Modeling scattering from azimuthally symmetric bathymetric features using wavefield superposition

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In this paper, an approach for modeling the scattering from azimuthally symmetric bathymetric features is described. These features are useful models for small mounds and indentations on the seafloor at high frequencies and seamounts, shoals, and basins at low frequencies. A bathymetric feature can be considered as a compact closed region, with the same sound speed and density as one of the surrounding media. Using this approach, a number of numerical methods appropriate for a partially buried target or facet problem can be applied. This paper considers the use of wavefield superposition and because of the azimuthal symmetry, the three-dimensional solution to the scattering problem can be expressed as a Fourier sum of solutions to a set of two-dimensional scattering problems. In the case where the surrounding two half spaces have only a density contrast, a semi-analytic coupled mode solution is derived. This provides a benchmark solution to scattering from a class of penetrable hemispherical bosses or indentations. The details and problems of the numerical implementation of the wavefield superposition method are described. Example computations using the method for a simple scattering feature on a seabed are presented for a wide band of frequencies. © 2007 Acoustical Society of America. [DOI: 10.1121/1.2785812]

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I. INTRODUCTION

The modeling of scattering from range-dependent bathymetry or surfaces is a subject of much interest in underwater acoustics. For example, bathymetric features can cause significant sonar returns or can cast an acoustic shadow behind the feature. At higher sonar frequencies, these effects can be caused by relatively small-sized features.

There have been many methods proposed for modeling scattering from targets and/or surfaces in two- and three-dimensional, range-dependent media. These include parabolic equation methods, coupled mode methods, boundary element methods, multipole expansion or wavefield superposition methods, T-matrix methods, and finite difference methods. These various methods all have their advantages and disadvantages in terms of accuracy, computational efficiency, and generality. The wavefield superposition method of this paper is straightforward to implement and the computations can be efficiently implemented. It uses a finite computational domain for the point sources representing the fields interior and exterior to the scattering region, even for distant source and receivers. The method naturally represents the fields in the domain exterior to the scattering region by using the appropriate wave number representation of the Green's function. The azimuthal symmetry of the features considered is exploited to reduce a full three-dimensional solution to solving a sequence of two-dimensional scattering problems. There are no approximations made in terms of the Green's function or the boundary conditions. A disadvantage of the method is that, even with the azimuthal assumption, it is often not feasible to model extended scattering regions. In this case, a parabolic equation method, for example, would be more appropriate. Second, because of the restriction to azimuthally symmetric objects, the method of this paper is not applicable to general three-dimensional scattering regions. However, the features which can be considered are still important approximations or benchmarks for small mounds, scours, seamounts, shoals, basins, etc. The same axially symmetric approximation or assumption used here has also been used by other models for bathymetric scattering. In other models, the bathymetry is taken to be invariant to one of the horizontal coordinates, once again allowing the three-dimensional solution to be constructed from a set of two-dimensional problems.

In this paper we describe our new approach to bathymetric scattering problems using the method of wavefield superposition. Second, in this paper, a new semi-analytic coupled mode approach is derived for the case of hemispherical bosses or scours in the case that the seabed has only a density jump with respect to the water column. Hemispherical bosses have been previously studied in acoustics and electromagnetics. The approach of this paper provides a set of solutions to a class of penetrable bosses or scours. The results from this method will provide a benchmark example for the more general wavefield superposition method of this paper. The theory of the wavefield superposition method is described. The theoretical problem of point source placement within the scattering region and irregular frequencies is also discussed. The numerical implementation details are described. A benchmark case using the semi-analytic coupled mode and wavefield superposition approaches is presented. Results from a wideband-frequency scattering computation for a simple feature on the seafloor are given both in the frequency domain and as a pulse computation using Fourier synthesis.
II. THEORY

For a scattering feature with azimuthal symmetry about the \( z \) axis, the full three-dimensional solution can be constructed from the Fourier sum of solutions to a set of twodimensional problems. This type of approach has been used by many authors for boundary element,\(^{1,25}\) finite-element methods,\(^{26}\) wavefield superposition position,\(^{27}\) thin-shell finite-element/wavefield superposition,\(^{28}\) and coupled modes.\(^{5-7}\) The discretization of a bathymetric surface might require \( N_b \) discrete points in the azimuthal coordinate and \( N_R \) in the radial coordinate. The resulting size of the matrix for the numerical computation of the unknown points sources is of the order \((N_b \times N_R) \times (N_b \times N_R)\). On the other hand, with the azimuthal transform approach there will be \( N_M \) systems of order \( N_b \times N_R \), where \( N_M \) is the required number of azimuthal Fourier components. In many cases, the computational cost of setting up and solving the \( N_M \times N_R \) linear systems is significantly less than that of solving the full three-dimensional system of equations.

For the pressure field, \( p \), and the Green’s function \( G \) in an azimuthally symmetric waveguide we can write

\[
p(r, z, \theta) = \sum_{m=0}^{\infty} p_m(r, z) \cos(m \theta)
\]

\[
p^{inc}(r, z, \theta) = \sum_{m=0}^{\infty} p^{inc}_m(r, z) \cos(m \theta)
\]

\[
G(r, z; r', z'; \theta') = \sum_{m=0}^{\infty} g_m(r, z; r', z') \cos(m(\theta - \theta')),
\]

where \((r, z)\) is a two-dimensional cylindrical coordinate system and \( \theta \) is the azimuthal angle about the axis of symmetry, the \( z \) axis. The terms \( g_m(r, z; r', z') \) are computed from the azimuthal transform of the Green’s function. In this paper we will be considering both the Green’s function for three-dimensional free space and the Green’s function for a half space and their azimuthal transforms. For example, the half-space Green’s function for the \( m \)th azimuthal order can be written for a source point \((r', z')\) in the upper half space \( z' > 0 \)

\[
g_m(r, z; r', z') = -\int_0^\infty J_m(pr) J_m(pr') \times \frac{\exp(i \gamma_1 |z - z'| + R(p) \exp(i \gamma_1 (z' + z))}{i \gamma_1} p \, dp,
\]

\( z > 0 \)

\[
= -\int_0^\infty J_m(pr) J_m(pr') \times \frac{T(p) \exp(-i \gamma_1 (z' + z'))}{i \gamma_1} p \, dp,
\]

\( z < 0 \)

where \( \gamma_1 = \sqrt{\omega^2/c^2 - p^2} \) and \( R(p) \) and \( T(p) \) are the reflection and transmission coefficients for the incident field in the upper half space. A similar expression exists for the case where the source points are in the lower half space with \( \gamma, R(p) \) and \( T(p) \) replaced by the appropriate expressions.

In this paper, the expression for the pressure field from a free-field point source is given by

\[
G(r, z, \theta; r', z', \theta') = \frac{1}{R} \exp(ikR)
\]

where \( k = \omega/c \) and

\[
R = \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z - z')^2}.
\]

The azimuthal transform of this function can be expressed as

\[
g_m(r, z; r', z') = \frac{1}{R} \exp(ikR) \cos(m \theta) d\psi
\]

where \( \psi = \theta - \theta' \), recalling from Eq. (4) that \( R \) depends on \( \theta \) and \( \theta' \) through their difference. There are a variety of analytical expressions for Eq. (5) either as a series of Legendre polynomials and spherical Hankel or Bessel functions or in terms of elliptic integrals.\(^{23}\) In this paper, we will simply discretize the integral and simultaneously evaluate the integral for several values of the azimuthal order \( m \).

For the half space Green’s function, we can rewrite it in the form, for \( z' > 0 \)

\[
g_m(r, z; r', z') = -\int_0^\infty J_m(pr) J_m(pr') \times \frac{\exp(i \gamma_1 |z - z'| + R(p) \exp(i \gamma_1 (z' + z))}{i \gamma_1} p \, dp,
\]

\( z > 0 \)

\[
= -\int_0^\infty J_m(pr) J_m(pr') \times \frac{T(p) \exp(-i \gamma_1 (z' + z'))}{i \gamma_1} p \, dp,
\]

\( z < 0 \)

where

\[
l_{dir} = \int_0^\pi \exp(ik_1 \sqrt{r^2 + r'^2 - 2rr' \cos \psi + (z - z')^2}) \times \cos(mp) d\psi
\]

\( z > 0 \)

\[
l_{ref} = R \int_0^\pi \exp(ik_1 \sqrt{r^2 + r'^2 - 2rr' \cos \psi + (z + z')^2}) \times \cos(mp) d\psi
\]

\( z < 0 \)

\[
l_{trans} = T \int_0^\pi \exp(ik_1 \sqrt{r^2 + r'^2 - 2rr' \cos \psi + (z - z')^2}) \times \cos(mp) d\psi
\]

and
boundary conditions along the water/seabed interface. In the case where we are considering a basin (i.e., a negative deformation of the seabed) the situation is almost exactly the same, except that now the interior values for the sound speed and density for $\Omega$ are those of the upper half space. If a deformation is both positive and negative with respect to the flat surface $z=0$ m, then there will be an area of the exterior regions which is not the simple half space (for example, an intrusion of the upper medium into the bottom half space). In this case, one must use a larger bounding surface and introduce additional unknown point sources along the actual surface deformation. We will not consider this case in this paper.

Taking $N$ sources with unknown amplitudes $\{a_i\}$ in the exterior region (representing the interior field) and $N$ sources in the interior region $b_i$ (representing the exterior field), the continuity equations across the bounding curve become, first, for the continuity of the pressure field

$$\sum_{i=1}^{N} a_i G_m^h(\mathbf{X}_k; \mathbf{X}_i + \mathbf{\delta}_i) = \sum_{i=1}^{N} b_i G_m^h(\mathbf{X}_k; \mathbf{X}_i - \mathbf{\delta}_i) + p_m^{inc}(\mathbf{X}_k), \quad k = 1, \ldots, N$$

and the normal derivative across the boundary

$$\frac{1}{\rho_{in}} \sum_{i=1}^{N} \frac{\partial G_m^h(\mathbf{X}_k; \mathbf{X}_i + \mathbf{\delta}_i)}{\partial n} = \frac{1}{\rho_{ext}(\mathbf{X}_k)} \left( \sum_{i=1}^{N} b_i \frac{\partial G_m^h(\mathbf{X}_k; \mathbf{X}_i - \mathbf{\delta}_i)}{\partial n} + \frac{\partial p_m^{inc}(\mathbf{X}_k)}{\partial n} \right),$$

$$k = 1, \ldots, N.$$  

In these equations we have indicated the discrete points along the curve $C$ as $\mathbf{X}_i = (r_i, z_i)$ and $\mathbf{X}_k = (r_k, z_k)$. The source points are displaced slightly from these points, along the normal, in the positive direction $+\mathbf{\delta}_i$ for the exterior sources and $-\mathbf{\delta}_i$ for the interior sources. The superscripts $f$ and $h$ with the Green’s function refer to the freespace and the water/seabed Green’s functions, respectively. In the case where the bathymetric feature is a positive deformation into the water column, then $\rho_{in} = \rho_2^2$ for a negative deformation $\rho_{in} = \rho_1$. The density term $\rho_{ext}(\mathbf{X}_k)$ is equal to $\rho_1$ for $\mathbf{X}_k$ such that $z_k \geq 0$ and is equal to $\rho_2$ otherwise. Thus the normal derivative (the radial derivative) of the incident field is discontinuous across $z=0$. It should also be noted that although we are stressing the modeling of scattering by three-dimensional bathymetric features, the region $\Omega$ can also model an embedded region of different sound speed and density. In this case the interior values of $c$ and $\rho$ would be different from those of the two surrounding half spaces.

III. A DERIVATION OF A COUPLED-MODE SOLUTION FOR A CLASS OF PENETRABLE HEMISPHERICAL BOSSES OR BASINS

In previous references, modal solutions were derived for a half-space problem in the case where the two half

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spaces had the same sound speeds but different densities. In Ref. 30 this allowed for the solution of scattering from an embedded cylinder in terms of a coupled mode solution. In this paper, we will derive a corresponding three-dimensional solution. In particular, we will describe the coupled mode problem to solve for each azimuthal order \(m\). Let us consider a sphere which is exactly half buried between two half spaces, each with sound speed \(c_{\text{ext}}\) and densities \(\rho_1\) and \(\rho_2\). The sphere has an internal sound speed \(c_{\text{in}}\) and density \(\rho_{\text{in}}\). In the case where \(c_{\text{in}} = c_{\text{ext}}\) and \(\rho_{\text{in}} = \rho_1\) we have a basin problem and for \(\rho_{\text{in}} = \rho_2\) we have a boss problem. The general solution in the interior of the sphere can be expanded in terms of the angular functions

\[ \psi^m_n(\phi, R) = N^m_n P^m_n(\cos \phi), \]  

where \(P^m_n\) denotes a Legendre function, and \(N^m_n\) is the normalization constant such that

\[ \int_0^\pi \frac{1}{\rho_n} (\psi^m_n(\cos \phi))^2 \sin \phi d\phi = 1. \]  

This angular function has the associated radial function, \(j_n(k_{\text{in}} R)\) where \(j_n\) denotes a spherical Bessel function.

In the exterior domain, the angular functions have the form for \(n = 0, 2, 4, \ldots\)

\[ \tau^m_n(\cos \phi) = P^m_{n+m}(\cos \phi) \quad 0 \leq \phi \leq \pi \]  

and for \(n = 1, 3, 5, \ldots\)

\[ \tau^m_n(\cos \phi) = \frac{D_n}{P_n} P^m_{n+m}(\cos \phi) \quad \pi/2 \leq \phi \leq \pi. \]  

These modal functions are then normalized such that

\[ \int_0^\pi \frac{1}{\rho_n}(\tau^m_n(\cos \phi))^2 \sin \phi d\phi = 1. \]  

The associated radial functions for these angular functions are \(h_n(k_{\text{ext}} R)\). The density term \(\rho(\phi)\) in Eq. (17) is simply \(\rho_1\) for \(0 \leq \phi \leq \pi/2\) and \(\rho_2\) for \(\pi/2 \leq \phi \leq \pi\). The specified form of the angular modal functions is used because for \(n + m = m\), \(m + 2\), \(m + 4\), \ldots the associated Legendre functions are even (and have zero derivative with respect to \(\phi\)) and thus are continuous across \(z = 0\), \(\phi = \pi/2\), and trivially satisfy

\[ \frac{\partial \tau}{\rho_1 \partial \phi} = \frac{\partial \tau}{\rho_2 \partial \phi} \]  

at \(\phi = \pi/2\).

The Legendre functions \(P^m_{n+m}(\cos \phi)\) are odd with respect to \(\phi\) at \(\phi = \pi/2\) for \(n = 1, 3, 5, \ldots\) and thus, in fact, the definition of \(\tau\) as in Eq. (16) is continuous (zero) at \(\phi = \pi/2\) and satisfies the boundary condition of Eq. (18). We can now set up a system of equations to be satisfied along the surface (in this coordinate system, \(R = a\) and \(0 \leq \phi = \pi\)) of the sphere

\[ \sum_{i=0}^N c_i \psi^m_i(\cos \phi) j_{i+m}(k_{\text{in}} R) = \sum_{i=0}^N d_i \tau^m_i(\cos \phi) h_{i+m}(k_{\text{ext}} R) \]

\[ + ik_{\text{ext}} \sum_{i=0}^N \tau^m_i(\cos \phi) j_{i+m}(k_{\text{ext}} R) h_{i+m}(k_{\text{ext}} R) \tau^m_i(\cos \phi) \]  

\[ = \frac{1}{\rho_{\text{in}}} \sum_{i=0}^N c_i \psi^m_i(\cos \phi) k_{\text{in}} j_{i+m}(k_{\text{in}} R) \]

\[ + ik_{\text{ext}} \sum_{i=0}^N \tau^m_i(\cos \phi) k_{\text{ext}} j_{i+m}(k_{\text{ext}} R) h_{i+m}(k_{\text{ext}} R) \tau^m_i(\cos \phi) \]

where the last terms of Eqs. (19) and (20) are the expansions of the incident field for a point source at \((R_s, \phi_s)\).

For the system of equations in Eq. (19), we project \(1/\rho_{\text{in}} \psi^m_i(\cos \phi) \sin \phi\) \(j_{j=0, \ldots, N}\) on both sides and integrate. For the systems of equations in Eq. (20) we project \(\tau^m_i(\cos \phi) \sin \phi\) on both sides and integrate. The last terms in Eqs. (19) and (20) are the expansion of the incident field from a point source located (in spherical coordinates) at \((R_s, \phi_s, \theta_s) = (0)\). The resulting system of equations has the matrix/vector form

\[ \begin{pmatrix} D_1 & R_1 \\ R_2 & D_2 \end{pmatrix} \begin{pmatrix} \tilde{c} \\ \tilde{d} \end{pmatrix} = \begin{pmatrix} \tilde{r}_1 \\ \tilde{r}_2 \end{pmatrix}. \]

Here \(D_1\) and \(D_2\) are diagonal matrices with entries \(j_{j+m}(k_{\text{in}} R)\) and \(h_{i+m}(k_{\text{ext}} R)\), respectively. The matrices \(R_1\) and \(R_2\) have the form

\[ R_1(p, q) = h_{q+m}(k_{\text{ext}} R) \frac{1}{\rho_{\text{in}}} \int_0^\pi \psi^m_q(\cos \phi) \tau^m_p(\cos \phi) \sin \phi d\phi. \]

\[ R_2(p, q) = j_{q+m}(k_{\text{in}} R) \frac{1}{\rho_{\text{in}}} \int_0^\pi \psi^m_q(\cos \phi) \tau^m_p(\cos \phi) \sin \phi d\phi. \]

\[ \tilde{r}_1 \text{ and } \tilde{r}_2 \text{ represent the known incident field terms. It is interesting to note that the solutions to this scattering problem provide a generalization to the case where the interface boundary condition is pressure release \((p_2 \rightarrow 0)\) or rigid \((p_2 \rightarrow \infty)\). We have verified numerically that in the limit of pressure release and rigid bosses (positive into the water column) that the correct analytic solutions are obtained.} \]

The solution of the system of equations of Eq. (21) yields the interior coefficients \(\tilde{c}\) and the exterior coefficients \(\tilde{d}\) which can then be used to compute the scattered \(p_{n}(r, z; m)\) and total pressure fields \(p(r, z; m)\). The scattered

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pressure field is defined as the difference between the total field and the incident field which would exist in the absence of the boundary feature. These two-dimensional fields are computed for \( M \) values of the azimuthal order \( m \) and the three-dimensional fields are then computed (for example, the scattered field) from
\[
P_{\infty}(r, z, \theta) = \sum_{m=0}^{M} \epsilon_m p_{\infty}(r, z; m) \cos(m \theta)
\]  
(24)

where \( \epsilon_m = 1 \) for \( m = 0 \) and \( \epsilon_m = 2 \) otherwise.

IV. NUMERICAL COMPUTATIONS

In this section we describe some of the details of the implementation of the coupled mode benchmark solution and of the wavefield superposition approach. This is followed by a comparison of the solutions from the two methods in the case of the seabed having only a density change. Some issues with the superposition approach are described and discussed. The method is then applied in the case of a deformation of a seabed having a different sound speed, density, and attenuation.

A. Numerical implementation

The numerical implementation of the coupled mode benchmark solution of Sec. III is straightforward. The modal projection integrals of Eqs. (22) and (23) are evaluated numerically. The columns of the matrix of Eq. (21) are normalized by the appropriate spherical Bessel or Hankel function, \( j_{\nu}(KR) \) or \( h_{\nu}(KR) \) in the case where the order is greater than 1.5 the value of the argument. This is done in order to prevent a column of values from being excessively small or large for large orders of the Bessel and Hankel functions.

In the case of a general seabed, the solution of the system of equations for the wavefield superposition source amplitudes, Eqs. (11) and (12), yields the coefficients \( \{a_i\} \) and \( \{b_i\} \) of the displaced point sources. The matrix corresponding to the system of linear equations involves either the freespace or two half-space Green's function (or their normal derivatives) evaluated at a large set of source/receiver positions. The expressions for the Green's function in the region exterior to the scattering region involve evaluations of wave number integrals [see, for example, Eq. (6)] and the azimuthal transform of the extracted singular terms [e.g., Eqs. (7) and (8)]. The extracted terms are not singular in the case of the wavefield superposition method as there is always a nonzero distance between the source and boundary points and thus these terms could be included in the wave number integral. However, by extracting analytic terms [as in Eq. (6)], it is hoped that the convergence of the wave number integrals, in terms of the required upper limit of the integral, is improved. In order to evaluate the azimuthal transforms of the analytically extracted terms, 201 discretization points were used and the integrals were evaluated for multiple values of \( \cos(m \theta) \) without recomputing the portion of the integrand which is independent of the azimuthal order \( m \). In the density-jump only case, the wave number integrals are zero and only the extracted terms contribute. In the numerical

evaluation of the wave number integrals for the general seabed case, there are various parameters to consider: the number of discretization points, the contour offset into the complex-\( p \) plane, and the upper limit of integration. The suitable choice of these types of parameters arises in all implementations of wave number integrals. For example, a discussion of these issues for fast field programs can be found in Ref. 31. In this paper, we consider integrals of the form
\[
\int_0^{U_{LOA}} J_m(p(t)r)J_m(p(t)r_c)K(p(t),z,z_c)p(t)\frac{dp}{dt} dt
\]  
(25)

for
\[
p(t) = t - i e \sin\left(\frac{\pi t}{t_{LOA}}\right)
\]  
(26)

and \( t_{LOA} = U_{LOA}/c \) (\( c \) is taken in our examples as 1500 m/s). This choice of the contour of \( p \) deforms the integration contour away from the real line except at the origin and at the upper limit of integration where it comes back to the real axis.

In all our computations we used 501 uniformly space values of \( t \) in Eqs. (25) and (26). This is a sufficient number to avoid any spatial aliasing issues which arise when the wave number sampling size is too large. In the case of the computation of the Green's function and its normal derivative for the system of equations for the amplitude coefficients, the upper limit of integration factor, \( U \) [Eq. (25)] varies with frequency. In the multiple frequency computation which is considered in the numerical examples, this upper limit factor \( U \) varied from 16 at \( f = 500 \) Hz to 6 at \( f = 2500 \) Hz. For this example, the contour offset factor \( e \) in Eq. (26) was set to 2 for all frequencies. In the numerical example with this frequency range, the maximum horizontal ranges of interest for the matrix Green's function were less than 2.3 m. We will also consider a problem where the dimensions of the scattering feature are increased by a factor of 100 and the frequency is 25 Hz. In this case, the same parameters as the 2500 Hz/small feature case are used with the exception of \( e \) which is decreased to 0.02.

Once the coefficients \( a_i \) and \( b_i \) of Eqs. (11) and (12) have been computed, they can be used in conjunction with the appropriate Green's function to compute the scattered and total fields. In the computations of the pressure field values for a grid of receiver positions, we simply used wave number integral representations with no subtraction of analytic expressions. In this case, the upper limit factor \( U \) for these wave number integrals and the contour offset \( e \) are increased by a factor of two.

The computations of the field values are done for each azimuthal order \( m \). The final solution is computed from the azimuthal sums of the form of Eq. (24). In cases where the incident pressure field is added to the solution (for example, to compute the total field in the exterior region), the incident field is computed only once in the form.
\[ p^{nc}(r, z; r_s, z_s) = \int_0^\infty J_0(p|r - r_s)|K(p; z_s)p\, dp \]  

(27)

where \( K(p; z_s) \) represents the appropriate kernel for the upper or lower half space. In other words, we compute the incident field with a three-dimensional expression with the origin at the source position.

The numerical code was implemented in MATLAB. The integrals were performed in a discretized form using matrix/vector operations where possible. For example, the wave number integrals of Eq. (6) (and also the set for \( z' < 0 \)) can be written in the separable form

\[ \int_0^\infty K_1(r, z; p)K_2(r', z'; p)F(p)dp. \]  

(28)

For example, the second expression of Eq. (6) can be written as

\[ \int_0^\infty J_m(pr)J_m(pr') \left( T(p)\exp(-i\gamma_2 z') \right) dp \]

\[ \times \left[ J_m(p)r' \left( \frac{\exp(i\gamma_1 z')}{i\gamma_1} - \frac{T_m \exp(i\gamma_2 z')}{i\gamma_2} \right) \right] dp. \]

(29)

This form allows us to efficiently evaluate the discretized version of the integral for the various combinations of source and receiver positions without reevaluating the integrand for each possible source/receiver pair. In addition, any integrand quantities which do not involve the azimuthal order \( m \) are only evaluated once during the computations for all the required azimuthal orders.

It was found that at high azimuthal orders the system of equations for the point sources sometimes became ill-conditioned. This is a result of the fact that the coefficients of the point sources for small values of the cylindrical radius \( r \) were poorly determined because their net contribution to the pressure field is very small. Because of this problem, a value of \( 10^{-8} \) was used at lower bound on the singular values in the solution of the system of equations.

As will be illustrated in the numerical examples, there is another source of ill-conditioning, due to the curve of the interior point sources and the corresponding interior Dirichlet problem. The solution for the interior point sources becomes ill conditioned (and this will happen at each azimuthal order) near frequencies corresponding to the eigenvalues of the interior Dirichlet problem as defined by the interior curve used for the source locations. In Ref. 32 it was suggested that one could use a combination of monopoles and dipoles to alleviate this problem. The approach we will use is to consider two curves of point sources each slightly offset from each other by a small separation \( \Delta \) (along the normal direction). Then, for example, we can consider one source to have the weighting \((0.5 + i(k\Delta))\) and the corresponding source from the other curve to have the weight \((0.5 - i(k\Delta))\) where \( k = 2\pi f/c \). These two sources in conjunction with each other approximate a monopole and dipole along the midway curve. In the construction of the system of equations, the contributions from the two sets of point source locations are evaluated and then combined with the weightings discussed above. The system size and the number of unknowns is the same as simply using a set of monopoles.

**B. Density-jump benchmark solution**

As a first numerical benchmark we will use both the benchmark modal approach and the superposition approach. We will consider a spherical region (radius = 1.0 m) with an interior sound speed \( c_i = 1500 \text{ m/s} \) and \( \rho = 1 \text{ g/cm}^3 \). The upper surrounding half space has a sound speed of 1500 m/s and a density of 1 g/cm\(^3\) and the lower half space a sound speed of 1500 m/s and a density of 5 g/cm\(^3\). This scenario represents a half-spherical basin intruding into the lower, high-density half space. A point source is located at 4 m above the seabed/water interface and along the \( z \) axis. In this case only the order \( m = 0 \) need be considered. The bounding chamber is simply a half circle in the \( (r, z) \) space in this case. There are 62 unknown coefficients for both the interior and exterior sources resulting in a \( 124 \times 124 \) system of equations. The interior point sources are positioned at 0.05 m from the boundary (i.e., at a radius of 0.95 m) and the exterior sources are located at a radius of 1.05 m. In Fig. 2 we show the amplitude of the scattered field (solid line) as a function of frequency for a point approximately along the \( z \) axis and at a height of 3.33 m above the seabed. As can be seen, the amplitude suffers a large jump at approximately 789.5 Hz which corresponds to \((2\pi f/c)0.95\) being equal to a zero of \( j_0 \). The significance of this term can be seen from the expansion of the free space Green's function.
\[ G^m(R, \phi; R_0, \phi_0) = \sum_{n=-\infty}^{\infty} P_n^m(\cos \phi)\cos n\phi_0 j_n(kR_\perp)h_n(kR_\perp). \] (30)

In this equation one can see that for an interior source representing an exterior field (in the case of a homogeneous exterior) that the coefficient of the nth Legendre function will be zero in the case where \( j_n(kR_\perp) = 0 \). Next, a second set of interior sources is added at \( R_0 = 0.93 \) m along with the sources at 0.95 m to approximate a set of monopoles and dipoles at \( R = 0.94 \) m (with a single set of unknown coefficients). As can be seen by the dashed curve, the new solution is now smooth as a function of frequency. [From our experience, it does not seem necessary to introduce a second set of exterior sources. This seems reasonable if one considers the expansion of the free space Green’s function, Eq. (30), where we do not need to be concerned with zeros of \( h_n(kR_\perp) \)].

A point source of 2500 Hz is now located at a range of 4 m and an angle of 10° from the \( z \) axis. In this case, the positioning of the point sources is the same as for the previous example, with an interior set of two circles of slightly displaced sources and an exterior set. However, in this case we use 122 exterior point sources representing the interior field and 122 double point sources in the interior, representing the exterior field. The size of the resulting system of equations is 244 x 244. The number of azimuthal terms used was given by the expression \( n_j = 2\pi / 1500 \times 1.5 + 9 \). As a benchmark solution we use the modal solution, using the same number of azimuthal terms as for the scattering-chamber approach. The number of modes considered for each azimuthal order is given by the expression, \( n_{mode} = 2\pi / 1500 \times 3 + 16 \). In Fig. 3 we show a comparison of the computed total fields along the line \( z = 0 \) (solid line is modal solution, dashed is superposition approach). As can be seen, the agreement is excellent. The points \( x = \pm 1 \) m are the boundaries of the basin. In Fig. 4 we show the total pressure field computed using the scattering chamber approach for a grid of positions. As can be seen, this “basin” has a significant effect upon the propagation and there is a strong focusing feature near the bottom. It is also important to note that there are no observable discontinuities in going across the boundary \( r = 1 \) where \( r \) denotes the distance from the origin. The values for \( x < 0 \) are computed by using \( \theta = \pi \) in the azimuthal sum.

The pressure field is predicted to be somewhat ill-behaved near \( x = \pm 1 \) m, \( z = 0 \) m. This is because there is a discontinuity in the boundary condition for \( p_n / \rho \theta \) across the line \( z = 0 \). In Fig. 5 we show the real part of the amplitudes for the exterior (the first 122 points) and interior (the last 122 points) for \( m = 2 \) and \( f = 2500 \) Hz. As can be seen, the coefficients become larger in amplitude and more erratic as a function of position toward \( z = 0 \). However, as seen from Fig. 4, they do combine to give a nicely continuous solution across the boundary \( r = 1 \) m.

### C. A general seabed feature

We now consider the more general situation of an axisymmetric feature with the upper half space consisting of

![Figure 5](image5.png)

FIG. 5. The real part of the solution for \( m = 2 \) for the point source amplitudes for the superposition approach at 2500 Hz. The values near 61 and 183 represent the solution near \( r = 1 \) m and \( z = 0 \) m.
one fluid $c_1$, $\rho_1$ overlying the seabed $c_2$, $\rho_2$. We consider the following analytical expression for the deformation of the seabed:

$$z(r) = \frac{a}{2} \left( 1 + \cos \left( \frac{2\pi r}{w} \right) \right).$$

(31)

We will consider $a=2$ and $w=4$; this is a “bump” which has a height of 2 m and a radial extent of 2 m. It is important to recall that the actual feature results from rotating this cross section about the $z$ axis, so that the true lateral extent in a horizontal slice is 4 m. A point source in the water column at 10 m range (here, $\sqrt{r^2+z^2}$) and at a height of 1 m above the seabed is considered. For the upper medium, $c_1=1500$ m/s, $\rho_1=1$ g/cm$^3$, and for the lower sediment $c_2=1800$ m/s and $\rho_2=1.5$ g/cm$^3$. The lower medium has an attenuation of 0.3 dB/λ which is implemented by using a complex-valued $c_2$. The two-dimensional slice of the scattering feature, the additional bounding curve in the bottom, and the source points are shown in Fig. 6. The number of azimuthal terms used in the computation varied with respect to frequency according to the relation $n_{\alpha}=6\pi f/1500+6$. This results in 37 (38, including $m=0$) azimuthal terms for the frequency of 2500 Hz. There are 121 interior and exterior point sources. The distance of these sources from the curve is important in obtaining accurate solutions and it was found that optimal position varied somewhat with frequency. For lower frequencies, the distance was greater than at higher frequencies. This is reasonable if one considers the longer wavelength at lower frequencies. In the following we compute the pressure fields for frequencies between 500 and 2500 Hz in steps of 25 Hz. Two sets of interior point sources (but with the same coefficient for each radially displaced pair) were used. For 500 Hz $\leq f < 1200$ Hz, the interior point sources were displaced 0.12 and 0.08 m. For frequencies greater than or equal to 1200 Hz, the displacements were 0.09 and 0.07 m. The external point sources were placed at the greater of the two distances away from the boundary. An example of this discretization is shown in Fig. 6 for the 500 Hz case. This sequence of distances yielded very good scattering results as a function of frequency. It is interesting to note that for the pair of interior point sources closest to the line $z=0$, one of the point sources is above $z=0$ and one is below. We use the Green’s function representation appropriate for the specified individual source location. The parameters used in the numerical implementation of the wave number integrals were described previously.

In Figs. 7 and 8 we show the computed pressure fields for six different frequencies for a vertical grid of receiver locations and then for a horizontal grid at a fixed vertical distance of $z=0.50$ m. In these figures there are no observable discontinuities of the fields across the scattering surface and the additional bounding curve in the bottom. It is interesting to note that since the feature is penetrable, some energy propagates into the bump and out the other side. In Fig. 8 the circle denotes the boundary of the bathymetric feature at this altitude above the seabed and hence the acoustic parameters within this circle correspond to those of the seabed. As one would expect, the interference patterns between the direct and reflected incident fields and the scattered field become more complicated for higher frequencies.

In Figs. 9 and 10, time domain computations are shown. The frequency results were transformed from the frequency domain to the time domain by using the expression

$$p(x,y,z,t) = \mathcal{F} \left\{ \sum_{n=1}^{N} p(x,y,z;f_n) \exp(-i2\pi f_n t) S(f_n) \right\}
$$

(32)

where for the source weighting we used...
the propagation of the pulse in a horizontal plane can be seen. In panel(c) the wavefront is just entering into the feature at this depth and begins to pull ahead of the wavefront on the outside. In panel(f) a backscattered wavefront within the feature can be observed.

The scattering feature, just considered, was 2 m in height. Even with this small height, 37 azimuthal orders were used for a frequency of 2500 Hz. A much larger version of the same scattering feature can be considered for lower frequencies. For example, at 25 Hz we can increase the height of the feature to 200 m as well as increasing its lateral extent. One would expect that one should obtain the same pressure fields (amplitude reduced by a factor of 100) if all the spatial dimensions of the original problem for 2500 Hz are increased by a factor of 100 (including the source distance). Thus to compute the field scattered by this large feature at 25 Hz, we simply scale the spatial dimensions of the parameters used in the 2500 Hz computation. The only parameter we change for the numerical evaluation of the wave number integrals is the contour offset which is reduced from 2 to 0.02 in Eq. (26). The resulting pressure field (multiplied by a factor of 100) is shown in the top plot of Fig. 11 and the corresponding 2500 Hz computation is shown in the bottom plot. As can be seen, the two results are in excellent agreement. This example illustrates: (a) the applicability of the method to model large feature scattering at low frequencies and (b) the numerical consistency between the high- and low-frequency computations.

V. SUMMARY

In this paper we have described an efficient and accurate approach for computing the pressure field scattered from a
three-dimensional seabed feature. The feature was taken to be azimuthally symmetric and the deformation to the surrounding flat seabed was taken to be of one polarity, positive—a “bump” and negative—a “basin.” The computational method solved a sequence of two-dimensional problems and then used Fourier synthesis to construct the full three-dimensional solution. A benchmark solution for the case of a hemispherical deformation with a density-contrast seabed was derived. This type of solution was used to show that the wave superposition/scattering chamber approach was very accurate for this case. The problem of irregular frequencies for the superposition method was addressed by the introduction of a second set of interior point sources. This more general method was then used to solve the scattering from a 2 m high, penetrable “bump.” The pressure field was accurately computed for a large range of frequencies for both vertical and horizontal planes. At the highest frequency, 37 azimuthal orders were computed. The multifrequency computations allowed for the computation of the pressure field in the time domain for an incident pulse. Finally, we concluded with an example of a low-frequency (25 Hz) computation for a 200 m high scattering feature.


