminating candidate choices that are increasingly likely.

Once these limits are determined, the Stern-Brocot tree can be used to quickly find fractions which approximate the limits. It can also be used to generate the fractions in between. The depth within the tree determines how many fractions are generated, higher resolution (larger denominator) fractions mean more fractions are generated. The techniques of Section 3.2 can be used to move through the tree, filling in the fractions between the limits. It is also possible to fill in the tree from the top, using the initial approximations of the limit points to truncate unwanted branches.

A better technique utilizes the Farey series. The Stern-Brocot tree is still used to find rational approximations of the limit ratios. From the lower limit, it is also used to generate the next larger fraction in the tree. These two fractions can then be used to recursively generate a Farey series of the desired order (maximum denominator). This is a fast way of generating all the fractions between the two limit ratios.

While the probability of Equation 1 is what determines how likely a particular choice of m and n are, it is the Stern-Brocot tree and the Farey series which provide methods to quickly find the most likely choices. Both are necessary to rapidly estimate the PRI of a three pulse radar train.

4 Conclusion

The pulse repetition interval (PRI) of a radar is an important parameter that can be used by an electronic support measures (ESM) system to classify the radar, identify the emitter type, and ascertain the current operating mode. The PRI is also used to deinterleave overlapping pulse trains from multiple radars.

There are many situations where only a limited number of radar pulses are detected, resulting in a sparse train of radar pulses. Three pulses are sufficient to provide an initial estimate of the PRI of such a pulse train. Using only three pulses increases the number of ambiguities, which are other estimates that are just as probable. As more pulses are considered, the ambiguities are resolved and the accuracy of the PRI measurement is improved. In ESM systems that use an initial three pulse PRI estimate, it is important that all ambiguous estimates are retained so that the correct value is not prematurely removed from further consideration. In this way, the final PRI measurement that is presented to the ESM operator is more likely to be the correct one.

Two useful approaches to three pulse PRI estimation are presented in this report, the Stern-Brocot tree and the Farey series. The Stern-Brocot tree is a binary tree containing all of the rational numbers in lowest terms. It has the usual ordering from least to greatest, and allows for rapid approximations. The Farey series of order N contains all fractions between 0 and 1 with denominators less than or equal to N, sorted by increasing size. It has the useful property that it can be recursively generated starting from just two consecutive elements. It can also be extended to fractions greater than 1.

Due to measurement errors, the probability of a time difference of arrival (TDOA) measurement given a particular PRI must be determined. A Gaussian error model was used to derive a closed form expression for the TDOA probability. Given a particular set of TDOA measurements, the best estimate of the underlying PRI is the one that maximizes this probability. Estimates that are almost as likely will also have high probabilities, and therefore cause ambiguities in PRI determination. The probability distribution for the given TDOA measurement gives a measure of how many PRI ambiguities there are.

Using this probabilistic information, the Stern-Brocot tree and the Farey series can be used to quickly determine all of the ambiguous PRI estimates. Future work will explore techniques that use more than three pulses to resolve these ambiguities and select the correct PRI estimate.

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