DEPARTMENT OF NATIONAL DEFENCE
CANADA

OPERATIONAL RESEARCH DIVISION

DIRECTORATE OF OPERATIONAL RESEARCH
(CORPORATE, AIR & MARITIME)

DOR(CAM) RESEARCH NOTE RN 9705

MAXIMUM RANK CORRELATION AS A SOLUTION CONCEPT IN THE *M* RANKINGS PROBLEM WITH APPLICATION TO MULTI CRITERIA DECISION ANALYSIS

by

E.J. Emond

SEPTEMBER 1997

OTTAWA, CANADA

National Defence Défense nationale

OPERATIONAL RESEARCH DIVISION

CATEGORIES OF PUBLICATION

ORD Reports are the most authoritative and most carefully considered publications of the DGOR scientific community. They normally embody the results of major research activities or are significant works of lasting value or provide a comprehensive view on major defence research initiatives. ORD Reports are approved personally by DGOR, and are subject to peer review.

ORD Project Reports record the analysis and results of studies conducted for specific sponsors. This Category is the main vehicle to report completed research to the sponsors and may also describe a significant milestone in ongoing work. They are approved by DGOR and are subject to peer review. They are released initially to sponsors and may, with sponsor approval, be released to other agencies having an interest in the material.

Directorate Research Notes are issued by directorates. They are intended to outline, develop or document proposals, ideas, analysis or models which do not warrant more formal publication. They may record development work done in support of sponsored projects which could be applied elsewhere in the future. As such they help serve as the corporate scientific memory of the directorates.

ORD Journal Reprints provide readily available copies of articles published with DGOR approval, by OR researchers in learned journals, open technical publications, proceedings, etc.

ORD Contractor Reports document research done under contract of DGOR agencies by industrial concerns, universities, consultants, other government departments or agencies, etc. The scientific content is the responsibility of the originator but has been reviewed by the scientific authority for the contract and approved for release by DGOR.

DEPARTMENT OF NATIONAL DEFENCE

CANADA

OPERATIONAL RESEARCH DIVISION

DIRECTORATE OF OPERATIONAL RESEARCH (CORPORATE, AIR & MARITIME)

DOR (CAM) RESEARCH NOTE RN 9705

MAXIMUM RANK CORRELATION AS A SOLUTION CONCEPT IN THE M RANKINGS PROBLEM WITH APPLICATION TO MULTI CRITERIA DECISION ANALYSIS

by

E.J. Emond

Recommended by: DW Mason, CORT

Approved by: JG Moscrip, DOR(CAM)

Directorate Research Notes are written to document material which does not warrant or require more formal publication. The contents do not necessarily reflect the views of ORD or the Canadian Department of National Defence.

OTTAWA, ONTARIO

SEPTEMBER 1997

ABSTRACT

The *m* rankings problem is the mathematical name for the committee consensus decision problem in which a number of objects or options are ranked by m equal committee members. This report documents research into a solution concept for the *m* rankings problem based on maximizing the average measure of agreement. Properties of this and other solution concepts are explored through a series of examples. The proposed solution concept compares favourably with other methods and lends itself directly to the more general multi criteria decision problem in which weights may be assigned to the individual rankings, reflecting their relative importance.

TABLE OF CONTENTS

MAXIMUM RANK CORRELATION AS A SOLUTION CONCEPT IN THE *M* RANKINGS PROBLEM WITH APPLICATION TO MULTI CRITERIA DECISION ANALYSIS

INTRODUCTION

1. This report documents current research into the *m* rankings problem which has potential application to multi criteria decision analysis. A new solution concept is proposed which is theoretically sound and allows for the inclusion of weights which model the relative importance of each ranker or ranking criterion. The method, although simple, is computationally intense. While any number of rankers or ranking criteria can be handled, the current practical limit for the number of ranked objects is eleven.

THE *M* RANKINGS PROBLEM

2. The *m* rankings problem deals with the situation in which we are presented with *m* complete rank orderings of *n* objects, with unlimited ties being allowed. Examples of objects which may be ranked are construction projects, candidates for a vacant position, statistical software packages, countries of the world, equipment proposals and breeds of dogs. Generally, any finite set of two or more elements may be ranked.

3. A rank ordering of *n* objects is a listing of the objects in which the order in the list implies preference, with the first object being the most preferred. A rank ordering provides quantitative information about the objects which is more basic or primitive than an actual measurement. When we have actual measurements on each object, we can easily determine not only the rank ordering but also the magnitude of the differences between each of the objects along whatever dimension is being considered. When we have only a rank ordering of objects we cannot determine the magnitude of differences between list members. To quote Reference 1 : "... the numerical processes associated with ranking are essentially those of counting, not of measurement."

4. As a rule, it is a mistake to use ordinal data in an analytical setting that imputes meaning to the differences between ranks beyond a simple counting of preferences. Although such analytical procedures do exist and are sometimes very useful, great care must be taken to ensure that any conclusions reached using them are sound. One of the most common violations is the unthinking and often unjustified use of ranks as if they were interval data or scores.

5. In the general *m* rankings problem, we have m complete rankings of the *n* objects. The rankings may result from the preferences of individual rankers or they may reflect different ways of prioritizing the objects. For example, a proposed list of possible new family car candidates may be ranked along several dimensions such as initial cost, comfort, maintainability, and style.

6. Although we require that each ranking includes all of the objects, unlimited ties are allowed. A tie between two or more objects in a ranking is taken to mean that the objects are equally preferred or are considered equal along the particular dimension underlying the given ranking. Thus a tie denotes a positive statement that two or more objects share equal rank rather than an implication of indifference. This subtlety will arise later with respect to the measurement of agreement between rankings.

NOTATION

7. Notation in a rank ordering context causes a certain amount of confusion. Rankers or different rankings are not a problem and are often denoted by capital letters or Roman numerals. However, it is an almost universal convention to identify the ranked objects in a given ranking by their ordinal label. This convention leads to confusion because the ordinal numbers are also used to label the various ranks. Consider for example the ranking 2,3,1,5,4. This can either mean that the first object is ranked second or that the first object is ranked third, depending on whether the list contains object labels or ranks. In this report we will avoid the problem by using the ordinal numbers only for ranks and referring to the objects themselves by subscripted letters such as O_1, O_2, and so on.

MEASURING AGREEMENT AND DISAGREEMENT BETWEEN RANKINGS

8. The basic statistical measure of agreement between two rankings is the rank correlation. In his classic work on the general theory of rank correlation, Kendall (Reference 1) provides a clear and convincing argument for the use of tau-b as the best measure of agreement between two rankings.

9. Kendall's tau-b rank correlation coefficient measures the amount of disarray between two rankings by counting the number of inversions of order between them. The reader is directed to Reference 1 for the exact formula for tau-b as well as several numerical examples.

10. "From the general point of view ... the appropriate form of coefficient, as a true measure of correlation between two sets of numbers is tau-b. For example, if we are measuring the agreement between two judges in arranging a set of candidates in order of merit (no objective order necessarily existing) we should use tau-b. Both judges may be wrong in relation to some objective order, and they may disagree with other judges, but that is not the point. We are measuring their agreement, not their accuracy."

11. Other possibilities for measuring agreement between rankings are Kendall's tau-a and Spearman's rho. Tau-a is used to measure agreement of a ranking with a known objective ranking and is therefore not appropriate in those cases where no 'correct' ranking exists. Spearman's rho is the rank equivalent of the product moment correlation coefficient commonly used with continuous random variables. It treats the ranks as if they were scores and sums the squares of the rank differences. As noted above, this treatment of rank differences as if they were continuous variates is theoretically questionable and inappropriate for preference data. Despite this drawback, Spearman's rho is simple to calculate and usually gives results which are similar to tau-b. It therefore continues to be used in many statistical packages. For a complete mathematical description of tau-a, tau-b, and Spearman's rho rank correlation coefficients, the reader is again directed to Reference 1.

12. Tau-b has the usual properties of a correlation coefficient. It has a maximum value of 1.0 when the two rankings are identical and a minimum of -1.0 when the two rankings are perfectly reversed. A value for tau-b of zero indicates the two rankings are uncorrelated as they would be if one or both was simply a random permutation of the integers. Tau-b is calculated by considering the relative order of each pair of objects in the two rankings. It is therefore an appropriate coefficient for rank data since it depends only on the sign of the rank differences and not on the magnitude. When ties are present, tau-b is normalized so that the maximum and minimum values of 1.0 and -1.0 are attainable.

13. As mentioned earlier, ties between two or more objects in a ranking are considered to be positive information that the objects are equally preferred or are equivalent according to the underlying ranking criterion. By convention, objects which are tied are assigned a rank equal to the simple average of the untied ranks. For example, if a ranking has two objects tied for first place, both would be assigned a rank of 1.5. It is important to note that this convention does not affect the value of tau-b in the sense that the use of a value of 1.6 instead of 1.5 would leave tau-b unchanged. The same cannot be said for Spearman's rho.

14. Finally, it is necessary to briefly discuss the case in which all objects in a ranking are tied. In accordance with our view of the meaning of ties, we assign a value of 1.0 to the correlation of such a ranking with itself. Note that in this case there is no reverse ranking which attains the value of -1.0. For all other rankings (with at least one non-tie), the correlation with the ranking with every object tied is zero. This result is a direct consequence of the definition of tau-b. It should be noted that these conventions result in no confusion or analytical difficulty in practice. Note that for Spearman's rho, the correlation of a ranking consisting of all ties with itself is also 1.0. However the Spearman correlation of such a ranking with any other ranking (with at least one non-tie) is a value between 0.5 and 1.0, depending on the number of ties in the second ranking.

<u>Example 1</u>

15. To illustrate the above discussion and as a vehicle to understand the motivation for a new solution concept for the *m* rankings problem, consider the following example taken from Reference 2 which describes the Consensus Decision Support Program (CDSP) algorithm for solving the *m* rankings problem. This example involves the preference ordering of six objects by $m=3$ rankers, denoted here by R1, R2, and R3.

Ranking	Preference Ordering
R1	O_2 O_5 O_4 O_3 O_1 O_6
R2	O_6 O_4 O_3 O_1 O_5 O_2
R3	O_3 O_4 O_5 O_2 O_1 O_6

16. For this example the CDSP algorithm, using the concept of majority coalitions, produces the following consensus ranking.

CDSP Solution: O_4 O_3 O_5 O_2 O_1 O_6

17. It is of interest to calculate the level of agreement between the solution ranking and the three individual input rankings in order to evaluate the degree of consensus. In this case the value of tau-b for the CDSP solution ranking and each of the input rankings is as follows.

Ranking	Correlation Coefficient (With CDSP Ranking)
R1	0.33
R2	0.07
R3	0.87

18. Based on the above values and as shown by inspection of the original R1, R2 and R3 rankings, the CDSP solution is positively correlated with all three rankings and agrees most closely with ranking R3 and least closely with ranking R2.

19. A natural question which arises based on the above discussion is whether there exists a solution ranking which gives a higher level of agreement with the input rankings as measured by the average value of the rank correlation. In order to answer this question we first need to generate all possible solution rankings which in this case number 4683 when ties of all orders are allowed. (Annex A discusses the technical problem of finding all possible solution rankings in general.) We then calculate the sum of the three rank correlation coefficients for each of the 4683 possibilities and find that solution (or solutions) which gives the highest total. For the above example, it turns out that the CDSP ranking is the unique ranking which gives maximum agreement as measured by the average rank correlation with the input rankings.

MAXIMUM AVERAGE TAU-B AS A SOLUTION CONCEPT

20. It is at once apparent that the idea of finding the ranking which produces the highest possible average value of rank correlation with the input rankings is a very attractive solution concept for the *m* rankings problem. In Reference 1 Kendall proves that a solution ranking based on the sum of the ranks for each object in the input rankings has the property that it maximizes the average value of Spearman's rho. However, this property is only true for the rather restricted case in which ties in the solution ranking are not allowed. As will be seen in the examples below, this restriction is very severe. In addition, Spearman's rho suffers from theoretical issues due to the inappropriate use of ranking data as if it were measurement data.

21. The idea of finding the ranking or rankings which produce the maximum average value of tau-b with the input rankings is doubly attractive since it is conceptually simple and theoretically sound in that tau-b is based only on ranking information. The following examples will illustrate the use of this new solution concept and compare the obtained solutions with other methods.

Example 2

21. This example is taken from Reference 1 and involves three rankings of eight objects.

Ranking	Preference Ordering
R1	O_3 O_2 O_6 O_1 O_7 O_5 O_4 O_8
R2	O_3 O_2 O_7 O_5 O_6 O_4 O_1 O_8
R3	O_7 O_3 O_6 O_2 O_5 O_4 O_1 O_8

22. Kendall suggests the following solution, based on the rank sums and therefore maximizing the (restricted) value of Spearman's rho.

Kendall Solution: O_3 O_2 O_7 O_6 O_5 O_1 O_4 O_8

23. The CDSP solution to this problem is as follows. Note the similarity to Kendall's solution, the only difference being a reversal of the final ordering of options O_1 and O_4.

CDSP Solution: O_3 O_2 O_7 O_6 O_5 O_4 O_1 O_8

24. By evaluation of all 545,835 possible rankings, the solution ranking based on maximum average tau-b rank correlation with the input rankings was obtained. Note that this ranking again is very similar to both the Kendall and CDSP solutions except for the occurrence of ties between options O_2 O_6 and O_7 and between options O_1 and O_4.

Max. Avg. Tau-b Solution: O_3 O_2-O_6-O_7 O_5 O_1-O_4 O_8

25. Finally, in order to illustrate the fact that Kendall's rank sum solution does not maximize the unrestricted average value of Spearman's rho with the input rankings, all possible rankings were again evaluated to determine the unrestricted Spearman's rho solution. In this particular case, the solution ranking is the same as the maximum average tau-b solution except for the placement of option O_6 behind the tied options O_2 and O_7.

Max. Unrestricted Spearman's Rho Solution: O_3 O_2-O_7 O_6 O_5 O_1-O_4 O_8

26. The maximum average tau-b solution in this example is very similar to both Kendall's solution and the CDSP solution as one would expect from any rational solution concept. Given that the CDSP and Kendall solutions disagree over the final order between options O_1 and O_4, the appearance of a tie between these two options in the maximum average tau-b solution is not surprising. However, the fact that the maximum average tau-b solution also has a tie between options O_2 , O_6 and O_7 is an indication that the final ordering between these three options is more debatable than either the Kendall or the CDSP solutions indicate.

Example 3

27. The following example is included in order to illustrate that both Kendall's rank sum solution as well as the related maximum average Spearman's rho solution are both vulnerable to one of the classic flaws of ranking methods pointed out by Kenneth Arrow in his famous work on the consistency of voting schemes. One of Arrow's axioms for a consistent ranking method is insensitivity to irrelevant alternatives. To illustrate this phenomenon, suppose we have three rankings of two objects as follows.

Ranking	Preference Ordering
R1	O_1 O_2
R2	O_1 O_2
R3	O_2 O_1

28. Inspection of the above rankings shows that rankers R1 and R2 both prefer option O_1 over option O_2, whereas ranker R3 prefers option O_2 over option O_1. It is clear that given equality of the rankers, any reasonable ranking method should produce a final ranking in this case which has option O_1 prevailing. This is the case for CDSP, Kendall's rank sum, maximum Spearman's rho, and maximum tau-b. Suppose however that ranker R3 decides to confuse the situation by insisting that irrelevant alternatives O_3 and O_4 be added to the objects to be ranked. Since these two new options are irrelevant, rankers R1 and R2 would rank them tied for last whereas ranker R3 can change the final outcome (at least in some ranking methods) by placing option O_1 at the end of his list. We would then have the following set of input rankings.

Ranking	Preference Ordering
R1	O_1 O_2 O_3-O_4
R2	O_1 O_2 O_3-O_4
R3	O_2 O_3-O_4 O_1

29. Both the CDSP and maximum average tau-b solutions for this expanded case are identical to the rankings R1 and R2, with option O_1 first, option O_2 second, followed by irrelevant options O_3 and O_4 tied for last place. However Kendall's rank sum solution and the maximum Spearman's rho solution have option O_2 first followed by option O_1, with irrelevant options O_3 and O_4 again tied for last place. This example clearly illustrates that both Kendall's rank sum solution and the maximum average Spearman's rho solution are sensitive to irrelevant alternatives and are thus undesirable as solution concepts for the m rankings problem. The CDSP solution is not sensitive to irrelevant alternatives because it is based on majority coalitions which are unaffected by the addition of irrelevant objects. Similarly the maximum average tau-b solution does not appear to be sensitive to irrelevant alternatives because it is based on preference counts.

30. The problem of irrelevant alternatives illustrates that any solution concept for the m rankings problem must be carefully examined for unacceptable properties. A very interesting example in this regard is the solution concept in which we find the ranking which maximizes the minimum value of the rank correlation coefficient with all the input rankings. This solution concept is very attractive in that it provides a solution which agrees to a maximum extent with every input ranking. Intuitively, we may label such a solution concept as 'least offensive'. However, this concept has a serious drawback when a committee has strong consensus with a small minority dissenting.

31. We will use an extreme case as an example of the problem. Suppose we have a ten person committee ($m = 10$) which must rank four options. Suppose further that nine of the ten committee members rank the options in the order O_1, O_2, O_3, O_4 and that the other member ranks the options in the reverse order, O_4, O_3, O_2, O_1. It is easy to see that any solution ranking which has positive

correlation with either of these two rankings must have negative correlation with the other. Therefore the maximum correlation must be zero and consequently the "least offensive" solution ranking will have zero correlation with both input rankings. In particular the 'all ties' ranking will be a solution, albeit one that is particularly unhelpful. This example illustrates that the 'least offensive' solution concept fails when there is a strong majority consensus and a weak minority opposite view, in that it may give a solution which is overly sensitive to the minority viewpoint.

Example 4

32. Example 4 is taken from a paper by Goddard (Reference 3) and originally came from a paper by Cook and Seiford. In this example, ten committee members have ranked 5 alternatives as follows.

Ranking	Preference Ordering
R1	O_1 O_5 O_3 O_2 O_4
R2	O_3 O_5 O_2 O_1 O_4
R3	O_5 O_2 O_1 O_4 O_3
R4	O_1 O_3 O_5 O_2 O_4
R5	O_4 O_1 O_5 O_3 O_2
R6	O_1 O_4 O_3 O_2 O_5
R7	O_2 O_5 O_4 O_1 O_3
R8	O_1 O_4 O_2 O_3 O_5
R9	O_2 O_5 O_1 O_4 O_3
R10	O_4 O_3 O_1 O_2 O_5

33. The solution obtained by Cook and Seiford for this problem is given below along with that obtained by Goddard. Note that both solutions agree except for the reversal of the final ordering of options O_3 and O_4.

Cook and Seiford : O_1 O_5 O_2 O_3 O_4

Goddard : O_1 O_5 O_2 O_4 O_3

34. The CDSP solution along with the maximum average tau-b solution are given below. The CDSP solution agrees with the Goddard solution except that options O_2 and O_5 are tied. The maximum average tau-b solution agrees with the others in that option O_1 is ranked first, however all the remaining options are tied. Given the tie between options O_2 and O_5 in the CDSP solution and the reversal of options O_3 and O_4 in the Cook and Seiford versus Goddard solutions, it is not surprising that these four options all come out in a four-way tie in the maximum average tau-b solution. In this case the maximum average tau-b solution again appears to provide a reasonable and conservative solution as compared with the others in that it declares ties where other concepts separate.

CDSP : O_1 O_2-O_5 O_4 O_3

Max. Avg. Tau-b : O_1 O_2-O_3-O_4-O_5

Example 5

35. The rankings for this example are taken from Reference 2 and relate to the ranking of 7 objects by a five member committee.

Ranking	Preference Ordering
R1	O_7 O_1 O_4 O_5 O_6 O_2 O_3
R2	O_5 O_3 O_7 O_4 O_1 O_6 O_2
R3	O_1 O_5 O_3 O_2 O_6 O_7 O_4
R4	O_6 O_3 O_7 O_4 O_1 O_5 O_2
R5	O_3 O_7 O_5 O_1 O_6 O_2 O_4

36. The CDSP solution as well as the maximum average tau-b solutions for this problem are given below. Note that in this case there are two solutions, i.e. two distinct rankings which result in the maximum average tau-b value with the five input rankings. The two tau-b solutions differ in their placement of options O_2, O_4 and O_6 at the end of list. They differ substantially from the CDSP solution in placing option O_5 in a tie with option O_1 behind options O_3 and O_7. This particular example shows the sensitivity of the various solutions when the input rankings disagree widely. Note that the maximum average tau-b with the input rankings was 0.32, indicating only modest overall agreement.

CDSP Solution	:	O_5 O_3 O_7 O_1 O_6 O_4 O_2
Max. Avg. Tau-b Solution 1	:	O_3 O_7 O_1-O_5 O_4-O_6 O_2
Max. Avg. Tau-b Solution 2	:	O_3 O_7 O_1-O_5 O_6 O_2-O_4

APPLICATION TO THE MULTI CRITERIA DECISION PROBLEM

37. The simplest form of the multi criteria decision problem is very similar in structure to the *m* rankings problem. There are a number of options or objects to be prioritized as well as a set of criteria on which each option is rated. As a minimum each object is ranked under each criterion although in some cases actual measurements may be available. The main difference between the *m* rankings problem and this form of the multi criteria decision problem is that in the latter we are also given weights for each criterion or ranker which reflect relative importance. Thus the *m* rankings problem may be considered to be a special case of the multi criteria decision problem in which each ranking is considered to be of equal importance, i.e. all the weights are equal.

38. A basic but flawed approach to the multi criteria decision problem involves the creation of a score (often the average rank) for each option under each criterion. These scores are then used to create an overall ranking of the options by combining the scores with the criterion weights in a simple weighted sum. Although this method seems reasonable on the surface, it involves the implicit assumption that the tradeoffs between criteria are linear over their entire range. Even worse, using the weighted average score to create a final ranking grossly oversimplifies a complex problem by mapping ratings in several different dimensions onto an interval in one dimension. Attempts to interpret the weights as conversion factors from the various criteria into a general utility measure have not met with success due to the nonlinearity of utility as well as the fact that most people do not understand the weights as utility conversions.

40. Just how do we interpret the criterion weights? It is certainly not clear that weights can be interpreted as utility conversion factors. Weights are usually described as a measure of the relative importance of the various ranking criteria, with very little additional explanation. Thus the weights are assumed to be an intuitive concept. It is up to the analyst to incorporate them into a model in such a way that the intuitive meaning is preserved while ensuring that the decision maker has a clear understanding of their effect on the problem solution.

41. Given the clear and mathematically sound solution concept for the *m* rankings problem in which the solution is the ranking which results in the maximum agreement with the input rankings as measured by tau-b, it is attractive to apply the same idea to the multi criteria decision problem. A natural, clear and intuitively appealing use of the criteria weights is to use them in the averaging of the individual correlation coefficients which measure the agreement of the solution ranking with the individual input rankings. Thus the proposed solution to the multi criteria decision problem is the rank ordering of the objects which maximizes the weighted average of the correlation coefficients with each of the input criteria rankings, where the weights in the weighted average are the criteria weights.

42. The proposed procedure is consistent with the *m* rankings solution since the weighted average in the case of equal weights is simply the common average. The use of the criteria weights to multiply the correlation coefficients is consistent with their intuitive meaning as measures of the relative importance of each criterion in determining the final ordering : larger weights will influence the solution to agree more closely with their associated criterion rankings relative to those criteria with smaller weights. Finally, the terms in the weighted average which determine the solution are all the same type of mathematical object, correlation coefficients measuring the agreement between rankings, and we therefore avoid the summing of apples and oranges which characterizes the usual attempts to produce a solution to the multi criteria decision problem.

<u>Example 6</u>

43. We will reconsider example 1 with the addition of ranker weights. Note that the CDSP and maximum average tau-b solutions were the same for example 1. However, with the addition of weights, CDSP is no longer appropriate for this problem. The rankings and original (no weights) solution are repeated below for convenience. The weight for ranking R1 has been set at 1.0 with ranking R2 at 0.7 indicating less importance and ranking R3 at 0.3 indicating least importance.

Ranking	Preference Ordering	Weight
R1	O_2 O_5 O_4 O_3 O_1 O_6	1.0
R2	O_6 O_4 O_3 O_1 O_5 O_2	0.7
R3	O_3 O_4 O_5 O_2 O_1 O_6	0.3

No Weights Solution (maximum average tau-b) : O_4 O_3 O_5 O_2 O_1 O_6

44. The maximum weighted average tau-b solution for the weighted problem is given below. Note that option O_2 has moved up into a first place tie in accordance with its high rating in ranking R1. Similarly, option O_3 has moved into fourth place, consistent with the low weight placed on ranking R3.

Maximum Weighted Average Tau-b Solution : O_2-O_4-O_5 O_3 O_1 O_6

Example 7

45. As a final example, we will consider a hypothetical decision problem involving the rating of seven different new family car options. Six criteria have been identified and each option has been ranked from most preferred to least preferred under each criterion. Note that for some of the criteria such as cost, actual measurement data have been replaced by rank data, consistent with the type of information available on the other criteria such as comfort and style. Weights have been assigned relative to the most important criterion, cost, which has been given a weight of 1.0.

Ranking	Rank Ordering of the 7 Options	Weight
Cost	O_1 O_2 O_3 O_4 O_5 O_6 O_7	1.0
Maintenance	O_3 O_5 O_6 O_4 O_7 O_2 O_1	0.7
Style	O_4-O_6 O_3 O_7 O_1-O_2 O_5	0.6
Comfort	O_6 O_4-O_7-O_5 O_3 O_1-O_2	0.4
Resale	O_7 O_5 O_4 O_6 O_1 O_3 O_2	0.2
Convenience	O_3 O_4 O_6 O_1-O_2-O_7 O_5	0.2

46. The maximum weighted average tau-b solution for this problem is given below. Options O_3, O_4, and O_6 are ranked together in a first place tie with the remaining four options ranked together in a last place tie. Given the large number of ties in the input data and the conflicting rankings along the different criteria, this result is reasonable. Although four options have been eliminated from contention, the decision maker still has to find a way to deal with the three way tie for first place.

Max. Wtd. Avg. Tau-b Solution : O_3-O_4-O_6 O_1-O_2-O_5-O_7

TECHNICAL CONSIDERATIONS

47. The use of maximum weighted average tau-b as a solution concept has the obvious advantages of simplicity and theoretical validity. Neither the number of rankers or ranking criteria nor the presence of weights causes any practical difficulties. However, as shown in Annex A to this report, the technical problem of exhaustively computing all possible rankings is exponentially dependent on the number of objects being ranked. The problem is one of sheer size, not of complexity, since a simple algorithm exists which generates all possible rankings of N objects directly from the rankings of N-1 objects.

48. Problems involving up to 8 objects have been easily handled on a Pentium workstation and it is feasible that up to 11 objects could be handled by exhaustive enumeration given adequate computing facilities. However, it is clear that in order to determine the maximum weighted average tau-b for 12 or more objects, exhaustive enumeration is impractical. A possible approach is to use dominance techniques to lop off large numbers of rankings. More research is needed into techniques of determining the maximum weighted average tau-b without using exhaustive enumeration if it is required to handle moderate to large numbers of objects.

CONCLUSIONS

49. The use of maximum average correlation coefficient of the solution ranking with the input rankings is a valid and useful solution concept for the *m* rankings problem. Kendall's tau-b rank correlation coefficient is ideally suited for this type of problem. As shown in several examples, the solutions produced are reasonable and comparable to other methods. The proposed solution concept is theoretically sound and avoids inconsistencies caused by treating rank data as measurement data.

50. The extension of the solution concept to a simple form of the multi criteria decision analysis problem is also very attractive. Using the criteria weights in a maximized weighted average correlation coefficient solution concept is natural and theoretically sound in that the weights influence the degree to which the solution agrees with each ranking as measured by the rank correlation. This use of the criteria weights is consistent with the intuitive meaning that is associated with them. The apples and oranges problem associated with a linear mapping of artificial scores in several incommensurate dimensions is avoided.

51. Any number of rankings can be handled with no difficulty. However, the proposed solution concept is sensitive to the number of objects being ranked. Exhaustive enumeration of the possible rankings of N objects becomes impractical around $N=12$. Therefore, further research is required into ways of determining the maximum weighted average ranking without exhaustive enumeration if larger numbers of objects are involved.

REFERENCES

1. Rank Correlation Methods, Fifth Edition, Maurice Kendall and Jean Dickinson Gibbons, 1990, Oxford University Press.

2. ORAE Project Report No. PR448, "A Consensus Procedure for Committee Decisions", E.J. Emond, Department of National Defence Canada, December 1987.

3. Ranking in Tournaments and Group Decisionmaking, Stephen T.Goddard, Management Science, Vol. 29, December 1983.

ENUMERATING ALL POSSIBLE RANKINGS

1. In order to enumerate all possible rankings for a given number N of options, an iterative procedure has been developed. Consider the case for N=2. There are three possible rankings of two objects : {1 , 2}, {2 , 1}, and { 1-2 }. Note that there is one ranking with one element (a two way tie) and two rankings with two elements. In order to enumerate the rankings for N = 3 we consider the three rankings for N=2 in turn.

2. A third object can be added to the ranking {1 , 2} by being added as tied with the existing two objects, resulting in the two-element rankings : {1-2 , 3} and {1 , 2-3}. In addition, a third object can be added in front of, between, or at the end of the ranking of two objects as follows : {3 , 1 , 2}, {1 , 3 , 2}, and {1 , 2 , 3}. The same procedure can be followed to generate an additional 5 rankings of three objects from the ranking {2 , 1}. Finally, a third object can be added to the ranking { 1-2 } as a three way tie or before or after the two way tie as follows : { 1-2-3 }, {3 , 1-2}, and { 1-2 , 3}.

3. In total then, the single element ranking { 1-2 } generates one single element ranking { 1-2-3 } and two rankings with two two-elements. The two-element rankings of two objects each generate 2 rankings of two objects and 3 rankings of three objects. The total number of objects for N = 3 is 13.

4. The above procedure can be used to generate all rankings for N from the list of rankings for N-1. All rankings of k elements for N options are generated either from the rankings of N-1 options involving k or k-1 elements. The formula below gives the number of rankings of N options involving k elements in the general case.

$$n_N(k) = k * (n_{N-1}(k-1) + n_{N-1}(k))$$

5. The table below gives the number of possible rankings for $N=2$ to $N=11$ as calculated using the above formula.

Number of Objects	Number of Possible Rankings
2	3
3	13
4	75
5	541
6	4 683
7	47 293
8	545 835
9	7 087 261
10	102 247 563
11	1 622 632 573

DOCUMENT CONTROL DATA
(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATOR (the name and address of the organization preparing the document. Organizations for whom the document was prepared e.g. Establishment Sponsoring a contractor's report, or tasking agency, are entered in Section 8).
Operational Research Division
Department of National Defence
Ottawa, Ontario K1A 0K2

2. SECURITY CLASSIFICATION (overall security classification of the document, including special warning terms if applicable)
UNCLASSIFIED

3. TITLE (the complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title)
Maximum Rank Correlation as a Solution Concept in the *M* Rankings Problem with Application to Multi Criteria Decision Analysis

4. AUTHORS (last name, first name, middle initial)
Emond, E.J.

5. DATE OF PUBLICATION (month Year of Publication of document)
SEPTEMBER 1997

6a. NO OF PAGES (total containing information. Include Annexes, Appendices, etc.)
27

6b. NO OF REFS (total cited in document)
3

7. DESCRIPTIVE NOTES (the category of document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)
Research Note

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include the address).
ORA 41303

9a. PROJECT OR GRANT NO. (if appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.)
DOR(CAM) RESEARCH NOTE RN9705

9b. CONTRACT NO. (if appropriate, the applicable number under which the document was written.)

10a. ORIGINATOR's document number (the official document number by which the document is identified by the originating activity. This number must be unique to this document.)
DOR(CAM) Research Note RN 9705

10b. OTHER DOCUMENT NOS. (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

11. DOCUMENT AVAILABILITY (any limitations on further dissemination of the document, other than those imposed by security classification.)
(x) Unlimited distribution
( ) Distribution limited to defence departments and defence contractors: further distribution only as approved
( ) Distribution limited to defence departments and Canadian defence contractors; further distribution only as approved
( ) Distribution limited to government departments and agencies; further distribution only as approved
( ) Distribution limited to defence departments; further distribution only as approved
( ) Other (please specify):

12. DOCUMENT ANNOUNCEMENT (any limitation to the bibliographic announcement of this document. This will normally correspond to the Document Availability (11). However, where further distribution (beyond the audience specified in 11) is possible, a wider announcement audience may be selected.)

UNCLASSIFIED

SECURITY CLASSIFICATION OF FORM

13. ABSTRACT (a brief and factual summary of the document. It may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall begin with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (S), (C), or (U). It is not necessary to include here abstracts in both official languages unless the test is bilingual).

The *m* rankings problem is the mathematical name for the committee consensus decision problem in which a number of objects or options are ranked by a *m* equal committee members. This report documents research into a solution concept for the *m* rankings between individual rankings. Properties of this and other solution concepts are explored through a series of examples. The proposed solution concept compares favourably with other methods and lends itself directly to the more general multi criteria decision problem in which weights may be assigned to the individual rankings, reflecting their relative importance.

14. KEYWORDS, DESCRIPTORS or IDENTIFIERS (technically meaningful terms or short phrases that characterize a document and could be helpful in cataloguing the document. They should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location may also be included. If possible keywords should be selected from a published thesaurus, e.g. Thesaurus of Engineering and Scientific Terms (TEST) and that thesaurus-identified . If it is not possible to select indexing terms which are Unclassified, the classification of each should be indicated as with the title.)

RANKINGS

CONSENSUS

COMMITTEE

DECISION

CORRELATION

MULTI CRITERIA

Canada

505 522