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DEPARTMENT OF NATIONAL DEFENCE
CANADA

OPERATIONAL RESEARCH DIVISION

DIRECTOR OPERATIONAL RESEARCH (CORPORATE, AIR & MARITIME)

DOR(CAM) RESEARCH NOTE RN9601

A MODEL OF COHESION FOR AIR DEFENCE

by

B.U. NGUYEN

JULY 1996

OTTAWA, CANADA

National Defence / Défense nationale

OPERATIONAL RESEARCH DIVISION

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B.U. NGUYEN

Recommended by: Pete Smith
A/MORT Team Leader

Approved by:
DOR(CAM)

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ABSTRACT

Different military units - personnel, equipment, forces and allies - protect and support each other in combat situations. We define cohesion as this type of protection and support. This paper quantifies the benefit of one aspect of cohesion, that is the optimal weapon assignment within a group of ships defending against attack from Anti-Ship Missiles. Extensions of the model to include different types of threat and geometry are discussed.

RÉSUMÉ

Différentes unités militaires, telles que l'équipement ou le personnel, se protègent et se soutiennent en situation de combat. On définit le concept de cohésion comme étant ce type de protection et de support. Ce document quantifie les bénéfices d'un aspect de cohésion, c'est-à-dire l'allocation optimale de munition à l'intérieur d'un groupe de vaisseaux lors d'attaques par des missiles. Les extensions du modèle sont décrites pour inclure différentes types d'attaque et la géométrie du scénario.

TABLE OF CONTENTS

LIST OF VARIABLES

a_i	Number of threats targeting ship *i*
A_{ij}	Contribution of defensive missiles (resource) from ship *j* to ship *i*
C	Cohesion matrix
C_{ij}	Measure of how ship *j* supports ship *i*
$C(a,b)$	Combinatoric factor of choosing *b* objects from a set of *a* objects
d_i	Number of defensive missiles allocated to ship *i*
$G(S,A)$	Number of ways *S* defensive missiles can engage *A* ASMs
H	Probability of intercepting one ASM
M	Probability of missing one ASM
p_s	Probability of ship survival
S_i	Number of defensive missiles (resource) of ship *i*
T	Number of ships
α	Probability of success of one ASM
δ_{ij}	Measure of importance of ship *i* with respect to ship *j*
Δ_{ij}	Kronecker delta

A MODEL OF COHESION FOR AIR DEFENCE

I. INTRODUCTION

1. In combat situations, different units - personnel, equipment, forces and allies - protect and support each other. For example, soldiers cover for each other in an attack, ships are protected by aircraft, etc. This paper quantifies the benefit from resource allocation (cohesion) within a group of ships defending against attack from Anti-Ship Missiles (ASMs). However, the procedure can be extended to more general scenarios which involve more than one type of threat and different military units.

2. A simple model of cohesion is proposed. This is done for several reasons. A simple model can be implemented easily on a small computer and the results can be generated rapidly. Changes to a simple model can be made easily. Also, intuitive understanding about cohesion can be obtained at an early stage of the analysis. A simple and optimal algorithm of cohesion would be a useful decision aid for commanders at sea where decision about resource allocation has to come rapidly when the task group is under attack. It could also be helpful in allocating resources when the number of incoming threats is large.

General Model of Cohesion

3. Assume that there are T vessels in a task group. We will define a cohesion matrix C such that the coefficient C_{ij} represents how much ship j supports ship i in the event of air attack. Thus, C is a T by T matrix. Denote S_i the individual resource of ship i, which will be specified later. The resources available to ship i under attack within the task group is the sum of all contributions from other ships including itself, denoted by d_i :

$$d_i = \sum_{j=1}^{T} C_{ij} S_j = \sum_{j=1}^{T} A_{ij} \tag{1}$$

where A_{ij} is the contribution of ship j to ship i and is equal to $C_{ij}\ S_j$. As resources are limited and finite, the total resulting sum of d_i must be equal to the total sum of individual resources S_i for each ship i. The simplest way that this condition can be satisfied is to impose the following constraint on the matrix C:

$$\sum_{i=1}^{T} C_{ij} = 1 \tag{2}$$

4. The constraint described in equation (2) ensures that the total resource allocated is equal to the total existing resource. The coefficient C_{ij} is of course dependent on the attack scenario as well as the resources of ship i and ship j. For example, let the number of ships, T, be equal to 3. Let the number of defensive missiles (DMs) represent the resource in this model. Ship 1 and ship 2 have no DMs, $S_1 = 0$, $S_2 = 0$. Ship 3 has three DMs, $S_3 = 3$. Assume that there is one ASM attacking each ship, one possible way - not necessarily optimal - to defend is that ship 3 allocates one DM each to ship 1 and ship 2. The equivalent cohesion matrix is equal to the following:

$$C = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 1/3 \end{pmatrix};\ d = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix};\ S = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \tag{3}$$

II. A QUASI-UNIFORM MODEL OF COHESION

5. Roughly speaking, how much one ship helps another must be a factor of how much the latter needs help and how much the former can help. In a military operation, a force commander might determine which ship would engage a particular threat in a sequential way. This specific model provides an optimal allocation of resources, that is, this allocation maximizes the expected number of unsuccessful Anti-Ship Missiles (ASMs). Equivalently, this model represents a model of weapon assignment of an air warfare commander of a group of perfectly coordinated ships when the attack size is known.

6. In this model, the long range missile layer will be studied. We represent S_i as the number of defensive missiles available on ship i. In a similar way C_{ij} represents a measure of how many defensive missiles ship j can provide for the defence of ship i. The more important ship i, the higher value of C_{ij}. That is,

$$C_{ij}=\frac{\delta_{ij}a_i}{\delta_{1j}a_1+\delta_{2j}a_2+\ldots+\delta_{Tj}a_T} \tag{4}$$

where a_i is the number of ASMs directed towards ship i , and δ_{ij} is a numerical value measuring the importance of ship i with respect to ship j. We assume that all δ_{ij} are equal to unity, which means that all ships are of equal importance. This can be seen explicitly from equation (1) by calculating the resources devoted to ship i, d_i:

$$d_i=\frac{a_i}{a_1+a_2+\ldots+a_T}\sum_{j=1}^{T}S_j \tag{5}$$

7. In the case of a high value unit, the numerical values of δ_{ij} can be different from unity. For example, if ship 1 is the most important ship, then δ_{1j} will be much greater than all the other δ_{ij}. Weakening the assumption $\delta_{ij} = 1$ is discussed later.

8. This specific model assumes that command and control for all the ships in question is tight - that is, with a high bandwidth, fast response time, and a central computer - and that the ships are in sufficiently close formation that jamming can not be carried out effectively. Then one can model the entire task group, regardless of the number of ships, as a single huge extended ship designed specifically for a scenario with an equivalent number of resources, as in Reference (1). This means that when an attack is coming from some quarter, it is viewed as an attack targeted on the appropriate side of the extended ship and countermeasures are deployed proportionally to the total number of attacks as described in equation (5).

9. From another point of view, this specific model describes a group of ships of similar importance in the sense that the total resource is shared equally depending on the requirement of each ship. This simple model will be used to investigate the benefits of cohesion versus a model where each ship defends by itself only. Although there are several different air defence algorithms in the literature, conditional such as the one in Reference (2), unconditional such as the algorithm in Reference (3), a simple model of air defence is used: an optimal and quasi-uniform strategy of Soland, Reference (3). Soland's algorithm is briefly described in the next chapter. The basic idea of a quasi-uniform algorithm is to spread the defensive weapons as uniformly as possible among the ASMs.

III. AIR DEFENCE ALGORITHM

Single Ship Defence

10. In general, vessels can defend themselves through several layers of defence, Reference (1). Roughly, the first layer usually consists of long range defensive missiles. If one or more threats get to the inner layers, vessels would employ gun systems or soft kill systems such as seduction chaff, References (4-5).

11. The efficiency of each layer can be characterised, in Soland's model, by a probability of hit of the defensive systems, and the number of available defensive weapons on board. In this study, the long range air defence will be focused. Denote the probability of hit (miss) by H (M), and the number of DMs on ship i by S_i. (H is assumed to be constant within the engagement envelope of the ship systems.)

12. The probability of hit H is a characteristic of the specific DM and is an input of the model. Likewise S_i , the number of DMs of ship i, is an input to the model. The variable a_i denotes the number of ASM attacking ship i. As the underlying theory of this paper is related to Reference (3), a brief description of Soland's quasi-uniform algorithm is in order here.

13. The optimal strategy, Reference (3), in which to use S_i DMs against a_i ASMs is to spread them as uniformly as possible among the a_i ($a_i > 0$, if $a_i = 0$ then there is no need to retaliate). If S_i / a_i is an integer I, each of the a_i ASMs is assigned I DMs. More generally, when S_i / a_i is not an integer, $a_i + a_i \lfloor S_i / a_i \rfloor - S_i$ of the a_i ASMs are assigned $\lfloor S_i / a_i \rfloor$ DMs each and the remaining $S_i - a_i \lfloor S_i / a_i \rfloor$ ASMs are assigned $\lceil S_i / a_i \rceil$ DMs. (The notation used here is $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and $\lceil x \rceil$ is the smallest integer greater than or equal to x.)

14. For example, if $S_1=3$, $a_1=2$ then $\lfloor S_1 / a_1 \rfloor = 1$, and $\lceil S_1 / a_1 \rceil = 2$. Therefore, using Soland's algorithm, one ASM is engaged by two DMs, and the other ASM is engaged by one DM.

15. The survival probability, $p(a_i, S_i)$, of target i resulting from such engagement is, Reference (6):

$$p(a_i,S_i)=(1-\alpha M^{k_i})^{a_i-r_i}\,(1-\alpha M^{k_i+1})^{r_i} \qquad (6)$$

where α is the probability that an ASM will destroy a target, $k_i = \lfloor S_i / a_i \rfloor$, and $r_i = S_i - a_i \lfloor S_i / a_i \rfloor$.

16. References (6-8) studied optimal strategies to counter ASM attacks under different assumptions. These references, basically, presented alternatives to maximize the probability of survival of the task group weighted by the importance of each ship. In this paper, we simply present an algorithm to maximize the expected number of unsuccessful ASMs, which we show enhances the probability of survival of the task group when all the ships are identical. Note that we assume that the attack size is known. (References (9-10) studied robustness issues connected with such an assumption.)

Defence of Perfectly Coordinated Multiple Ships

17. Denote:

$$S=\sum_{i=1}^{T} S_i \tag{7}$$

$$A=\sum_{i=1}^{T} a_i \tag{8}$$

18. Using Soland's unconditional firing algorithm, and assuming that all the ships are perfectly coordinated, we show that if the total number, S, of DMs available is shared among the ships in such a way that the allocation, a_i S/A, is weighted by the number of ASMs directed at each ship, then the expected number of unsuccessful ASMs (either it is shot down or it simply misses its target) will be maximised.

19. Proof: Denote a_i d_i the number of DMs allocated to ship i. Assuming that each ship defends itself using Soland's algorithm, then the expected number of unsuccessful ASMs, J is equal to the following:

$$J=\sum_{i=1}^{T} a_i\ (1-\alpha M^{d_i}) \tag{9}$$

subject to the constraint:

$$S=\sum_{i=1}^{T} a_i d_i \tag{10}$$

This is a simple calculus problem. J is maximum if there exists a real value λ such that the following is true, for all $i = 1,\ldots,T$:

$$\frac{\partial J}{\partial d_i} + \lambda \frac{\partial S}{\partial d_i} = 0 \qquad (11)$$

Evaluation of equation (11) shows that all d_i are equal. Therefore, equation (11) gives $d_i = d = S/A$.

20. A more rigorous proof showing that $J(d_i = d = S/A)$ is a maximum and not a minimum is presented in Annex A.

21. In general, S/A is not an integer. Using the same argument as the one presented in Reference (3), the discrete optimal strategy is the following.

a. $\lceil S/A \rceil$ DMs for each of the first $R = S - \lceil S/A \rceil A$ ASMs, where the first R ASMs are selected in sequential order from a_1 , then from a_2 , etc., and where $a_1 \geq a_2 \geq a_3 \ldots \geq a_T$.

b. $K = \lfloor S/A \rfloor$ DMs for each of the last A-R ASMs.

22. From the above assignment, the probability of survival of the task group as a whole is equal to the following:

$$p_s = p(A,S) = (1-\alpha M^K)^{A-R}\ (1-\alpha M^K)^R \qquad (12)$$

23. It is interesting to note that this probability of survival is also optimal. A proof of that is presented in Annex B.

IV. EXAMPLES

24. Consider a scenario with $T=2$, $a_1 = 2$, $a_2 = 1$, $s_1=3$, $s_2= 6$, and $\alpha = 1/2$.

25. In an optimal self defence scenario, the expected number of unsuccessful ASMs, J_{SD} is equal to the following:

$$J_{SD}=3-\alpha(M+M^2+M^6) \qquad (13)$$

26. In a cohesive defence, the expected number of unsuccessful ASMs, J_{CD} can be written as:

$$J_{CD}=3(1-\alpha M^3) \qquad (14)$$

27. Figure 1 displays the two expected values (self defence and cohesive defence). It can be seen from Figure 1 that for all values of M the cohesive defence produces a higher expected value of unsuccessful ASMs. On the other hand, cohesive defence requires more sophisticated hardware and software, and possibly slower reaction time.

28. Annex C shows that there are $C(S+A-1,A-1)$ (combinatorics of S in A) ways to assign S DMs to counter the A ASMs. In this example, there are $C(11,2) = 55$ ways to assign the DMs. When the total number of ASMs, A, and the number of available DMs, S, are large, the number of combinations can be increasingly large. Therefore, such an optimal assignment is helpful in making a rapid decision.

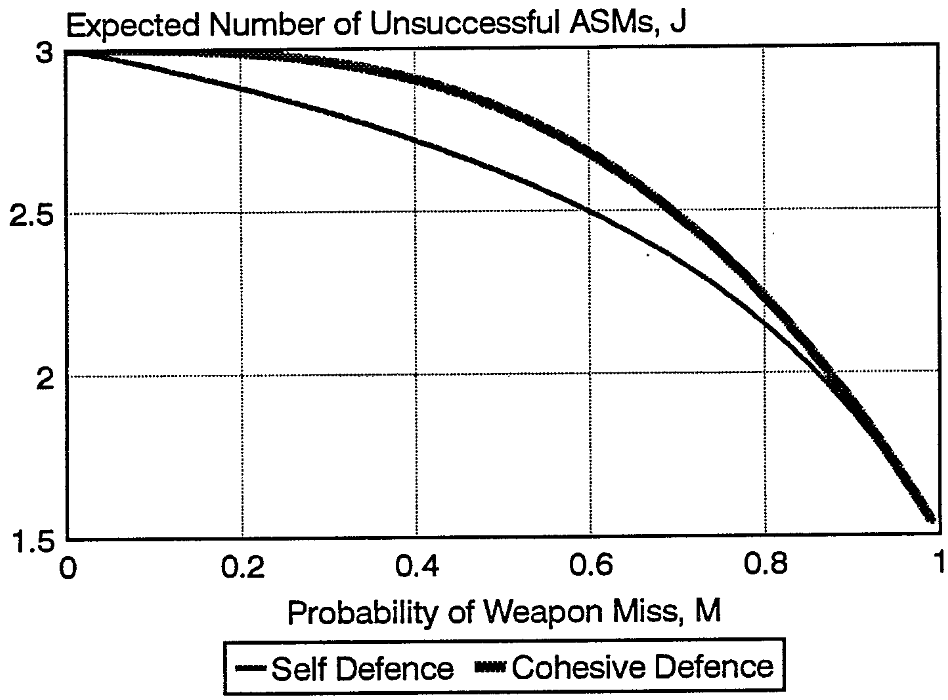

Figure 1. Expected Number of Unsuccessful ASMs

VI. FURTHER STUDIES

29. The above approach allows various refinements to be incorporated in the future. Although a specific model was built based on a few simple assumptions to analyze cohesion of a task group under air attack, these assumptions may be amended when more detail is required in the analysis.

30. For instance, geometry was not included in this re-allocation study. In practice, it does matter which direction the threats are coming from, and it is difficult to assess which ship is targeted by which ASM.

31. The first problem can be included in the general cohesion model described in the introduction. If a number of ASMs are directed towards some ship i at a specific bearing, another ship j can help the targeted ship i, if those ASMs are within the defensive range of ship j. This can be quantified by knowing the defensive range of ship j, the positions of ships i and j, and the bearings of the ASMs. One way of achieving this goal is to make the coefficient δ_{ij} of equation (4) equal to one if the ASMs are within the defensive zone of ship j, and equal to zero if the ASMs are outside the defensive zone of ship j. If a target i is more important then δ_{ij} will be large for all j. The re-allocation procedure can then be carried out as described in the general cohesion model. The outcome will be similar to the first proposed model, but will be more accurate in details as it carries information about the geometry of the ASM attacks.

32. This brings us to the second problem. Given the direction of the threats, how does one know which ship is targeted? If there are T ships, one can divide the plane into T sectors. By doing this, one can tentatively identify the number of attacks a_i in the *ith* sector which would correspond roughly to the *ith* ship.

33. When the task group does not form a compact group, this model can be broken down to include only close neighbours of the ship i under attack. For example the ship labels can be ordered in a way that ship 1 is closest to ship 2 and ship T, ship 2 is closest to ship 3 and ship 1. The cohesion matrix would then relate only ship i to its neighbours. For example, instead of equation (4), one could have the following:

$$C_{ij} \propto \frac{a_i}{a_{i-1}+a_i+a_{i+1}} \tag{15}$$

34. Generally, the task group does not defend itself with a single layer of defence. Usually it has at least three layers of defence composed of missiles, guns, and softkill systems. This study focused only on the first layer. However, if we had enough data about the other layer(s) of defence, we could perform the same procedure of re-allocation for the task group. In principle, one could build three matrices C to describe the cohesion of the task group for each layer and between each layer, or equivalently add one or two indices to the cohesion matrix. In practice, though, when the ASMs are not successfully intercepted at the first defence layer of ship i, they would be too close to ship i to be engaged by defensive systems of another ship j. In other words, when the threat is within some distance from a ship i, it is up to that ship to carry out self defence against that threat. Thus, there would be no cohesion in that limit, and therefore the cohesion matrix becomes trivial, i.e., C is equal to the identity matrix.

35. While the ASMs are serious threats, it would be natural to extend this model to include more than one type of threat. If the defence mechanism is different for different threats, and provided that the computer is powerful enough, one can construct a matrix C with an extra index k, C_{ijk} , which would describe how ship j helps ship i under an attack of type k .

VII. DISCUSSION AND CONCLUSION

Discussion

36. When considering large simultaneous air attacks, this algorithm can prove useful for two reasons. First, it is simple and hence can be run quickly by a computer. While an experienced commander might do a better job, it is not certain that he can make decisions as quickly as a computer when the simultaneous number of ASMs is high. The second reason is that, cohesion as illustrated in the examples, is optimal in the sense that it always diminishes the number of ASMs that penetrate the defences, which means a higher probability of survival for all ships in the task group.

37. (It is interesting that the eigenvalues and eigenvectors of the cohesion matrix C share a similar analogy as normal modes in classical mechanics. The corresponding eigenvalue will be the factor of enhancement of resources for each ship. For instance, in the case of self defence the eigenvalue is equal to 1.)

38. The framework for the cohesion model presented here can also be studied for different air defence algorithms. In this paper, Soland's quasi-uniform strategy, Reference (3), is used because of its simplicity. Other algorithms could be investigated in the same way. For example, conditional algorithms, Reference (2), and algorithms with or without impact point prediction, References (7-8).

Conclusion

39. A simple cohesion model is presented. It is simple to implement and yet provides an optimal expected value of unsuccessful ASMs. This model could serve as a first step towards definition of cohesion. It could also be implemented in the threat engagement and weapon analysis (TEWA) module in a simulation such as SAADS, Reference (4), to evaluate the benefit of cohesion when the number of ASMs is large.

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Annex A: Proof of Maximality of J

1. To show that $J(d_i = S/A)$ is a local maximum (instead of a minimum) under the constraint that S is a constant, we must show that the following determinants start out positive and alternate in sign, References (11-12).

$$H_2 = \det\begin{pmatrix} 0 & \frac{\partial S}{\partial d_1} & \frac{\partial S}{\partial d_2} \\ \frac{\partial S}{\partial d_1} & \frac{\partial^2 J}{\partial d_1^2} & \frac{\partial^2 J}{\partial d_1 \partial d_2} \\ \frac{\partial S}{\partial d_2} & \frac{\partial^2 J}{\partial d_1 \partial d_2} & \frac{\partial^2 J}{\partial d_2^2} \end{pmatrix} \tag{16}$$

$$H_3 = \det\begin{pmatrix} 0 & \frac{\partial S}{\partial d_1} & \frac{\partial S}{\partial d_2} & \frac{\partial S}{\partial d_3} \\ \frac{\partial S}{\partial d_1} & \frac{\partial^2 J}{\partial d_1^2} & \frac{\partial^2 J}{\partial d_1 \partial d_2} & \frac{\partial^2 J}{\partial d_1 \partial d_3} \\ \frac{\partial S}{\partial d_2} & \frac{\partial^2 J}{\partial d_1 \partial d_2} & \frac{\partial^2 J}{\partial d_2^2} & \frac{\partial^2 J}{\partial d_2 \partial d_3} \\ \frac{\partial S}{\partial d_3} & \frac{\partial^2 J}{\partial d_1 \partial d_3} & \frac{\partial^2 J}{\partial d_2 \partial d_3} & \frac{\partial^2 J}{\partial d_3^2} \end{pmatrix} \tag{17}$$

$\ldots, H_T.$

2. There is no H_1 because if there were only one ship, there would not be any resource re-allocation.

3. Calculations give:

$$\frac{\partial S}{\partial d_i} = a_i > 0 \qquad (18)$$

$$\frac{\partial^2 J}{\partial d_i \partial d_j} = -\gamma a_i \Delta_{ij} \qquad (19)$$

where

$$\gamma = \alpha \log^2(m)\, m^{S/A} \qquad (20)$$

and where Δ_{ij} is the Kronecker delta ($\Delta_{ij} = 1$ if $i = j$; $\Delta_{ij} = 0$ otherwise).

4. Substituting the values of the above derivatives into the determinants H_i, the following results are obtained:

$$H_2 = \gamma(a_1^2 a_2 + a_1 a_2^2) > 0 \qquad (21)$$

$$H_3 = -\gamma^2(a_1^2 a_2 a_3 + a_1 a_2^2 a_3 + a_1 a_2 a_3^2) < 0 \qquad (22)$$

5. In general,

$$H_i = (-1)^i (\gamma)^{i-1} (a_1^2 a_2 a_3 \ldots a_i + a_1 a_2^2 a_3 \ldots a_i + \ldots + a_1 a_2 a_3 \ldots a_i^2) \qquad (23)$$

for $i = 2,3 \ldots T$.

6. The conditions stated in References (11-12) are then automatically satisfied. Therefore, $J(d_i = S/A)$ is a maximum.

7. An alternative proof can be obtained by varying J around $d = S/A$. Denote d_i by $d + \beta_i$, and J^* by $J(d_i{=}d)$. A Taylor expansion of J^*-J can be written in the following way:

$$\begin{aligned} J^* - J &= \sum_i a_i(1 - \alpha M^d) - a_i(1 - \alpha M^{d_i}) \\ &= \alpha\left(\sum_i a_i \beta_i^2\right) M^d \log^2(M) + O(\beta^3) \end{aligned} \tag{24}$$

where $O(\beta^3)$ indicates terms of order β^3 or higher.

8. Equation (24) is positive definite for small β. Therefore, we can conclude that J^* is a local maximum.

Annex B: Optimal Probability of Survival

1. This Annex describes a simple proof that shows the probability of survival of the task group as a whole is maximized in our model when the expected number of unsuccessful ASMs is maximized.

2. Proof: The probability of survival of the task group, p_s , can be written in the following way:

$$p_s = \prod_{i=1}^{T} (1 - \alpha M^{d_i})^{a_i} \tag{25}$$

where d_i is the number of DMs assigned to each of a_i ASMs.

3. The d_i satisfy the following constraint:

$$S = \sum_{i=1}^{T} a_i d_i \tag{26}$$

4. The probability of survival, p_s , achieves a maximum when the following differential equations are satisfied.

$$\frac{\partial p_s}{\partial d_i} + \mu \frac{\partial S}{\partial d_i} = 0 \tag{27}$$

for all $i = 1, \ldots, T$. (μ is a real value number and must be independent of the index i.)

5. Equation (27) is equivalent to the following:

$$\mu = p_s \frac{\alpha \log(M) M^{d_i}}{1-\alpha M^{d_i}} \tag{28}$$

6. Equation (28) implies that all the d_i are equal. Therefore, from the constraint, equation (26) gives $d_i = S/A$ for all $i = 1,\ldots,T$. The location of this maximum coincides with the one for maximal expected value of unsuccessful ASMs.

7. The discrete version of the proof can be found in Reference (3).

Annex C: Distribution of *S* Defensive Missiles Over *A* Targets

1. Consider *S* identical DMs and *A* identical ASMs. This Annex will count all the number of possibilities that the *S* DMs can be assigned among the *A* ASMs.

2. A fixed number *k* of ASMs is chosen among *A* ASMs. The number of possibilities for such a selection is *C(A,k)*. There are different ways to distribute the DMs among the *k* ASMs. Denote x_i the number of DMs attacking the *ith* ASM. The total number of DMs attacking *k* ASMs must be *S*. Thus,

$$x_1+\ldots+x_k=S \tag{29}$$

For each x_i, there can be at least one DM and up to *S* DMs directed towards each ASM and which satisfies equation (29). Therefore, the generating function which keeps track of all the possibilities represented by equation (29) is the following.

$$\frac{x^k}{(1-x)^k} \tag{30}$$

3. Although, the number of DMs is limited, the geometric series can still be used since terms with power higher than *S* do not contribute. Using a Taylor's expansion, the above generating function can be conveniently expressed as:

$$\frac{x^k}{(1-x)^k}=\sum_l C(k-1+l,l)\ x^{k+l} \tag{31}$$

4. The number of sub-cases, for a given number k of ASMs, is the coefficient of the generating function with power equal to S. Explicitly,

$$C(S-1,S-k)=C(S-1,k-1) \tag{32}$$

5. For each number k ASMs, there are a certain number of sub-cases which can be represented as above. Therefore, the total number of scenarios possible is the sum of all these possibilities.

$$G(S,A)=\sum_k C(A,k)\ C(S-1,k-1) \tag{33}$$

6. To evaluate $G(S,A)$, the following observations can be made:

$$(1+z)^A=\sum_k C(A,k)\ z^k \tag{34}$$

and

$$C(A,k)=C(A,A-k) \tag{35}$$

7. The number $G(S,A)$ is then the coefficient of the A-1 th power in z of the product of the two polynomials, $(1+z)^A$ and $(1+z)^{S-1}$. Using the same kind of binomial expansion as in equation (31), G can be seen to be equal to $C(S+A-1,A-1)$.

8. An alternative counting method using differential equation can be found in References (13-14).

DOCUMENT CONTROL DATA
(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATOR (the name and address of the organization preparing the document. Organizations for whom the document was prepared e.g. Establishment Sponsoring a contractor's report, or tasking agency, are entered in Section 8).

Directorate of Operational Research (Corporate, Air & Maritime)
National Defence Headquarters
MGen George R. Pearkes Building
Ottawa, Ontario K1A 0K2

2. SECURITY CLASSIFICATION (overall security classification of the document, including special warning terms if applicable)

UNCLASSIFIED

3. TITLE (the complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title)

A Model of Cohesion for Air Defence

4. AUTHORS (last name, first name, middle initial)

Nguyen, Bao U.

5. DATE OF PUBLICATION (month Year of Publication of document)

JULY 1996

6a. NO OF PAGES (total containing information. Include Annexes, Appendices, etc.)

iii+23 + 2 (DCD Form)

6b. NO OF REFS (total cited in document)

14

7. DESCRIPTIVE NOTES (the category of document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

DOR(CAM) Research Note No. 9601

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include the address).

Director General Operational Research, National Defence Headquarters, Ottawa, Ottawa, Ontario K1A 0K2

9a. PROJECT OR GRANT NO. (if appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.)

9b. CONTRACT NO. (if appropriate, the applicable number under which the document was written.)

N/A

10a ORIGINATOR's document number (the official document number by which the document is identified by the originating activity. This number must be unique to this document.)

DOR(CAM) RN 9601

10b OTHER DOCUMENT NOS. (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

N/A

11. DOCUMENT AVAILABILITY (any limitations on further dissemination of the document, other than those imposed by security classification.)

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anti-ship missile (ASM)
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