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Harold Wilson

Defence Research Establishment Atlantic
Centre de Recherches pour la Défense Atlantique

Canada

DEFENCE RESEARCH ESTABLISHMENT ATLANTIC

9 GROVE STREET

P.O. BOX 1012
DARTMOUTH, N.S.
B2Y 3Z7

TELEPHONE
(902) 426-3100

CENTRE DE RECHERCHES POUR LA DÉFENSE ATLANTIQUE

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Approved by R.E.Erickson: *Signature on File*
Head / Electromagnetics Section

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Abstract

This report presents a boundary-integral method for calculating the magnetic field of a superconducting body in the London model (i.e. taking account of field penetration into the superconductor). The method used is to calculate the standard vector potential for the magnetic field both outside and inside the superconductor. A method for handling circulating currents in multiply connected superconductors is included. Simplified forms of the integral equations are given for the cases of (i) small penetration depths and (ii) thin films.

Résumé

Ce rapport présente une methode à limites intégrales pour calculer le champ magnétique d'un corps supraconducteur dans la modèle London (c.-à-d. tenant compte del la pénétration du champ dans le supraconducteur). La Méthode utilisée consiste en calculer le potentiel vectoriel normal pour le champ magnétique à l'extérieur et à l'intériere du supraconducteur. Nous incluons une méthode pour manier des courants circulants dans des supraconducteurs reliés à plusiers reprises. Nous présentons des formes simplifiées des équations intégrales pour les cas (i) des profondeurs de pénétration menues, et (ii) des couches minces.

Executive Summary

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Background

The Surveillance and Ship Silencing Group at Esquimalt Defence Research Detachment (EDRD) of Defence Research Establishment Atlantic is investigating the feasibility of using SQUID magnetic gradiometers (SQUID=Superconducting Quantum Interference Device) to detect and localize targets in airborne antisubmarine warfare. A major part of this program has been investigations of noise in moving gradiometers. One of the sources of noise is believed to be SQUID signals that originate in changes in the superconductors when small temperature fluctuations occur. Temperature fluctuations are caused by motion of stratified liquid helium in the sensor and by small pressure changes. The temperature changes in turn result in a small change in the magnetic behaviour of the superconductors.

This report describes a set of integral equations that have been developed to allow investigation of the effect of changes in one temperature-dependent parameter of superconductors, namely the penetration depth in the London model.

Principal Results

The report derives exact boundary-integral equations for the magnetic vector potential for arbitrary shapes of superconductors. It also provides simplifications that will provide accurate simple results in the situations where these equations are most likely to be applied (i.e. where the penetration depth is small, and where the superconductor is a thin film).

Significance of Results and Future Research

A quantitative estimate of the effects of penetration-depth fluctuations on sensor performance could be made by applying these equations to thin-film SQUIDs and coupling coils. This may be important in understanding noise in the superconducting gradiometer being built for EDRD by CTF Systems, Inc., of Port Coquitlam, B.C. which uses thin-film DC SQUIDs to sense the magnetic field.

I. Introduction

Calculations of the magnetic configuration of superconductors (that is the currents in the superconductors and the magnetic fields generated by the currents) are useful in the design and understanding of superconducting systems. In the superconducting-gradiometer program at EDRD, the concern has been primarily the magnetic fields in bulk and thin-film SQUIDs, superconducting circular wires and cylindrical SQUID shields. Other applications include large superconducting magnetic shields, and thin-film strips. In the past, these superconducting structures have always had dimensions much greater than the penetration depth λ, and the effects of magnetic-field penetration into the superconductors could be ignored. However, in the past few years, the introduction of Nb thin-film devices with thicknesses as small as ~5λ and high-T_C superconductors with large λ (up to 0.5 μm) means that there are likely to be cases where the non-zero penetration depth will make a significant difference to the current density. This will be especially important when estimating the effect of temperature fluctuations in superconducting sensors since the penetration depth increases when the temperature of a superconductor rises.

There are, of course many ways to calculate the magnetic fields and currents in a superconducting system. It is often possible to solve directly the differential equations for the magnetic field, the scalar potential or the vector potential via finite-difference or finite element methods. An alternative formulation given by Refs.[1,2,3], and the one described here, is to formulate the problem as a boundary-integral equation. Ref.[1] solves directly for the surface currents in a superconducting cylinder. Ref.[2] describes the boundary integral equation for the magnetic scalar potential, while [3] shows how to formulate the problem with zero penetration depth using the magnetic vector potential. Comparison of Refs.[2] and [3] shows that the scalar boundary-integral formulation is algebraically simpler than the vector-potential approach. In addition, it requires the calculation of only one field, while the vector-potential approach may involve 1, 2 or 3 fields depending on the symmetry of the problem. The major failing of the scalar-potential approach is that it cannot be applied in cases where there are electric currents <u>inside</u> the volume where the scalar potential is calculated. This is obvious : with a scalar potential the magnetic field is given by

$$\vec{B} = \vec{\nabla}\Phi \tag{1a}$$

and the current density is

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{\nabla}\Phi = 0 \ . \tag{1b}$$

In other words, the scalar-potential method is useful only in source-free regions where the line integral of $\vec{B}$ around all closed paths vanishes, and it cannot be applied to multiply-connected superconductors (i.e. superconductors with holes such as SQUIDs and hollow tubes) nor to the study of the effect of magnetic-field penetration into superconductors. The vector-potential approach does not suffer from this limitation.

The price paid for using vector potentials is increased complexity, especially when the superconducting system under analysis is not highly symmetric. One might therefore want to continue with scalar potentials, but take care of the non-zero penetration depth by moving the surface a distance λ into the superconductors. In some multiply-connected situations with sufficient symmetry, it may be possible to set up the problem so that there are no closed paths enclosing currents, and again a scalar potential could be applied. However, a general formulation must use vector potentials.

The purpose of this report is to provide boundary-integral equations for the magnetic vector potential taking into account field penetration into the surface of the superconductor using the simple description of superconductor electrodynamics provided by the London model (see e.g. Ref.[4]). The London model gives a differential equation for the magnetic vector potential and the electric current density inside a superconductor which is valid when the magnetic fields and the current densities are small compared to their critical values (i.e. the field strength where the superconductivity is lost (Type I superconductors) or where flux vortices penetrate the material (Type II)).

II. The Magnetic Vector Potential in the London Model

Ref.[4] provides a detailed, modern description of the London model of superconductors. The basic idea is that θ, the phase of the superconducting wavefunction, is related to the current density $\vec{J}$ and the vector potential $\vec{A}$ by

$$\hbar\vec{\nabla}\theta = 2e\lambda^2\mu_0\vec{J} + 2e\vec{A} \tag{2}$$

where the London penetration depth λ is determined by the density and effective mass of the superconducting pairs. To complete the definition of the problem, we have the gauge condition

$$\vec{\nabla}\cdot\vec{A} = 0 \quad . \tag{3}$$

Conservation of charge means that

$$\vec{\nabla}\cdot\vec{J} = 0 \quad , \tag{4}$$

so the equation for θ is

$$\nabla^2\theta = 0 \quad . \tag{5}$$

For an isolated superconductor, the boundary conditions on $\vec{J}$ and θ are vanishing normal derivative at the surface of the superconductor :

$$\vec{J}\cdot\hat{n} = 0 \qquad ; \qquad \frac{\partial\theta}{\partial n} = \vec{\nabla}\theta\cdot\hat{n} = 0 \quad . \tag{6}$$

(6) shows that, on the boundary, $\vec{A}\cdot\hat{n} = 0$. Taking the curl of (2) gives the differential equation for $\vec{A}$:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A} + \frac{\vec{A}}{\lambda^2} = \frac{\phi_0}{2\pi\lambda^2}\vec{\nabla}\theta \quad . \tag{7}$$

$\phi_0 = h/2e$ is the quantum of magnetic flux. (7) is the equation for the vector potential in the London model including the effect of superconductor phase. Since $\vec{\nabla}\cdot\vec{A} = 0$, (7) can also be written as

$$-\nabla^2\vec{A} + \frac{\vec{A}}{\lambda^2} = \frac{\phi_0}{2\pi\lambda^2}\vec{\nabla}\theta \quad . \tag{7a}$$

The boundary conditions on A at the surface of a superconductor are :

(i) Continuity of $\vec{A}$,

(ii) Continuity of $\vec{B}=\vec{\nabla}\times\vec{A}$, (8)

(iii) For isolated bodies, vanishing normal component : $\vec{A}\cdot\hat{n}=0$.

Note that the solution to (7) could also be written as

$$\vec{A}=\vec{A}_1+\vec{A}_2 \quad ,$$

where $\vec{A}_1$ is the solution to the source-free equation (7) :

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}_1+\frac{\vec{A}_1}{\lambda^2}=0$$

and

$$\vec{A}_2=\frac{\phi_0}{2\pi}\vec{\nabla}\theta \quad .$$

The derivations presented below will use the total vector potential $\vec{A}$ rather than treating the two parts separately.

The single parameter describing the superconductor in these equations is the penetration depth λ. In a typical low-T_C superconductor like Nb or Pb, λ~40 nm, although it may be significantly higher in evaporated thin films. YBCO, the most common high-T_C material, has λ in the range 150-700 nm. The London model of superconductors is very simple and it is not expected to describe superconductors very well in high magnetic fields where the current densities are more than a small fraction of their maximum values, but it is a straightforward way to investigate the effects of field penetration without the complication of more complete treatments of the superconductors such as Ginzburg-Landau theory.

To solve the problem, $\vec{A}$ inside the superconductor must be matched to $\vec{A}$ in the exterior. Outside the superconductor, the equation for $\vec{A}$ is

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}=0 \quad ; \quad \vec{\nabla}\cdot\vec{A}=0 \tag{9}$$

with an external applied magnetic field

$$\vec{B}_0(\vec{r})=\vec{\nabla}\times\vec{A}_0(\vec{r}) \quad . \tag{10}$$

III. Calculating the Phase θ

(7) shows that the gradient of the wavefunction phase θ acts like a source term for the magnetic vector potential. This section discusses the calculation of θ.

(i) Simply-connected superconductor

When there are no holes in a superconductor, the phase is constant, so $\vec{\nabla}\theta = 0$ and (7) becomes a homogeneous equation :

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}+\frac{\vec{A}}{\lambda^2}=-\nabla^2\vec{A}+\frac{\vec{A}}{\lambda^2}=0 \quad .$$

This is the form of the equation for $\vec{A}$ inside a superconductor that is given by most references.

(ii) Circular symmetry, one hole

With circular symmetry, using the cylindrical coordinates (ρ,ϕ,z), the gradient of $\theta(\vec{r})$ is

$$\vec{\nabla}\theta(\vec{r})=\frac{2\pi n}{\rho}\hat{\phi} \quad ; \quad n=integer \quad . \tag{11}$$

Circular symmetry greatly simplifies boundary-integral calculations because the angle coordinate ϕ can be removed from the problem, so this is likely to be the most common situation encountered in calculations of the magnetic field of a superconductor. Physically, it means that the magnetic flux through the hole is $\Phi=n\phi_0$.

(iii) No symmetry, one hole.

In this situation, $\theta(\vec{r})$ can be found only by solving (5) and (6),(i.e. Laplace's equation with zero normal derivative), and requiring that the total phase change be $\Delta\theta=2\pi n$ around a path that encloses the hole (i.e. the magnetic flux in the hole is $\Phi=n\phi_0$). A boundary-integral equation for $\theta(\vec{r})$ is presented below.

Figure 1 sketches a body with a single hole. S is the surface of the body , $\hat{n}(\vec{s})$ is the outward-pointing unit-vector normal to S, and V denotes the volume enclosed by surface S. Start by defining a surface Σ that cuts from the hole to the external region; A and B label the two sides of Σ. Across Σ, θ jumps by $2\pi n$, but $\vec{\nabla}\theta(\vec{r})$ is continuous. The unit vector normal to Σ is $\hat{\nu}$ and it points from side A toward side B. Now consider the superconducting body to be simply connected with its surface defined by S, Σ_A and Σ_B.

We now apply the usual boundary-integral equation approach using the Green's function

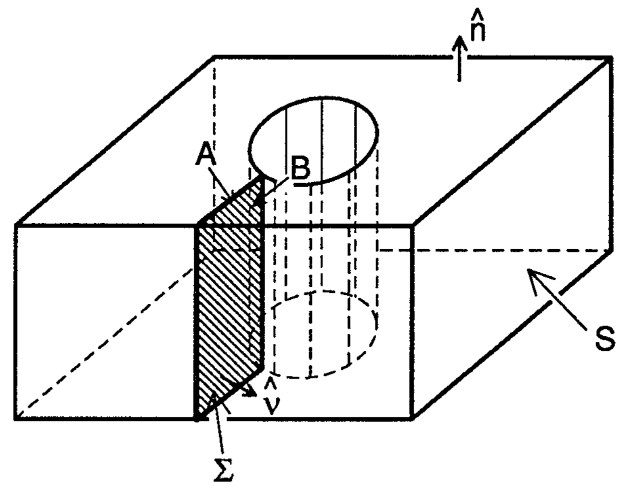

Figure 1 *A superconducting body with one hole showing the symbols and unit vectors used in the boundary-integral equation for the phase $\theta(\vec{r})$.*

$$G(\vec{r},\vec{s}) = \frac{1}{4\pi|\vec{r}-\vec{s}|} \quad . \tag{12}$$

The first step is to substitute G and θ in Green's theorem :

$$\int_V \left[G(\vec{r},\vec{s})\nabla^2\theta(\vec{s}) - \theta(\vec{s})\nabla_s^2 G(\vec{r},\vec{s})\right] d^3s \;=\; \int_{S+\Sigma_A+\Sigma_B} \left[G(\vec{r},\vec{s})\vec{\nabla}\theta(\vec{s}) - \theta(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\right]\cdot \hat{n}\, d^2s \quad . \tag{13}$$

On surface Σ_A, $\hat{n} = \hat{v}$, and on Σ_B, $\hat{n} = -\hat{v}$. Ref.[5] shows that the volume integral on the left side of (13) is

$$\textit{Volume integral} = \alpha(\vec{r})\theta(\vec{r}) \quad ; \quad \alpha(\vec{r}) = \frac{\textit{(solid angle of interior)}}{4\pi} \quad . \tag{13a}$$

For points inside V, $\alpha=1$; for points on the surface S, $0<\alpha<1$ ($\alpha=1/2$ where the boundary is smooth) ; and outside the body, $\alpha=0$. The surface integral on the right side of (13) is evaluated by applying the boundary conditions on surfaces S and Σ :

$$S \;:\; \hat{n}\cdot\vec{\nabla}\theta = 0 \;;$$

$$\Sigma \;:\; \theta_B - \theta_A = 2\pi n \quad \text{and} \quad \vec{\nabla}\theta_A = \vec{\nabla}\theta_B \;.$$

Then the surface integral in (13) becomes

$$Surface\ integral = -\int_S \theta(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{n}\,d^2s + \int_\Sigma G(\vec{r},\vec{s})\vec{\nabla}\theta_A(\vec{s})\cdot\hat{\nu}\,d^2s - \int_\Sigma \theta_A(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{\nu}\,d^2s$$

$$-\int_\Sigma G(\vec{r},\vec{s})\vec{\nabla}\theta_B(\vec{s})\cdot\hat{\nu}\,d^2s + \int_\Sigma \theta_B(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{\nu}\,d^2s$$

$$= -\int_S \theta(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{n}\,d^2s + 2\pi n\int_\Sigma \vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{\nu}\,d^2s \quad . \qquad (13b)$$

Equating (13a) and (13b), gives the boundary-integral equation for $\theta(\vec{r})$:

$$\alpha(\vec{r})\theta(\vec{r}) = -\int_S \theta(\vec{s})\vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{n}\,d^2s + 2\pi n\int_\Sigma \vec{\nabla}_s G(\vec{r},\vec{s})\cdot\hat{\nu}\,d^2s$$

$$= \frac{1}{4\pi}\int_S \theta(\vec{s})\frac{(\vec{s}-\vec{r})\cdot\hat{n}}{|\vec{r}-\vec{s}|^3}d^2s - \frac{n}{2}\int_\Sigma \frac{(\vec{s}-\vec{r})\cdot\hat{\nu}}{|\vec{r}-\vec{s}|^3}d^2s \quad . \qquad (14)$$

To calculate $\theta(\vec{r})$ everywhere in the body,

(i) Specify θ at one point on the surface,

(ii) Solve (14) to get $\theta(\vec{r})$ everywhere on the surface, and

(iii) Calculate θ for $\vec{r}$ on the interior by evaluating the surface integrals on the right side of (14).

IV. The Boundary - Integral Equation for $\vec{A}$

The boundary-integral equation for the magnetic vector potential $\vec{A}$ is derived by modifying the analysis given by Ref.[6]. We want to solve the equations

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r}) + \kappa^2\vec{A}(\vec{r}) = \vec{F}(\vec{r}) \quad ; \quad \vec{\nabla}\cdot\vec{A} = 0 \quad . \qquad (15)$$

where the source term $\vec{F}$ is defined throughout a volume V and $\vec{A}$ is to be determined on the surface S which bounds V. Ref.[6] presents the solution for the case κ=0 ; we make no restriction on κ which may have any complex value. Start by introducing a vector $\vec{Q}$:

$$\vec{Q}(\vec{r},\vec{s})=\vec{a}g(\vec{r},\vec{s}) \quad ; \quad g(\vec{r},\vec{s})=\frac{e^{-\kappa|\vec{r}-\vec{s}|}}{4\pi|\vec{r}-\vec{s}|} \quad , \tag{16}$$

where $\vec{a}$ is an arbitrary, constant vector. Then $\vec{Q}$ satisfies the following identities

$$\left.\begin{aligned} \vec{\nabla}_s\cdot\vec{Q}(\vec{r},\vec{s}) &= \vec{\nabla}_s g(\vec{r},\vec{s})\cdot\vec{a} \\ \vec{\nabla}_s\times\vec{Q}(\vec{r},\vec{s}) &= \vec{\nabla}_s g(\vec{r},\vec{s})\times\vec{a} \\ \vec{\nabla}_s\times\vec{\nabla}_s\times\vec{Q}(\vec{r},\vec{s})+\kappa^2\vec{Q}(\vec{r},\vec{s}) &= \vec{\nabla}_s\left[\vec{\nabla}_s g(\vec{r},\vec{s})\cdot\vec{a}\right]+\delta(\vec{r}-\vec{s}) \end{aligned}\right\} \quad . \tag{16a}$$

Next, rewrite eq.(3), p.250 of Ref.[6] using $\vec{A}$ and $\vec{Q}$:

$$\begin{aligned} &\int_V \left\{\vec{A}(\vec{s})\cdot\left[\vec{\nabla}_s\times\vec{\nabla}_s\times\vec{Q}(\vec{r},\vec{s})+\kappa^2\vec{Q}(\vec{r},\vec{s})\right]-\vec{Q}(\vec{r},\vec{s})\cdot\left[\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{s})+\kappa^2\vec{A}(\vec{s})\right]\right\}d^3s = \\ &\qquad \int_S \left\{\vec{Q}(\vec{r},\vec{s})\times\vec{\nabla}\times\vec{A}(\vec{s})-\vec{A}(\vec{s})\times\vec{\nabla}_s\times\vec{Q}(\vec{r},\vec{s})\right\}\cdot\hat{n}(\vec{s})d^2s \quad . \end{aligned} \tag{17}$$

The first term in the volume integral on the left side of (17) can be converted to a surface integral by noting that, because $\vec{\nabla}\cdot\vec{A}=0$,

$$\vec{A}(\vec{s})\cdot\left[\vec{\nabla}_s\left(\vec{a}\cdot\vec{\nabla}_s g(\vec{r},\vec{s})\right)\right]=\vec{\nabla}_s\cdot\left[\vec{A}(\vec{s})\left(\vec{a}\cdot\vec{\nabla}_s g(\vec{r},\vec{s})\right)\right] \quad . \tag{18}$$

Then, substituting for $\vec{Q}$ in (17), and applying Gauss's Theorem, the volume integral becomes

$$\mathit{Volume\ integral}=\vec{a}\cdot\left[\alpha(\vec{r})\vec{A}(\vec{r})-\int_V g(\vec{r},\vec{s})\vec{F}(\vec{s})d^3s+\int_S \vec{\nabla}_s g(\vec{r},\vec{s})\left(\vec{A}(\vec{s})\cdot\hat{n}\right)d^2s\right] \quad , \tag{19a}$$

where $\alpha(\vec{r})$ is defined in (12a). After substituting for $\vec{Q}$ and rearranging the cross products, the right side of (17) becomes

$$\begin{aligned} \mathit{Surface\ integral} &= \int_S \left\{g(\vec{r},\vec{s})\vec{a}\times\vec{\nabla}\times A(\vec{s})-A(\vec{s})\times\left(\vec{\nabla}_s g(\vec{r},\vec{s})\times\vec{a}\right)\right\}\cdot\hat{n}\,d^2s \\ &= \vec{a}\cdot\left[\int_S g(\vec{r},\vec{s})\left(\vec{\nabla}\times A(\vec{s})\right)\times\hat{n}\,d^2s-\int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(A(\vec{s})\times\hat{n}\right)d^2s\right] \quad . \end{aligned} \tag{19b}$$

The boundary-integral equation for $\vec{A}$ is obtained by equating (19a) and (19b) and noting that these equations hold for any $\vec{a}$:

$$\alpha(\vec{r})\vec{A}(\vec{r}) = \int_V g(\vec{r},\vec{s})\vec{F}(\vec{s})d^3s - \int_S \vec{\nabla}_s g(\vec{r},\vec{s})\left(\vec{A}(\vec{s})\cdot\hat{n}\right)d^2s + \int_S g(\vec{r},\vec{s})\left(\vec{\nabla}\times A(\vec{s})\right)\times\hat{n}\,d^2s - \int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(A(\vec{s})\times\hat{n}\right)d^2s \tag{20}$$

This equation is true for any field satisfying (15).

In the London model of superconductors, the source $\vec{F}$ is proportional to the gradient of the superconductor phase (see (7)), and the first surface integral in (20) vanishes since $\vec{A}\cdot\hat{n}=0$ (see (8)). Thus the boundary-integral equation for $\vec{A}$ in the London model for $\vec{r}$ on the surface of, or inside, a superconductor with penetration depth λ is

$$\alpha(\vec{r})\vec{A}(\vec{r}) = \frac{\phi_0}{2\pi\lambda^2}\int_V g(\vec{r},\vec{s})\vec{\nabla}\theta(\vec{s})d^3s + \int_S g(\vec{r},\vec{s})\left(\vec{\nabla}\times\vec{A}(\vec{s})\right)\times\hat{n}\,d^2s - \int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(\vec{A}(\vec{s})\times\hat{n}\right)d^2s\,. \tag{21}$$

When λ is comparable to the dimensions of the superconducting body, evaluating the integrals in (21) can be difficult.

For $\vec{r}$ outside the superconductor, (9) gives the equation for the vector potential:

$$\vec{\nabla}\times\vec{\nabla}\times\vec{A}(\vec{r})=0 \quad ; \quad \vec{\nabla}\cdot\vec{A}=0 \ .$$

To develop a boundary-integral equation, the analysis of $\vec{A}$ on the interior is used with $\kappa=1/\lambda=0$. It is convenient to write surface integrals for the exterior problem in terms of the same normal unit-vector $\hat{n}$ that was used in the boundary integrals for $\vec{A}$ inside the superconductor. Since $\hat{n}$ points <u>out</u> of the superconductor, it points <u>into</u> the exterior region, and consequently the signs of the surface integrals for the exterior problem appear reversed. Where the solid angle subtended by the interior of the body was used in the interior solution (13a), the exterior solution uses

$$1-\alpha(\vec{r}) = \frac{(\textit{solid angle of exterior})}{4\pi} \ .$$

The boundary of the exterior region includes a surface at infinity, and integrals over this infinitely-distant surface give $\vec{A}_0(\vec{r})$, the vector potential of the external applied magnetic field. Following the procedures in the last section, the boundary integral equation for $\vec{A}$ is

$$[1-\alpha(\vec{r})]\vec{A}(\vec{r}) = \vec{A}_0(\vec{r}) - \frac{1}{4\pi}\int_S \frac{\left[\vec{\nabla}\times\vec{A}(\vec{s})\right]\times\hat{n}}{|\vec{r}-\vec{s}|}d^2s + \frac{1}{4\pi}\int_S \frac{(\vec{r}-\vec{s})\times\left[\vec{A}(\vec{s})\times\hat{n}\right]}{|\vec{r}-\vec{s}|^3}d^2s \ . \tag{22}$$

$\vec{A}$ and $\vec{\nabla}\times\vec{A}$ on the superconductor surface are the unknown vectors in this problem. The two coupled boundary-integral equations, (21) and (22), are sufficient for a solution.

V. Small Penetration Depth

In most cases, the penetration depth λ is <<(characteristic distances of the problem) : i.e. the penetration depth will be small compared to (i) the size of the body and (ii) the radius of curvature. In such a case, the boundary-integral equation (21) can be simplified by an approximate evaluation of the integrals.

The first two terms on the right side of (21) are easy to do, so we start this analysis with the third term. Near a point $\vec{r}$ on a smooth part of the surface, the surface is defined by the normal to the surface $\hat{n}$, two orthogonal unit vectors $\hat{u}_1$ and $\hat{u}_2$ tangential to the surface, and the curvature tensor $\boldsymbol{W}$ ($\boldsymbol{W}\hat{n}=0$ and the radius of curvature in a direction $\hat{u}$ tangent to the surface is $1/\hat{u}\cdot\boldsymbol{W}\hat{u}$). Surface points, the normal and the differential area element are described by two coordinates σ_1 and σ_2 ($\vec{\sigma}=\sigma_1\hat{u}_1+\sigma_2\hat{u}_2$) :

$$\vec{s}=\vec{r}+\vec{\sigma}+\frac{1}{2}\hat{n}(\vec{\sigma}\cdot\boldsymbol{W}\vec{\sigma}) \quad , \tag{23a}$$

$$\hat{n}(\vec{\sigma})=\hat{n}-\boldsymbol{W}\vec{\sigma} \quad , \tag{23b}$$

$$d^2s=d\sigma_1 d\sigma_2 \quad . \tag{23c}$$

The gradient tensor of the vector potential is $\boldsymbol{C}$ so $\vec{A}(\vec{s})$ is

$$\vec{A}(\vec{s})=\vec{A}(\vec{r})+\boldsymbol{C}\vec{\sigma} \quad . \tag{23d}$$

Since $\vec{\nabla}\cdot\vec{A}=0$, $\boldsymbol{C}$ is traceless (i.e. $\hat{u}_1\cdot\boldsymbol{C}\hat{u}_1+\hat{u}_2\cdot\boldsymbol{C}\hat{u}_2+\hat{n}\cdot\boldsymbol{C}\hat{n}=0$). The normal derivative of $\vec{A}$ is $\partial\vec{A}/\partial n=\boldsymbol{C}\hat{n}$. The gradient of $g(\vec{r},\vec{s})$ is

$$\begin{aligned}\vec{\nabla}_s g(\vec{r},\vec{s}) &= \frac{e^{-|\vec{r}-\vec{s}|/\lambda}}{4\pi}\left(\frac{1}{|\vec{r}-\vec{s}|^3}+\frac{1}{\lambda|\vec{r}-\vec{s}|^2}\right)(\vec{r}-\vec{s}) \\ &= \frac{-e^{-\sigma/\lambda}}{4\pi}\left(\frac{1}{\sigma^3}+\frac{1}{\lambda\sigma^2}\right)\left(\vec{\sigma}+\frac{1}{2}\hat{n}(\vec{\sigma}\cdot\boldsymbol{W}\vec{\sigma})\right) \quad .\end{aligned} \tag{23e}$$

Substituting (23a-e) into the third term on the right side of (21), and doing the integral over σ_1 and σ_2 gives the following lowest-order approximation :

$$\int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(\vec{A}(\vec{s})\times\hat{n}\right)d^2s=\frac{\lambda}{2}\left[-\boldsymbol{W}\vec{A}(\vec{r})+\frac{1}{2}Tr(\boldsymbol{W})\vec{A}(\vec{r})-\hat{n}\left(\hat{n}\cdot\frac{\partial\vec{A}}{\partial n}\right)\right] \quad . \tag{24}$$

Figure 2 shows that the side of the thin film that includes $\vec{r}$ is indicated by subscript 1 and the other side by 2. The surface normal is $\hat{n} = \hat{n}_1 = -\hat{n}_2$, and the vector potentials are $\vec{A}_1$ and $\vec{A}_2$ on the two surfaces. The first surface integral on the right side of (21) becomes the sum of integrals over the 'top' and 'bottom' of the thin film near surface point $\vec{r}$:

$$\int_S g(\vec{r},\vec{s})\left(\vec{\nabla}\times A(\vec{s})\right)\times\hat{n}\,d^2s = \left(\vec{\nabla}\times\vec{A}_1\right)\times\hat{n}\int_{\sigma=0}^{\infty}\frac{e^{-\sigma/\lambda}}{4\pi\sigma}2\pi\sigma d\sigma - \left(\vec{\nabla}\times\vec{A}_2\right)\times\hat{n}\int_{\sigma=0}^{\infty}\frac{e^{-\sqrt{\sigma^2+h^2}/\lambda}}{4\pi\sqrt{\sigma^2+h^2}}2\pi\sigma d\sigma$$

$$= \frac{\lambda}{2}\left[\left(\vec{\nabla}\times\vec{A}_1\right)\times\hat{n} - e^{-h/\lambda}\left(\vec{\nabla}\times\vec{A}_2\right)\times\hat{n}\right] \quad . \tag{27}$$

The second surface integral on the right side of (21) becomes

$$\int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(A(\vec{s})\times\hat{n}\right)d^2s = -\int_{\substack{\sigma_1=0\\ \sigma_2=0}}^{\infty}\frac{e^{-\sigma/\lambda}}{4\pi}\left(\frac{1}{\sigma^3}+\frac{1}{\lambda\sigma^2}\right)\left[-\hat{n}\left(\vec{\sigma}\cdot\vec{A}_1(\vec{\sigma})\right)\right]d\sigma_1 d\sigma_2$$

$$-\int_{\substack{\sigma_1=0\\ \sigma_2=0}}^{\infty}\frac{e^{-\sqrt{\sigma^2+h^2}/\lambda}}{4\pi}\left(\frac{1}{\left(\sigma^2+h^2\right)^{3/2}}+\frac{1}{\lambda\left(\sigma^2+h^2\right)}\right) \tag{28}$$

$$\left[h\vec{A}_2(\vec{\sigma})+\hat{n}\left(\vec{\sigma}\cdot\vec{A}_2(\vec{\sigma})\right)\right]d\sigma_1 d\sigma_2 \quad .$$

Rewriting this in terms of the vector-potential gradient tensor ***C*** defined in (23d), gives

$$\int_S \vec{\nabla}_s g(\vec{r},\vec{s})\times\left(A(\vec{s})\times\hat{n}\right)d^2s = -\frac{\lambda}{2}\left[\hat{n}(\hat{n}\cdot\boldsymbol{C}_1\hat{n}) - \hat{n}(\hat{n}\cdot\boldsymbol{C}_2\hat{n})H_1(h/\lambda)\right] - \vec{A}_2 H_2(h/\lambda) \quad , \tag{29}$$

where functions H_1 and H_2 are integrals :

sensors, a small $\delta\lambda$ may result in significant output noise. The boundary-integral approach presented here is an effective way to investigate the effect of variations in λ.

References

[1] M. Kobayashi and Y.Sugiyama, "Distributions of screening currents in circular superconducting cylinders", IEEE Trans. Magn., vol.26, pp297-298, Jan.1990

[2] H. Wilson, "Currents Induced on a Finite Superconducting Cylinder", IEEE Trans. Magn., vol.30, pp177-188, March 1994

[3] Y.N. Zhilichev, "Superconducting Cylinder in a Static Longitudinal Magnetic Field", IEEE Trans. Magn., vol.29, pp113-118, Jan. 1993

[4] T. Van Duzer and C.W. Turner, Principles of Superconductive Devices and Circuits, North Holland : Elsevier, 1981 Sec 3.03-3.06

[5] C.W. Steele, Numerical Computation of Electric and Magnetic Fields, Van Nostrand Reinhold, 1987

[6] J.A.Stratton, Electromagnetic Theory, Section 4.14, pp250-253, McGraw-Hill, 1941

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