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Nonlinear filtering and quantum physics

A Feynman path integral perspective

Bhashyam Balaji

Defence R&D Canada – Ottawa

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Principal Author

Original signed by Bhashyam Balaji

Bhashyam Balaji

Approved by

Original signed by Anthony Damini

Anthony Damini
Head/Radar System

Approved for release by

Original signed by Dr. Brian Eatock

Dr. Brian Eatock
Chair/Document Review Panel

Abstract

In this report, it is shown that Euclidean quantum mechanics is closely related to the continuous nonlinear filtering problem. The key is the configuration space Feynman path integral representation of the fundamental solution of a Fokker-Planck type of equation termed the Yau Equation of continuous-continuous filtering. A corollary is the equivalence between nonlinear filtering problem and a time-varying Schrödinger equation previously pointed out by S-T. Yau and Stephen Yau. The path integral formulation is shown to lead to a better conceptual understanding of the origin of this relationship.

Résumé

Dans ce rapport, nous montrons qu'il existe un lien étroit entre la mécanique quantique euclidienne et le problème du filtrage non linéaire en temps continu. L'explication de ce lien est la représentation, par l'intégrale de chemin de Feynman sur l'espace de configuration, de la solution fondamentale d'une équation de type Fokker-Planck appelée l'équation du filtrage continu-continu de Yau. Un corollaire est que l'on peut établir un rapport d'équivalence entre le problème du filtrage non linéaire et l'équation de Schrödinger dépendante du temps relevée par S.T. Yau et Stephen Yau. On constate que la formulation de l'intégrale de chemin permet de mieux comprendre l'origine de ce rapport.

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Executive summary

Nonlinear filtering and quantum physics

Bhashyam Balaji; DRDC Ottawa TM 2009-196; Defence R&D Canada – Ottawa; November 2009.

The general approach to solution of the nonlinear filtering problem is based on solving the Fokker-Planck-Kolmogorov forward equation (FPKfe) for the continuous-discrete problem and the Yau equation (YYe) for the continuous-continuous filtering problem. This is more general than Kalman filtering since the signal and measurement models need not be linear. Furthermore, unlike the extended Kalman filter, the signal and/or measurement models are not linearized.

The calculation of the kernel, or fundamental solution, of the FPKfe and the YYe is the nontrivial step in the solution of the filtering problem. This is because from the fundamental solution the general solution can be obtained by integration.

It has been noted that both the continuous-discrete and the continuous-continuous filtering problems can be formulated and solved using Feynman path integrals. Specifically, the fundamental solution for the FPKfe and the YYe can be written as a Feynman path integral. The advantages of the Feynman path integral formulation are many, including the possibility of development of reliable and efficient algorithms for higher-dimensional problems.

Since Feynman path integrals arise in both filtering theory and quantum physics, it is natural to expect that there is a close relationship between the two subjects. In this report, the Feynman path integral formula for the fundamental solution of the FPKfe and the YYe is used to point out a close relationship between nonlinear filtering and Euclidean quantum mechanics. Also included are remarks that clarify the origin of this mathematical equivalence and the significant conceptual differences between the two subjects. In addition, the report also includes a review of relevant aspects of classical physics, canonical and path integral formulation of quantum mechanics.

Sommaire

Nonlinear filtering and quantum physics

Bhashyam Balaji ; DRDC Ottawa TM 2009-196 ; R & D pour la défense Canada – Ottawa ; novembre 2009.

La solution du problème du filtrage non linéaire passe généralement par la résolution de l'équation de Fokker-Planck-Kolmogorov (FPK), pour ce qui est du filtrage continu-discret, et de l'équation de Yau (YY) pour ce qui est du filtrage continu-continu. C'est une solution plus générale que celle offerte par le filtre de Kalman, puisqu'il n'est pas nécessaire que les modèles de signal et de mesure soient linéaires. De plus, contrairement au filtre de Kalman étendu, l'un et l'autre de ces modèles ne sont pas linéarisés.

Le calcul du noyau, ou de la solution fondamentale, de l'équation FPK et de l'équation YY constitue l'étape non triviale de la résolution du problème du filtrage, car on peut déduire la solution générale de la solution fondamentale au moyen de l'intégration.

Il est à noter que l'on peut formuler et résoudre les problèmes du filtrage continu-discret et du filtrage continu-continu au moyen des intégrales de chemin de Feynman. Plus précisément, on peut exprimer la solution fondamentale de l'équation FPK et de l'équation YY sous la forme d'une intégrale de chemin de Feynman. Cette formulation présente de nombreux avantages, y compris la possibilité de concevoir des algorithmes fiables et efficaces pour résoudre les problèmes à plusieurs dimensions.

Comme la théorie du filtrage et la physique quantique traitent l'une et l'autre des intégrales de chemin de Feynman, il est naturel de penser que les deux domaines soient étroitement liés. Dans ce rapport, nous nous servons de la formulation de la solution fondamentale des équations FPK et YY par l'intégrale de chemin de Feynman pour faire ressortir l'existence d'un lien étroit entre le filtrage non linéaire et la mécanique quantique euclidienne. Nous faisons en outre des commentaires pour expliquer l'origine de cette équivalence mathématique et les différences théoriques importantes qui existent entre les deux domaines. Enfin, nous examinons les aspects importants des formulations de la mécanique quantique selon la physique classique, l'analyse canonique et l'intégrale de chemin.

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1 Introduction

The problem of the evolution of a Langevin state, or a signal of interest, described by a continuous-time stochastic dynamical model arises frequently in practice. In particular, in classical filtering problems one deals with macroscopic objects whose state variables are phenomenologically well-described by classical deterministic laws modified by external disturbances that can be modelled as random noise. Thus, the state of the system is described by a noisy version of a deterministic nonlinear dynamical system termed the state model: the dynamics are governed by a system of first-order differential equations in the state variable with an additional contribution due to noise that is random. The noise in the state model is referred to as the signal noise. If the noise is Gaussian (or, more generally, multiplicative Gaussian) the state process is a Markov process. Since the process is stochastic, the state process is completely characterized by a probability density function. The Fokker-Planck-Kolmogorov forward equation (FPKfe) describes the evolution of this probability density function (or equivalently, the transition probability density function) and leads to the complete solution of the state evolution problem [1].

However, in many applications, the signal, or state variables, cannot be directly observed. Instead, what is measured is a related stochastic process called the measurement process. The measurement process can often be modelled as yet another nonlinear continuous-time stochastic dynamical system called the measurement model. Loosely speaking, the observations, or measurements, are discrete-time samples drawn from a different system of noisy first order differential equations. The noise in the measurement dynamical system is referred to as measurement noise. The continuous-continuous "filtering" problem is the estimation of the continuous-time signal or state process given the noisy observations of a related continuous-time stochastic process (the measurement or observation process). For a recent discussion, see [2].

The conditional probability density provides the complete solution to the filtering problem. That is, it embodies all the probabilistic information about the state process contained in the observations and in the initial condition. Here the Bayesian point of view is being adopted. From the conditional probability density, an optimal estimate may be computed for any loss function. For instance, the minimum variance estimate is the conditional mean. The solution of the nonlinear filtering problem is said to be universal if the initial distribution can be arbitrary [1].

The traditional approach to the solution of the continuous-continuous universal nonlinear filtering problem requires the solution of the Duncan-Mortensen-Zakai (DMZ) equation, a stochastic differential equation (SDE) describing the unnormalized conditional probability density. The DMZ equation can be gauge transformed to the time-varying partial differential

1. A classic discussion of nonlinear filtering theory can be found in the text by Jazwinski [3]. For a more up-to-date discussion, see [4].

equation termed the robust DMZ equation. However, since the robust DMZ equation is a partial differential equation (PDE) with the coefficients depending on the measurements the PDE cannot be solved off-line, or in real time. In [5], it was proved that solving the robust DMZ equation is equivalent to solving a PDE, termed the Yau Equation (YYe) of continuous-continuous filtering, whose coefficients are independent of the measurements. Hence, the YYe can be solved off-line and in a memoryless way.

Recently, it has been noted that the Feynman path integral [2] can be used to solve the general nonlinear filtering problem. In fact, it was shown that the path integral formulation of the continuous-continuous filtering problem directly leads to the YYe [4]. The path integral formula for the fundamental solution of the YYe was also derived in [4]. The advantages of the Feynman path integral formulation are many, including a completely independent and self-contained formulation and solution of the general nonlinear filtering problem (as opposed to that in traditional filtering theory literature which is based on measure theoretical techniques termed the Feynman-Kăc formalism), and simple, efficient and accurate algorithms that can be implemented in real-time[2].

The Feynman path integral has proven to be a very powerful tool in modern theoretical physics, often leading to results that are not evident using other methods. In this report, one such instance is provided in the application to the filtering problem. In particular, it is demonstrated that there is a very close relationship between nonlinear filtering theory and Euclidean quantum mechanics. Specifically, the fundamental solution of the YYe may be viewed as the expectation of a certain operator in a Euclidean quantum mechanical system. This follows from the path integral formula for the fundamental solution of the YYe derived in [4]. This result generalizes the equivalence between nonlinear filtering and the Euclidean Schrödinger equation derived in [7] via more complicated means. That result was limited to a filtering system where the signal model drift is the same as in the finite-dimensional Yau filter [3]. The result obtained here does not require that assumption—the signal model is assumed to have additive noise (or somewhat more generally, orthogonal diffusion vielbein so that the diffusion matrix is proportional to the identity matrix), and the measurement model noise is also additive. Also, no explicit time dependence is assumed in the model.

The outline of our report is as follows. In Section 2, notation and some of the important results for the real-time solution of the continuous-continuous nonlinear filtering problem are reviewed. In Section 3, summaries of the results obtained using the Feynman path integral formulation, specifically the path integral expressions for the fundamental solution for the FPKfe and the YYe are given. In Section 4, the equivalence between Euclidean quantum mechanics and nonlinear filtering is presented. In the following section it is shown that the Feynman path integral formulation leads to the equivalence between nonlinear filtering and the time-varying Schrödinger equation obtained in [7]. In Section 6, some important

2. For a classic review of path integrals, see [6].

3. Their explicit solution imposed an additional condition on the measurement model drift, but that is not relevant for our discussion here.

conceptual issues are discussed which further clarify the relationship between the YYe and Euclidean quantum mechanics. The conclusions and directions of future work are presented in Section 7. In Appendix A, the path integral formula for the fundamental solution of the YYe that forms the basis of our work is verified.

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2 Basic Results of Nonlinear Filtering

A very brief summary of the results of filtering theory is provided in this section. For the purposes of this report, the important result is that the solution of the filtering problem requires the solution of a Fokker-Planck type of equation called the Yau equation. A more complete discussion with references can be found in [4] and [2]. It is worth contrasting the simplicity of the Feynman path integral solution discussed in the next section with the traditional discussion summarized in this section.

2.1 From the DMZ SDE to the Robust DMZ PDE

The signal and observation model considered is the following:

$$\begin{cases} d\mathbf{x}(t) &= f(\mathbf{x}(t))dt + e(\mathbf{x}(t))d\mathbf{v}(t), \quad x(0) = x_0, \\ d\mathbf{y}(t) &= h(\mathbf{x}(t))dt + d\mathbf{w}(t), \quad y(0) = 0. \end{cases} \tag{1}$$

Here the state (or signal) $\mathbf{x}$ and state process noise $\mathbf{v}$ are $\mathbb{R}^n-$valued stochastic processes, and the measurements $\mathbf{y}$ and measurement noise $\mathbf{w}$ are $\mathbb{R}^m-$valued stochastic processes, respectively, and $e \in \mathbb{R}^{n\times n}$. These are defined in the Itô sense. In statistical physics literature, one defines $\boldsymbol{\nu}(t) = d\mathbf{v}(t)/dt$ and $\boldsymbol{\mu}(t) = d\mathbf{w}(t)/dt$. The processes $\mathbf{v}$ and $\mathbf{w}$ are assumed to be independent Brownian processes with variances $\hbar_\nu$ and $\hbar_\mu$ respectively. Also, f is referred to as the signal model drift, h as the measurement model drift, e as the diffusion vielbein, and ee^T, where T is the transpose, as the diffusion matrix. In this report, the additive noise case is considered where e is the identity matrix, although the same result holds if one assumes that the diffusion vielbein is orthogonal. Finally, no explicit time dependence is assumed in the model.

The unnormalized conditional probability density, $\sigma(t, x)$, of the state given the observations $Y(t) = \{y(t) : 0 \leq s \leq t\}$ satisfies the DMZ stochastic differential equation:

$$d\sigma(t, x) = \mathscr{L}_Y \sigma(t, x)dt + \sum_{i=1}^{m} \mathscr{L}_i \sigma(t, x)dy_i(t), \quad \text{where } \sigma(0, x) = \sigma_0(x). \tag{2}$$

Here

$$\mathscr{L}_Y(\sigma(t, x)) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_i(x)\sigma(t, x)) + \frac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 \sigma}{\partial x_i^2}(t, x) - \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)\sigma(t, x), \tag{3}$$

and $\mathscr{L}_i$ is the zero-degree differential operator (or multiplication by) $h_i(x), i = 1, \ldots, m$, and $\sigma_0(x)$ is the probability density at the initial time $t_0 = 0$. Under the gauge transformation

$$u(t, x) = \exp\left(-\sum_{i=1}^{m} h_i(x)y_i(t)\right)\sigma(t, x), \tag{4}$$

the DMZ SDE is transformed into the following time-varying PDE called the robust DMZ equation [8]:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) =& \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}(t,x) + \sum_{i=1}^{n}\left(-f_i(x) + \sum_{j=1}^{m} y_j(t)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u}{\partial x_i}(t,x) \\
& -\Bigg(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) - \frac{1}{2}\sum_{i=1}^{m} y_i(t)\Delta h_i(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(t) f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\
& -\frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{n} y_i(t) y_j(t)\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)\Bigg) u(t,x), \\
u(0,x) =& \sigma_0(x).
\end{aligned} \tag{5}
$$

Here Δ is the Laplacian. The solution of a PDE when the initial condition is a delta function is called its fundamental solution.

2.2 The Yau Equation of Continuous-Continuous Filtering

S-T. Yau and Stephen Yau made a major advance in the real-time solution of the general nonlinear filtering problem as described in [5]. Let the samples of the measurement process be available at times $\{\tau_0, \tau_1, \ldots, \tau_{l-1}, \tau_l, \ldots\}$. They began by observing that if $u_l(t,x)$ satisfies the equation

$$
\begin{aligned}
\frac{\partial u_l}{\partial t}(t,x) =& \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 u_l}{\partial x_i^2}(t,x) + \sum_{i=1}^{n}\left(-f_i(x) + \sum_{j=1}^{m} y_j(\tau_l)\frac{\partial h_j}{\partial x_i}(x)\right)\frac{\partial u_l}{\partial x_i}(t,x) \\
& -\Bigg(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x) - \frac{1}{2}\sum_{i=1}^{m} y_i(\tau_l)\Delta h_i(x) + \sum_{i=1}^{m}\sum_{j=1}^{n} y_i(\tau_l) f_j(x)\frac{\partial h_i}{\partial x_j}(x) \\
& -\frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{n} y_i(\tau_l) y_j(\tau_l)\frac{\partial h_i}{\partial x_k}(x)\frac{\partial h_j}{\partial x_k}(x)\Bigg) u_l(t,x), \\
u_l(\tau_{l-1},x) =& u_{l-1}(\tau_{l-1},x),
\end{aligned} \tag{6}
$$

in the time interval $\tau_{l-1} \leq t \leq \tau_l$, then the function $\tilde{u}_l(t,x)$ defined as

$$
\tilde{u}_l(t,x) = \exp\left(\sum_{i=1}^{m} y_i(\tau_l) h_i(x)\right) u_l(t,x) \tag{7}
$$

satisfies the parabolic partial differential equation termed the Yau Equation (YYe)

$$
\frac{\partial \tilde{u}_l}{\partial t}(t,x) = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2 \tilde{u}_l}{\partial x_i^2}(t,x) - \sum_{i=1}^{n} f_i(x)\frac{\partial \tilde{u}_l}{\partial x_i}(t,x) - \left(\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) + \frac{1}{2}\sum_{i=1}^{m} h_i^2(x)\right)\tilde{u}_l(t,x) \tag{8}
$$

in the same time interval. The converse of the statement is also true. In [9], they further showed that it is sufficient to use the previous observation $y_i(\tau_{l-1})$, i.e., $u_l(t,x)$ satisfies Equation 6 if and only if $\tilde{u}_l(t,x)$ defined as

$$\tilde{u}_l(t,x) = \exp\left(\sum_{i=1}^{m} y_i(\tau_{l-1})h_i(x)\right) u_l(t,x) \tag{9}$$

satisfies Equation 8 in the time interval $\tau_{l-1} \leq t \leq \tau_l$. Equation 7 (Equation 9) and Equation 8 are referred to as the post-measurement (pre-measurement) forms of the YYe.

Equation 6 can also be obtained by setting $y(t)$ to $y(\tau_l)$ in Equation 5. It was proved in [5] that the solution of Equation 6 approximates very well the solution of the robust DMZ equation (Equation 5), i.e., it converges to $u(t,x)$ in both the pointwise sense and the L^2 sense. Thus, solving Equation 5 is equivalent to solving Equation 8. In fact, in [10] it was proved that $\tilde{u}(t,x)$ converges to $\sigma(t,x)$.

There are two important points to note. The coefficients of the robust DMZ equation (Equation 5) contain the measurements. Consequently, the partial differential equation to be solved for continuous-continuous filtering is unknown prior to measurements. In other words, the robust DMZ equation has to be solved on-line; its solution cannot be pre-computed. By contrast, in the YYe (Equation 8), measurements are absent from the partial differential equation. The measurements only enter the initial condition at each measurement step. This implies that the YYe can be be solved off-line; there is no need for on-line solution of PDEs [4]. This makes real-time solution feasible even if the state dimension is large. The nonlineaer filtering algorithm based on the YYe is termed the Yau algorithm

In addition, it is noted that all other known algorithms for continuous-continuous filtering assume boundedness of the measurement model drift (see discussion in [10] and [11]). This is a highly restrictive assumption and implies, for instance, that they cannot even handle the linear model which the Kalman filter handles very well. The Yau algorithm is shown to converge to the true solution under much weaker conditions on the drift (see [5],[9], [11]). In that sense, the Yau algorithm is the *only known practical nonlinear filtering algorithm for the general filtering problem*. Also note that from a path integral perspective, the Yau algorithm is the most natural one.

The simplicity of the YYe has already led to some important advances. In particular, it has led to a demonstration of an equivalence between solving a class of nonlinear filtering problems and the time-varying Schrödinger equation[7]. As a result, the Yau PDE can be reduced to a system of ODEs explicitly solvable using the power series method. This result will be shown to be a corollary of the main result of this report.

4. This assumes that the measurement steps are equidistant or known.

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3 Path Integral Formula for the Fundamental solution of the YYe

In recent papers, it was demonstrated that the path integral method leads to an independent formulation and solution of the universal nonlinear filtering problem [12, 4]. Specifically, the path integral formula for the transition probability density in continuous-discrete and continuous-continuous filtering was investigated. One of the results of the aforementioned work is the path integral formula for the fundamental solution of the FPKfe and the YYe. Some of the relevant results from these papers are reviewed below.

When the diffusion matrix is $\hbar_\nu I$, the transition probability satisfies the FPKfe (see, for instance, [1])

$$\begin{aligned}\frac{\partial p}{\partial t}(t,x) &= -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[f_i(x)p(t,x)\right] + \frac{\hbar_\nu}{2}\sum_{i=1}^{n}\frac{\partial^2 p}{\partial x_i^2}(t,x), \\ &\equiv \mathscr{L}p(t,x).\end{aligned} \tag{10}$$

The fundamental solution of this FPKfe can be shown to be given by (see [13] and [12])

$$P(t,x|t_0,x_0) = \int_{x(t_0)=x_0}^{x(t)=x}[\mathscr{D}x(t)]\exp\left(-\frac{1}{2\hbar_\nu}\sum_{i=1}^{n}\int_{t_0}^{t}dt\left[\left(\dot{x}_i(t)-f_i(x(t))\right)^2+\hbar_\nu\frac{\partial f_i}{\partial x_i}(x(t))\right]\right). \tag{11}$$

In continuous-continuous filtering, it is necessary to incorporate the measurement stochastic process as well as the signal process. That is, the measurement process ensemble must also be considered. The inclusion of measurement noise means that each system in the ensemble leads to a different time-dependent vector $y(t)$. Although only one realization of the measurement stochastic process is observed, it is still meaningful to talk about an ensemble average of the measurement process (in addition to an ensemble average over the state process). Thus, the quantity of interest in continuous-continuous filtering is

$$P(t_r x_r; y(t_r)|t_{r-1}, x_{r-1}; y(t_{r-1})) = \left\langle\langle\delta^n(\mathbf{x}(t_r)-x_r)\delta^m(\mathbf{y}(t_r)-y(t_r))\rangle_{\boldsymbol{\mu}}\right\rangle|_{\mathbf{x}(t_{r-1})=x_{r-1},\mathbf{y}(t_{r-1})=y_{r-1})}, \tag{12}$$

where $\langle\cdot\rangle_{\boldsymbol{\mu}}$ denotes averaging with respect to measurement noise $\boldsymbol{\mu}(t)$. It has been shown in [4] that the transition probability density conditional on the measurements is given by [5]

$$P(t_r,x_r;y(t_r)|t_{r-1},x_{r-1};y(t_{r-1})) = \exp\left(\frac{1}{\hbar_\mu}\sum_{k=1}^{m}h_k(x(t_r))\left[y_k(t_r)-y_k(t_{r-1})\right]\right)\tilde{P}(t_r,x_r|t_{r-1},x_{r-1}) \tag{13}$$

5. Here the "post-measurement" form is used; the pre-measurement form does not affect the conclusion[4].

where

$$\tilde{P}(t_r, x_r|t_{r-1}, x_{r-1}) = \int_{x(t_{r-1})=x_{r-1}}^{x(t_r)=x_r} [\mathscr{D}x(t)] \exp\left(-\frac{1}{\hbar_\nu} S(t_{r-1}, t_r)\right), \tag{14}$$

and

$$S(t_{r-1}, t_r) = \frac{1}{2}\int_{t_{r-1}}^{t_r} dt \left[\sum_{i=1}^{n} (\dot{x}_i(t) - f_i(x(t)))^2 + \hbar_\nu \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x(t)) + \frac{\hbar_\nu}{\hbar_\mu} \sum_{k=1}^{m} h_k^2(x(t))\right]. \tag{15}$$

It can be shown that $\tilde{P}(t_r, x_r, t_{r-1}, x_{r-1})$ is the fundamental solution of the YYe [4]. For completeness, a verification of this result is presented in the Appendix.

The transition probability density is the complete solution to the continuous filtering problem. Thus if the initial distribution is $u(t_0, x)$, then the evolved conditional probability distribution is given by

$$u(t, x) = \int P(t, x; y(t)|t_0, x_0; y(t_0)) u(t_0, x_0) \{d^n x_0\}. \tag{16}$$

4 YYe and Euclidean Quantum Mechanics

In this section it is shown that there is a general equivalence between the YYe and Euclidean quantum mechanics.

4.1 A General Equivalence

Consider the path integral formula for the fundamental solution of the Yau Equation

$$\tilde{P}(t, x|t_0, x_0) = \int_{x(t_0)=x_0}^{x(t)=x} [\mathscr{D}x(t)] \exp\left(-\frac{1}{\hbar_\nu} S\right), \tag{17}$$

where the action S is

$$S = \frac{1}{2}\int_{t_0}^{t} dt \left[\sum_{i=1}^{n}\left[\dot{x}_i^2(t) + f_i^2(x(t)) - 2\dot{x}_i(t) f_i(x(t))\right] + \hbar_\nu \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x(t)) + \frac{\hbar_\nu}{\hbar_\mu}\sum_{k=1}^{m} h_k^2(x(t))\right]. \tag{18}$$

Upon integration, the third term above yields

$$-\sum_{i=1}^{n}\int_{t_0}^{t} dt \dot{x}_i(t) f_i(x(t)) = -\sum_{i=1}^{n}\int_{x(t_0)}^{x(t)} dx_i(t) f_i(x(t)). \tag{19}$$

Here the Feynman convention is used implicitly, i.e., symmetrized arguments of f_i, so that the ordinary rules of calculus apply. Thus, the action simplifies to

$$S = \frac{1}{2}\int_{t_0}^{t} dt \left[\sum_{i=1}^{n}\left[\dot{x}_i^2(t) + f_i^2(x(t))\right] + \hbar_\nu \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x(t)) + \frac{\hbar_\nu}{\hbar_\mu}\sum_{k=1}^{m} h_k^2(x(t))\right] - \sum_{i=1}^{n}\int_{x(t_0)}^{x(t)} dx_i(t) f_i(x(t)). \tag{20}$$

We can therefore write the fundamental solution as

$$\int_{x(t_0)=x_0}^{x(t)=x} [\mathscr{D}x(t)] \exp\left(\frac{1}{\hbar_\nu}\sum_{i=1}^{n}\int_{x(t_0)=x_0}^{x(t)=x} dx_i(t) f_i(x(t))\right) \exp\left(-\frac{1}{\hbar_\nu}\int_{t_0}^{t} dt \mathscr{L}\right), \tag{21}$$

where the "Lagrangian" $\mathscr{L}$, is

$$\begin{aligned}\mathscr{L} &= \frac{1}{2}\sum_{i=1}^{n}\dot{x}_i^2(t) + \frac{1}{2}\sum_{i=1}^{n}\left[f_i^2(x(t)) + \hbar_\nu \frac{\partial f_i}{\partial x_i}(x(t))\right] + \frac{\hbar_\nu}{2\hbar_\mu}\sum_{k=1}^{m} h_k^2(x(t)), \\ &= T - V,\end{aligned} \tag{22}$$

and

$$T = \frac{1}{2} \sum_{i=1}^{n} \dot{x}_i^2(t), \tag{23}$$

$$-V = \frac{1}{2} \sum_{i=1}^{n} \left[f_i^2(x(t)) + \hbar_\nu \frac{\partial f_i}{\partial x_i}(x(t)) \right] + \frac{\hbar_\nu}{2\hbar_\mu} \sum_{k=1}^{m} h_k^2(x(t)).$$

Consider a Euclidean quantum mechanical system with the Hamiltonian

$$\mathscr{H} = T + V. \tag{24}$$

Then, performing the path integral quantization of this system will lead to the following expression of the transition probability amplitude:

$$\langle x, t | x_0, t_0 \rangle = \int_{x(t_0)=x_0}^{x(t)=x} [\mathscr{D}x(t)] \exp\left(-\frac{1}{\hbar_\nu} \int_{t_0}^{t} dt \mathscr{L} \right). \tag{25}$$

Now, let us consider the following quantum-mechanical matrix element:

$$\left\langle x, t \middle| T \left[\exp\left(\frac{1}{\hbar_\nu} \sum_{i=1}^{n} \int_{x(t_0)=x_0}^{x(t)=x} d\hat{x}_i(t) f_i(\hat{x}(t)) \right) \right] |x_0, t_0 \right\rangle = \tag{26}$$

$$\int_{x(t_0)=x_0}^{x(t)=x} [\mathscr{D}x(t)] \exp\left(\frac{1}{\hbar_\nu} \sum_{i=1}^{n} \int_{x(t_0)=x_0}^{x(t)=x} dx_i(t) f_i(x(t)) \right) \exp\left(-\frac{1}{\hbar_\nu} \int_{t_0}^{t} dt \mathscr{L} \right).$$

Here $\hat{x}(t)$ is the quantum operator corresponding to $x(t)$. In the general case, the value of the line integral depends on the path. Therefore, it cannot be "factored out". After all, it is a part of the path integral, which is a weighted sum over all paths. This is not a "gauge transformation" in the traditional sense (since it is not outside of the path integral). This expectation value can be evaluated perturbatively, or more generally, non-perturbatively.

However, it can be viewed as an expectation of the signal model drift line integral operator, i.e.,

$$\exp\left(\frac{1}{\hbar_\nu} \sum_{i=1}^{n} \int_{x(t_0)=x_0}^{x(t)=x} d\hat{x}_i(t) f_i(\hat{x}(t)) \right), \tag{27}$$

in the quantum mechanical system with the Lagrangian given by Equation 22.

Note that there are several differences between this operator and the Wilson lines that arise in quantum field theories. First of all, this is a quantum mechanical system, not a quantum field theory. Secondly, in field theory the coordinates are a parameter, whereas here (as in quantum mechanics) the coordinates (i.e., x) are quantized and not a parameter.

This can be summarized as follows: *The fundamental solution of the Yau Equation, Equation 8, is the same as the matrix element of the operator in Equation 27 for the Euclidean quantum mechanical system with the Hamiltonian as in Equation 24.*

4.2 A Special Case

The equivalence of nonlinear filtering and quantum mechanics can be made even more direct for a certain class of the signal model drift. This known result is shown here using path integral methods. Specifically, suppose the line integral is path independent. Then, the line integral can be factored out of the path integral since its value does not depend on the path. It is straightforward to see when this is possible.

In the one-dimensional case, it is always possible to write the (smooth) drift as a gradient:

$$f(x) = \frac{d}{dx} g(x), \tag{28}$$

since $g(x)$ is simply given by the integral of $f(x)$.

In the general $n-$dimensional case, path independence of the line integral requires that the drift be a gradient:

$$f(x) = \nabla \phi(x). \tag{29}$$

For clarity, matrix notation is employed here. A special case is the linear symmetric case. Specifically, if L is a symmetric matrix the drift is

$$\begin{aligned} f(x) &= \sum_{j=1}^{n} [L_{ij} x_j + l_i], \\ &= \nabla \phi(x), \end{aligned} \tag{30}$$

where

$$\phi(x) = \frac{1}{2} \sum_{i,j=1}^{n} x_i L_{ij} x_j + \sum_{i=1}^{n} l_i x_i. \tag{31}$$

then the state model drift for the Yau filtering system with L a symmetric matrix is also a gradient of a scalar.

In all such cases, Equation 26 reduces to

$$\exp\left(\frac{1}{\hbar_\nu}\left[\phi(x) - \phi(x_0)\right]\right) \int_{x(t_0)=x_0}^{x(t)=x} [\mathscr{D}x(t)] \exp\left(-\frac{1}{\hbar_\nu}\int_{t_0}^{t} dt \mathscr{L}\right). \tag{32}$$

In fact,

$$\tilde{P}(t, x|t_0, x_0) = \exp\left(\frac{1}{\hbar_\nu}\left[\phi(x) - \phi(x_0)\right]\right) \langle x, t | x_0, t_0 \rangle . \tag{33}$$

From the above equation it follows that a gauge transformation relates the fundamental solution of the Yau Equation and the corresponding Euclidean quantum mechanical system.

The discussion in this section can be summarized as follows: *When the signal model drift is a gradient of a scalar field, the fundamental solution of the Yau Equation is, up to a gauge transformation, the same as the transition probability amplitude of a Euclidean quantum mechanical system with Lagrangian given by Equation 22.*

5 Nonlinear Yau Filtering System and a Time-Dependent Schrödinger equation

5.1 The Work of S-T. Yau and Stephen Yau

As discussed in Section 2, in order to obtain the unnormalized conditional probability $\sigma(t, x)$, it is sufficient to solve the YYe, repeated below for convenience:

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial t}(t, x) = \dfrac{1}{2}\Delta \tilde{u}(t, x) - \displaystyle\sum_{j=1}^{n} f_j(x) \dfrac{\partial \tilde{u}}{\partial x_j}(t, x) - \left(\displaystyle\sum_{j=1}^{n} \dfrac{\partial f_j}{\partial x_j}(x) + \dfrac{1}{2} \displaystyle\sum_{j=1}^{m} h_j^2(x) \right) \tilde{u}(t, x), \\ \tilde{u}(\tau_{i-1}, x) = \sigma_i(x). \end{cases} \tag{34}$$

In [7], the authors considered the filtering system signal model drift

$$f(x) = Lx + l + \nabla \phi, \tag{35}$$

where L is an anti-symmetric matrix. The symmetric part of the linear term can be incorporated into ϕ. By considering the quantity $\tilde{\nu}(t, x)$ defined by

$$\begin{aligned} \tilde{u}(t, x) &= e^{\phi(x)} \tilde{\nu}(t, x), \\ &= e^{\phi(x)} \nu(t, \tilde{x}), \qquad \tilde{x}(t) = B(t)x(t) + b(t), \end{aligned} \tag{36}$$

the authors showed that it is sufficient to solve the following Schrödinger equation

$$\begin{cases} \frac{\partial \nu}{\partial t}(t, \tilde{x}) &= \frac{1}{2} \nabla \nu(t, \tilde{x}) - \frac{1}{2} q \left(B^{-1}(t)\tilde{x} - B^{-1}(t) b(t) \right) \nu(t, \tilde{x}), \\ \nu(\tau_{i-1}, x) &= \sigma_i(x) e^{-\phi(x)}, \end{cases} \tag{37}$$

where

$$q(x) \equiv \nabla^2 \phi(x) + |\nabla \phi|^2(x) + 2(Lx + l) \cdot \nabla \phi + \sum_{i=1}^{m} h_i^2(x) + \text{tr } L, \tag{38}$$

and

$$B(t) = e^{-Lt}, \qquad b(t) = -\int_0^t e^{-Ls} l ds. \tag{39}$$

Here,

$$\frac{dB}{dt}(t) = -B(t)L, \qquad \text{and} \qquad \frac{db}{dt}(t) = -B(t)l, \tag{40}$$

and $B(t)$ is an orthogonal matrix.

5.2 Path Integral Derivation

We now derive the result discussed in the previous section relating the nonlinear Yau filtering system to a Schrödinger equation using path integral methods. In matrix notation, the Lagrangian, Equation 22, for the Yau filter state model is

$$\mathscr{L} = \frac{1}{2}\left(\dot{x} - Lx - l - \nabla\phi\right)^2 + \frac{\hbar_\nu}{2}\nabla\cdot f + \frac{\hbar_\nu}{2\hbar_\mu}\sum_{k=1}^{m} h_k^2(x). \tag{41}$$

Since

$$\nabla\cdot f = \mathrm{tr}\ L + \nabla^2\phi, \tag{42}$$

it follows that

$$\begin{aligned}\mathscr{L} = {} & \frac{1}{2}(\dot{x}(t) - Lx(t) - l)^2 + \frac{1}{2}(\nabla\phi)^2(x(t)) - \dot{x}(t)\cdot\nabla\phi(x(t)) + (Lx(t)+l)\cdot\nabla\phi(x(t)) \\ & + \mathrm{tr}\ L + \nabla^2\phi(x(t)) + \frac{\hbar_\nu}{2\hbar_\mu}\sum_{k=1}^{m} h_k^2(x(t)), \\ = {} & \frac{1}{2}\left(\dot{x}(t) - Lx(t) - l\right)^2 - \dot{x}(t)\cdot\nabla\phi(x(t)) + \frac{1}{2}q(x(t)),\end{aligned} \tag{43}$$

where $q(x)$ is as defined in Equation 38. The $\dot{x}\cdot\nabla\phi(x)$ term can be integrated out of the path integral and simply yields a phase factor

$$\exp\left(\frac{1}{\hbar_\nu}\left[\phi(x) - \phi(x_0)\right]\right). \tag{44}$$

Now, in terms of $\tilde{x}(t)$

$$\begin{aligned} x(t) &= B^{-1}(t)[\tilde{x}(t) - b(t)], \\ \dot{x}(t) &= B^{-1}(t)[\dot{\tilde{x}}(t) + L\tilde{x}(t) + B(t)l - Lb(t)], \\ Lx(t) &= B^{-1}(t)[L\tilde{x}(t) - Lb(t)] \end{aligned} \tag{45}$$

so that

$$\dot{x}(t) - Lx(t) - l = B^{-1}(t)\dot{\tilde{x}}(t), \qquad \text{and} \tag{46}$$

$$(\dot{x}(t) - Lx(t) - l)^2 = \dot{\tilde{x}}^2(t). \tag{47}$$

Therefore, in terms of $\tilde{x}(t)$, the Lagrangian is

$$\mathscr{L} = \frac{1}{2}\dot{\tilde{x}}^2(t) + \frac{1}{2}q\left(B^{-1}(t)[\tilde{x} - b(t)]\right). \tag{48}$$

Combining Equation 48 with Equation 44, the fundamental solution becomes

$$\tilde{P}(t,x|t_0,x_0) = \exp\left(\frac{1}{\hbar_\nu}[\phi(x)-\phi(x_0)]\right)\int_{\tilde{x}(t_0)=\tilde{x}_0}^{\tilde{x}(t)=\tilde{x}}[\mathscr{D}\tilde{x}(t)]\exp\left(-\frac{1}{\hbar_\nu}\int_{t_0}^{t}dt\mathscr{L}\right), \tag{49}$$
$$= \exp\left(\frac{1}{\hbar_\nu}[\phi(x)-\phi(x_0)]\right)\tilde{\nu}(t,x|t_0,x_0),$$

where $\mathscr{L}$ is given by Equation 48, and this equation defines $\tilde{\nu}(t,x|t_0,x_0)$. Note that the Jacobian of the transformation from the measure $[\mathscr{D}x(t)]$ to $[\mathscr{D}\tilde{x}(t)]$ is unity since $B(t)$ is an orthogonal matrix (because L is an anti-symmetric matrix).

It follows from the standard path integral formulation of quantum mechanics [6] that $\tilde{\nu}(t,x|t_0,x_0)$ is just the path integral representation of the fundamental solution of the following Schrödinger equation:

$$\frac{\partial\nu}{\partial t}(t,\tilde{x}) = \frac{1}{2}\nabla^2\nu(t,\tilde{x}) - \frac{1}{2}q\left(B^{-1}(t)\tilde{x} - B^{-1}(t)b(t)\right)\nu(t,\tilde{x}). \tag{50}$$

This is precisely the equation obtained in [7]. For the initial condition, since

$$\tilde{u}(t,x) = \int \tilde{P}(t,x|t_0,x_0)\sigma(t_0,x_0)d^nx_0, \tag{51}$$
$$= \exp\left(\frac{1}{\hbar_\nu}\phi(x)\right)\tilde{\nu}(t,x)$$
$$= \exp\left(\frac{1}{\hbar_\nu}\phi(x)\right)\int\tilde{\nu}(t,x|t_0,x_0)\exp\left(-\frac{1}{\hbar_\nu}\phi(x_0)\right)\sigma(t_0,x_0)d^n\tilde{x}_0,$$

it follows that

$$\tilde{P}(t,x|t_0,x_0) = \exp\left(\frac{1}{\hbar_\nu}\left[\phi(x)-\phi(x_0)\right]\right)\tilde{\nu}(t,x|t_0,x_0), \tag{52}$$

which is the same as the result in Equation 49. Thus the result in [7] has been independently established.

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6 Additional Remarks

A few comments on our results are in order.

1. The general equivalence proposed in Section 4.1 is between the fundamental solution of the FPKfe and a certain matrix element of a quantum mechanical system. It is noted that all previous equivalences were between fundamental solutions of FPKfe and a Schrödinger equation. While the previous equivalences were obtainable using operator methods, it is not clear how the proposed general equivalence obtained here can be derived using operator methods.
2. The Feynman path integral methods used here are very different from the measure-theoretic methods used to study the nonlinear filtering problem (see, for instance, [14]). The Feynman path integral approach developed in [4, 2] and this report leads to an independent way of tackling the universal nonlinear filtering problem. In particular, unlike in the standard filtering theory approaches, the DMZ equation (or its variants) is not taken as an input. On the contrary, the YYe is obtained directly as a consequence of the path integral approach. It is also noted that the Euclidean quantum physics referred to in the equivalence to nonlinear filtering developed in this report is a quantum mechanical one, not a quantum field theoretical one. Furthermore, note that $\hbar_\nu$ here is analogous to the Planck's constant, $\hbar$, in quantum physics.
3. Although a formal equivalence has been developed between Euclidean quantum mechanics and universal nonlinear filtering, it is important to point out that there is a profound difference between classical and quantum probabilities. In the "real time" (as opposed to Euclidean time) quantum mechanics, the transition probability amplitude is not a probability; it may not even be real. The probability amplitude is to be multiplied with its complex conjugate to obtain a probability density. In contrast, the transition probability density in filtering theory is a classical probability density. We have shown that, in a mathematical sense, matrix elements in a Euclidean quantum mechanical system equal the transition probability density of a classical stochastic process. This equivalence is purely mathematical, not conceptual.
4. Further consideration of the path integral formulas clarifies the relationship between nonlinear filtering and quantum mechanics. The reason for some of the mathematical equivalence between nonlinear filtering and quantum physics is the semi-group property. That is, in stochastic processes the Chapman-Kolmogorov semi-group property is a fundamental property of Markov processes [3]. The Chapman-Kolmogorov semi-group property allows the transition probability density in stochastic process to be written as follows. Let us partition the time interval $[t_0, t]$ into N equi-spaced time intervals so that $t_i = t_0 + i\epsilon$ where $\epsilon = (t - t_0)/N$. Then, from the Chapman-Kolmogorov

semi-group property it follows that

$$P(t,x|t_0,x_0) = \int \{d^n x(t_1)\cdots d^n x(t_{N-1})\} P(t,x|t_{N-1},x(t_{N-1}))\cdots P(t_1,x(t_1)|t_0,x_0), \tag{53}$$
$$= \int \left\{\prod_{i=0}^{N} d^n x(t_i)\right\}\left\{\prod_{i=1}^{N} P(t_i,x(t_i)|t_{i-1},x(t_{i-1}))\right\}\delta^n(x(t)-x)\delta^n(x(t_0)-x_0),$$
$$= \int_{x(t_0)=x_0}^{x(t)=x} \left\{\prod_{i=1}^{N-1} d^n x(t_i)\right\}\prod_{i=1}^{N} P(t_i,x(t_i)|t_{i-1},x(t_{i-1})),$$

where $x(t_0) = x_0$ and $x(t_N) = x$ and the delta function condition in the definition of $P(t,x|t_0,x_0)$ is written as integration limits. On the other hand, in quantum physics, the semi-group property is a basic property of the time-evolution operator. It is the semi-group property of the time-evolution operator that leads to the path integral formula for the transition probability amplitude.

5. The unitarity of the time evolution operator plays a crucial role in quantum mechanics. This is because unitarity is essential for conservation of probability. It is noted that the unitarity of the time evolution operator implies that the Hamiltonian operator is Hermitian [6].

   The evolution of the probability distribution for a continuous Markov processes is described by the FPKfe. The FPKfe operator is not a Hermitian operator. This is not inconsistent with the conservation of probability. This is because the FPKfe is a continuity equation, and so the probability is conserved (as long as the boundary terms vanish). This is yet another instance of the profound difference between classical and quantum probabilities.

6. In the continuous-continuous filtering case, the YYe plays the role that the FPKfe plays in continuous-discrete filtering. However, the YYe is not a continuity equation. This is not a contradiction since the YYe is related to the unnormalized conditional probability density, whereas the FPKfe evolves (and preserves the normalization of) the normalized probability density.

7. In this report, it has been assumed that the noise is additive and the model is not explicitly time-dependent. In the more general case, a simple equivalence, as derived in this report, is not possible. For instance, in order to obtain the YYe, it was necessary to assume that the measurement model was not explicitly time dependent (see [4]); this is not valid in the general case. Also, when the noise is multiplicative, quantization is not

6. In the standard formulation of quantum mechanics, Hermiticity of the Hamiltonian is required in order to ensure that the eigenvalues of the Hamiltonian (and hence the possible energies) are real and that the time evolution is unitary (i.e., conserves probability). These properties can also be ensured even if the Hamiltonian is not Hermitian provided that the Hamiltonian is space-time (or $\mathscr{PT}$) reflection symmetric. For a pedagogical discussion of non-Hermitian quantum mechanics, see [15].

as straightforward. This can be traced to the well-known operator ordering ambiguity in quantum physics (since the position and momentum operators do not commute). Furthermore, manipulations of standard calculus are no longer possible in the path integral. However, the path integral result is the same even for the multiplicative noise case if the diffusion matrix is proportional to the identity matrix.

Finally, note that the results of this report, which apply to the YYe, also apply to the FPKfe itself, which is simply the case $h(x) = 0$, in a straightforward manner.

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7 Conclusions and Future Work

The main conclusion of this report is that certain continous-continuous nonlinear filtering problems are related to Euclidean quantum mechanics. Specifically, the transition probability density for nonlinear filtering problems with additive noise and with square diffusion vielbein and explicitly time-independent drift is the fundamental solution of the YYe, and is the same as the expectation of a certain operator in a quantum mechanical system. A corollary of this general result is a derivation of the relationship between the YYe in the Yau filtering system and a time-varying Schrödinger equation which was first derived earlier. This equivalence leads to some useful results from the methods used in quantum physics.

There are many possible directions for future work:

1. The path integral formula can be exactly solvable, or lead to simpler methods of obtaining solutions, in some simple cases. In fact, for the model studied in [7], an explicit path integral solution is described in a report currently in preparation.
2. The equivalence to Euclidean quantum mechanics immediately leads to many filtering problems that can be solved exactly, as will be explored in future papers. The remarkable fact is that many exactly solvable Euclidean quantum mechanical problems correspond to filtering problems that are not finite-dimensional (i.e., Lie algebra of $\mathscr{L}_0$ and $h_i(x), i = 1, \ldots, m$ is not finite-dimensional, see [4]) . Thus, *simplicity in filtering theory does not imply finite dimensionality.*
3. The path integral formulation naturally leads to a perturbative solution of the nonlinear filtering problem. Such a perturbative solution which is analogous to extended Kalman filtering is called extended Yau filtering (EYF). Thus, it is possible to perturb about the Yau filter (a generalization of the linear Kalman filter), rather than the linear Kalman filter. Since the EKF is a special case of the EYF, such a formulation will be superior to the EKF. However, it is noted that the fundamental solution is defined nonperturbatively. This is important since sometimes the perturbative approaches, like EKF, fail.
4. Perhaps the most important advantage of the path integral lies in numerical methods of computation [16]. In fact, path integrals are the only known way to carry out nonperturbative computation in quantum chromodynamics (QCD). Note that QCD, the gauge theory of strong interactions, is a quantum field theory, not merely quantum mechanical and the QCD Lagrangian is highly nonlinear. The excellent numerical results in QCD suggest that currently known path integral methods should give very good performance for the simpler case of (large dimensional) universal nonlinear filtering. It is sufficient to note that the Dirac-Feynman approximation, the crudest approximation of the path integral, already yields excellent results (see [2]), and is adequate for smaller dimensional problems.

It is clear that many lines of investigation are suggested by the path integral methods and it is planned that results of those investigations will be presented in future papers.

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Annex A: Verification of the Path Integral Formula

The results obtained in this report are based on the the path integral formula Equation 13 satisfies the YYe. This formula is now proved in this section. It closely follows the method used by Feynman to verify the path integral formula for the Schrödinger equation in quantum mechanics [6].

According to the Chapman-Kolmogorov semi-group property

$$\tilde{P}(t+\epsilon, x|t_0, x_0) = \int \tilde{P}(t+\epsilon, x|t, x')\tilde{P}(t, x'|t_0, x_0)\{d^n x'\}, \tag{A.1}$$

where $\tilde{P}(t, x'|t_0, x_0)$ is given by Equation 14. When ϵ is infinitesimal

$$\begin{aligned} &\tilde{P}(t+\epsilon, x|t, x') = \\ &\exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\left[x_i - x_i' - \epsilon f_i(\bar{x})\right]^2 - \epsilon\frac{1}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(\bar{x}) - \epsilon\frac{1}{2\hbar_\mu}\sum_{k=1}^{m} h_k^2(\bar{x})\right), \end{aligned} \tag{A.2}$$

where $\bar{x} = \frac{1}{2}(x + x')$. The dominant contribution occurs when the following condition is satisfied:

$$x - x' - \epsilon f(\bar{x}) \approx 0. \tag{A.3}$$

We may write in this region

$$\begin{aligned} x &= x' + \epsilon f(\bar{x}) + \eta, \qquad \text{or} \\ x &= x' + \epsilon f(x) + \eta, \end{aligned} \tag{A.4}$$

where the equalities are valid to $O(\epsilon)$. Substituting the above Equation into Equation A.2 (and keeping terms up to $O(\epsilon)$)

$$\begin{aligned} \tilde{P}(t+\epsilon, x|t_0, x_0) &= A\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\eta_i^2\right) \\ &\quad\times\left(1 - \frac{\epsilon}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{\epsilon}{2\hbar_\mu}\sum_{k=1}^{m}h_k^2(x)\right)\tilde{P}(t, x'|t_0, x_0)\left\{d^n x'\right\}, \\ &= A\int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\eta_i^2\right) \\ &\quad\times\left(1 - \frac{\epsilon}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{\epsilon}{2\hbar_\mu}\sum_{k=1}^{m}h_k^2(x)\right)\tilde{P}(t, x'|t_0, x_0)\left(1 - \frac{\epsilon}{2}\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x)\right)\left\{d^n \eta\right\}, \\ &= A\int_{-\infty}^{\infty}\{d^n\eta\}\exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\eta_i^2\right)\left(1 - \epsilon\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x) - \frac{\epsilon}{2\hbar_\mu}\sum_{k=1}^{m}h_k^2(x)\right) \\ &\quad\tilde{P}(t, x - \epsilon f(x) - \eta|t_0, x_0). \end{aligned} \tag{A.5}$$

It is noted that the Jacobian of the the transformation from x_0 to η to $O(\epsilon)$ is included in the second step.

The constant A is fixed by

$$A \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\eta_i^2\right)\{d^n\eta\} = 1. \tag{A.6}$$

Hence, it follows that

$$A \int_{-\infty}^{\infty} \eta_i\eta_j \exp\left(-\frac{1}{2\hbar_\nu \epsilon}\sum_{i=1}^{n}\eta_i^2\right)\{d^n\eta\} = \hbar_\nu\epsilon\delta_{ij}. \tag{A.7}$$

The left hand side of Equation A.5 is

$$\tilde{P}(t,x|t_0,x_0) + \epsilon\frac{\partial\tilde{P}}{\partial t}(t,x|t_0,x_0). \tag{A.8}$$

The second order expansion (in η) of $P(t,x'|t_0,x_0)$ yields

$$\begin{aligned}\tilde{P}(t,x'|t_0,x_0) &= \tilde{P}(t,x-\epsilon f(x,t)-\eta|t_0,x_0), \\ &= \tilde{P}(t,x|t_0,x_0)\left(1-\epsilon\sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}(x,t)-\epsilon\frac{1}{2\hbar_\mu}\sum_{k=1}^{m}h_k^2(x)\right) \\ &\quad -\sum_{i=1}^{n}\left(\epsilon f_i(x)+\eta_i\right)\frac{\partial\tilde{P}}{\partial x_i}(t,x|t_0,x_0)+\frac{1}{2}\sum_{i,j=1}^{n}\eta_i\eta_j\frac{\partial^2\tilde{P}}{\partial x_i\partial x_j}(t,x|t_0,x_0).\end{aligned} \tag{A.9}$$

We are interested only in terms of $O(\epsilon)$. The only nonvanishing terms are terms linear in ϵ and quadratic in η. Then it is easy to verify that the diffusion part of the YYe follows from the term quadratic in η and the rest follows from the term linear in ϵ, so that

$$\begin{cases}\dfrac{\partial\tilde{P}}{\partial t}(t,x|t_0,x_0) = \dfrac{\hbar_\nu}{2}\displaystyle\sum_{i=1}^{n}\frac{\partial^2\tilde{P}}{\partial x_i^2}(t,x|t_0,x_0)-\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left[f_i(x)\tilde{P}(t,x|t_0,x_0)\right]-\frac{1}{2\hbar_\mu}\sum_{k=1}^{m}h_k^2(x)\tilde{P}(t,x|t_0,x_0), \\ \tilde{P}(t_0,x|t_0,x_0) = \delta^n(x-x_0).\end{cases} \tag{A.10}$$

Hence, $\tilde{P}$ satisfies the YYe with a delta-function initial condition.

Annex B: Euclidean Quantum Physics: A Brief Review

For completeness, a brief discussion of relevant aspects of classical and quantum physics are summarized. For a more detailed discussion of these topics summarized below, see [17], [18], and [6].

B.1 Classical Dynamics: Lagrangian and Hamiltonian Formulations

We begin by recalling the experimental fact that the fundamental laws of classical dynamics are described by second-order differential equations for the coordinates in time. That is, the equations of motion relate the acceleration for all time in terms of velocities and positions, so that the state of a system is completely determined (and into the future) if all the coordinates and velocities are simultaneously specified. In the Newtonian formulation of classical mechanics such second-order differential equations are considered. For our purposes, we consider the motion of an unconstrained point particle whose position in $n-$dimensional Cartesian coordinates, $x_i, i = 1, \ldots, n$ moving under the influence of a conservative force is given by

$$m\frac{d^2x_i}{dt^2} = F(x_1, \ldots, x_n), \qquad \text{(B.1)}$$
$$= -\nabla V(x_1, \ldots, x_n).$$

Here $V(x)$ (for notational convenience we denote the n components x_i collectively by x) is termed the potential. Equation B.1 is referred to as the "model dynamical system". The complete solution of this dynamical problem is then a matter of integrating this vector differential equation, given simultaneous specification of position and velocity at any one instant, typically the initial condition.

While this formulation is adequate for simpler systems, it is found to be considerably more convenient and useful to reformulate the dynamical laws in terms of a scalar called the Lagrangian, which is a functional in position and velocity. Specifically, if the s generalized coordinates[7] and s velocities in an arbitrary dynamical system are collectively denoted by q and $\dot{q}$, then according to the principle of least (or more generally, extremum) action, the dynamical laws follow from extremizing the "action" S defined as follows:

$$S = \int_{t_0}^{t} L(q(\tau), \dot{q}(\tau), \tau)d\tau. \qquad \text{(B.2)}$$

7. By definition the generalized coordinates are unconstrained.

Invariance	Conserved Quantity	Expression	Model
Time Translation	Energy	$E(p,q) = \sum_{i=1}^{s} p_i \dot{q}_i - L$	$E = T + V$
Spatial Translation	Momentum	$p_i = \frac{\partial L}{\partial q_i}$	$p_i = m\dot{x}_i$, $V(x) = 0$
Rotation	Angular Momentum	$L_{ij} = x_i p_j - x_j p_i$	$L_{ij} = m(x_i \dot{x}_j - x_j \dot{x}_i)$

Table B.1: *Symmetry and Conservation Laws*

Using the standard methods of the Calculus of Variations, the Euler-Lagrange equations are:

$$\frac{\delta S}{\delta q_i} = 0 \qquad \Rightarrow \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, s). \tag{B.3}$$

For this model, the Lagrangian is

$$\begin{aligned} L(x, \dot{x}) &= \frac{1}{2} m \sum_{i=1}^{n} \left(\frac{dx_i}{dt}\right)^2 - V(x), \\ &= T - V, \end{aligned} \tag{B.4}$$

where T and V are known as the kinetic and potential energy, respectively. The Lagrangian is a functional of the position and velocities. The Lagrangian resulting in particular equations of motion is not unique. For example, the addition of the total time derivative of any function of the coordinates and time to the Lagrangian results in the same dynamical law.

The Lagrangian formulation has many useful features. First of all, the Lagrangian is a scalar, so it is much easier to work with it than with the vector equations of motion. The Lagrangian formalism makes it easy to incorporate the fundamental principle of relativity—i.e., invariance of the equations of motion under Galilean transformation in the nonrelativistic case, or Lorentz transformations in the relativistic case. It makes it possible to more easily incorporate constraints.

Perhaps the most significant advantage of the Lagrangian is the relation between symmetries and conservation laws that follows from Noether's theorem. This is summarized in Table B.1. Specifically, invariance of the Lagrangian under time translation (where L is not an explicit function of time), or homogeneity of time, implies conservation of energy. The conservation of canonical momentum of a mechanical system follows from the homogeneity of space, or invariance of L under spatial translations. Finally, the conservation of angular momentum follows from the isotropy of space, i.e., invariance of L under rotation in space. The existence of such conserved quantities enables considerable simplification of the solution of the dynamical problem.

Conversely, it is easy to write the Lagrangain invariant under these symmetry transformations. From Noether's theorem, quantities that are conserved are readily apparent. It also enables a systematic way of studying a generalization of the dynamical laws that satisfy conservation laws. In fact, in the modern approach it is the Lagrangian, rather than the equations of motion, that is used as the starting point in the analysis of a dynamical system.

A related formulation is the Hamiltonian formulation in terms of the Hamiltonian, H, which is related to L via a Legendre transformation:

$$H(q,p) = \sum_{i=1}^{s} p_i \dot{q}_i - L, \qquad p_i \equiv \frac{\partial L}{\partial \dot{q}_i}. \tag{B.5}$$

For our model dynamical system, the Hamiltonian is

$$\begin{aligned} H(x,p) &= \sum_{i=1}^{n} \frac{p_i^2}{2m} + V(x), \\ &= T + V. \end{aligned} \tag{B.6}$$

Hamilton's equations, or canonical equations follow from the equation of the differential dH:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}(q,p), \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}(q,p). \tag{B.7}$$

The Hamiltonian is the generator of time translation. It thus readily follows that if $\Phi(q,p)$ is any function of coordinates and momenta (or "phase space"), then

$$\frac{d\Phi}{dt}(q,p) = \{H(q,p), \Phi(q,p)\}, \tag{B.8}$$

where the Poisson bracket of two functions on the phase space, Φ_1 and Φ_2, is defined as

$$\{\Phi_1(q,p), \Phi_2(q,p)\} \equiv \sum_{i=1}^{s} \left(\frac{\partial \Phi_1}{\partial q_i}(q,p) \frac{\partial \Phi_2}{\partial p_i}(q,p) - \frac{\partial \Phi_1}{\partial p_i}(q,p) \frac{\partial \Phi_2}{\partial q_i}(q,p) \right). \tag{B.9}$$

A notable property of the Poisson bracket is that it is a Lie bracket, i.e., it is bilinear, antisymmetric, and satisfies the Jacobi identity.

The Hamiltonian formulation illuminates the geometric aspects of the dynamical system. In particular, while the Lagrange equations are invariant under point transformations of the form

$$Q_i = Q_i(q,t), \tag{B.10}$$

Hamilton's equations are invariant canonical transformations, or symplectic transformations:

$$\sum_{i=1}^{s} p_i dq_i - H dt = \sum_{i=1}^{s} P_i dQ_i - H' dt + dF, \qquad \text{with Generating Function } F, \tag{B.11}$$

or

$$dp_i \wedge dq_i = dP_i \wedge dQ_i \equiv \Omega. \tag{B.12}$$

The quantity Ω is referred to as the symplectic 2-form. In geometrical language, dynamical system evolution is evolution on a symplectic manifold, which naturally leads to the Poincaré-Cartan integral invariants:

$$\Omega^{\wedge p} \equiv \Omega \wedge \cdots \wedge \Omega. \tag{B.13}$$

The invariance of the $2s$ dimensional volume element is Liouville's theorem.

B.2 Canonical Quantization

It is sufficient for our purposes to consider the quantization of the model classical system described by the Hamiltonian in Equation B.6. The canonical quantization proceeds as follows[18]:

- In quantum mechanics, observables, such as position, momentum, energy, are promoted to Hermitian operators. Thus, for H given by Equation B.6

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}). \tag{B.14}$$

The eigenvalues of the Hermitian operators are the only values that are possible for the observable. The 'quantization' of observables, such as energy, then follows from the fact that the eigenspectrum of the operators is discrete.

- The fundamental Poisson bracket in the classical system is replaced by the commutator. The commutator for any two operators $\hat{f}$ and $\hat{g}$, denoted as $[\hat{f}, \hat{g}]$, is defined as

$$[\hat{f}, \hat{g}]\Phi \equiv \hat{f}(\hat{g}\Phi) - \hat{g}(\hat{f}\Phi), \tag{B.15}$$

for an arbitrary infinitely differentible, or C^∞, function Φ. Quantization implies that the fundamental Poisson bracket become the canonical commutation relations:

$$\begin{aligned} [\hat{q}_i, \hat{p}_j] &= i\hbar\{q_i, q_j\}, \\ &= i\hbar\delta_{ij}, \end{aligned} \tag{B.16}$$

and

$$\begin{aligned} [\hat{q}_i, \hat{q}_j] &= 0, \\ [\hat{p}_i, \hat{p}_j] &= 0. \end{aligned} \tag{B.17}$$

- The states of a quantum system, represented by $|\psi\rangle$ using the Dirac notation, are now given by rays in a Hilbert space (which is a complete, normed inner product space). Two vectors differing only by a phase are said to belong to the same ray.
- Unlike the canonical commutation relations, the position and momentum operator depend on the representation. Two common representations are the position and momentum space representations which diagonalize the position and momentum operators respectively:

$$\hat{x}_i|x'\rangle = x'_i|x_0\rangle, \tag{B.18}$$

$$\hat{p}_i|p'\rangle = p'|p'\rangle. \tag{B.19}$$

In the position representation, position and momentum operators are given by the replacements

$$\hat{x}_i \rightarrow x_i, \qquad \hat{p}_i \rightarrow -i\hbar\frac{\partial}{\partial x_i}, \tag{B.20}$$

as can be verified by noting that this satisfies the canonical commutation relations. Another way to understand this is to note that the momentum operator is related to the spatial translation operator, or momentum is the generator of spatial translation. It is noted that

$$\langle q|p'\rangle = \frac{1}{\sqrt{(2\pi\hbar)^n}} \exp\left(i\sum_{i=1}^{n} p_i q_i\right), \qquad \langle p'|p''\rangle = \delta(p' - p''). \tag{B.21}$$

– The time translation operator is

$$i\hbar\frac{\partial}{\partial t}, \tag{B.22}$$

and it is also known as the generator of time translation, which is the Hamiltonian $\hat{H}$. The evolution of the state $|\psi\rangle$ is given by

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}|\psi(t)\rangle. \tag{B.23}$$

This is known as the Schrödinger equation in the Schrödinger representation where the state evolves with time while the operators do not. Returning to our model example, in the position representation, the Hamiltonian is

$$\begin{aligned} \hat{H} &= \sum_{i=1}^{n} \frac{\hat{p}_i^2}{2m} + V(\hat{x}), \\ &= -\frac{\hbar^2}{2m}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + V(x). \end{aligned} \tag{B.24}$$

Thus, the Schrödinger equation becomes

$$i\hbar\frac{\partial\psi}{\partial t}(t,x) = \left[-\frac{\hbar^2}{2m}\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + V(x)\right]\psi(x,t), \qquad \text{where} \qquad \psi(t,x) \equiv \langle x|\psi(t)\rangle. \tag{B.25}$$

– A formal solution, when H is not explicitly time dependent, is

$$|\psi(t)\rangle = U(t,t_0)|\psi(t_0)\rangle, \qquad \text{where} \qquad U(t,t_0) = \exp\left(-i(t-t_0)\hat{H}\right). \tag{B.26}$$

This suggests an alternative formulation of quantum mechanics, termed the Heisenberg representation, where the state vectors are constant in time and it is the operators that evolve with time:

$$f(\hat{q}(t),\hat{p}(t)) = U^{-1}(t,t_0)f(\hat{q}(t_0),\hat{p}(t_0))U(t,t_0), \qquad \text{or} \quad \frac{d}{dt}f(\hat{q}(t),\hat{p}(t)) = [f(\hat{q}(t),\hat{p}(t)),H]. \tag{B.27}$$

The relation to the Poisson bracket formulation of classical physics is apparent (see Equation B.8).

– The fundamental problem in quantum mechanics is the computation of the transition probability amplitude

$$\langle x, t | x_0, t_0 \rangle = \langle x | \exp\left(-i\hat{H}(t - t_0)\right) | x_0 \rangle. \tag{B.28}$$

The physical interpretation of the wave function $\psi(t, x)$ is that it is a probability amplitude, i.e., the probability density is given by $|\psi(t, x)|^2$. This is the origin of the profound difference between classical and quantum probability. In particular, unlike probabilities, probability amplitudes need not be real.

– Finally, the discussion of canonical quantization assumed that the classical Lagrangian is in Cartesian coordinates. The general coordinate case is more involved and this can be traced to ordering ambiguities of quantum operators (i.e., $\hat{x}$ and $\hat{p}$ do not commute).

B.3 Path Integral Quantization

We now discuss the path integral approach to quantization of a classical system. It is the path integral formulation that enables to obtain the results of this report. For our purposes, we consider the expression for the probability amplitude for our model system [8]. That is, we are interested in the probability amplitude for a particle to move from point x_0 at time t_0 to point x at time t, where $x \in \mathbb{R}^n$. The Hamiltonian H is given by

$$\hat{H} = \sum_{i=1}^{n} \frac{\hat{p}_i^2}{2m} + V(\hat{x}). \tag{B.29}$$

The free-particle transition probability amplitude, or propagator is

$$\begin{aligned} \left\langle x \middle| \exp -i \sum_{i=1}^{n} \frac{\hat{p}_i^2}{2m} t \middle| x_0 \right\rangle &= \int \left(\prod_{i=1}^{n} \frac{dp_i}{2\pi} \right) \langle x | p \rangle \exp\left(-i \sum_{i=1}^{n} \frac{p_i^2}{2m} t \right) \langle p | x_0 \rangle , \\ &= \int \left(\prod_{i=1}^{n} \frac{dp_i}{2\pi} \right) e^{i \sum_{i=1}^{n} p_i (x_i - x_i')} \exp\left(-i \sum_{i=1}^{n} \frac{p_i^2}{2m} t \right) , \\ &= \left(\frac{m}{2\pi i t} \right)^{n/2} \exp\left(i \frac{m}{2t} \sum_{i=1}^{n} (x_i - x_i')^2 \right) . \end{aligned} \tag{B.30}$$

For a general $V(x)$, an explicit expression for the propagator cannot be written down. However, for infinitesimal time ϵ, the time evolution operator

$$U(\epsilon) \equiv \exp\left(-iH\epsilon\right) \tag{B.31}$$

can be approximated by $W(\epsilon)$ defined as

$$\begin{aligned} W(\epsilon) &\equiv \exp\left(-i\frac{\epsilon}{2}V\right) \exp\left(-iH_0\epsilon\right) \exp\left(-i\frac{\epsilon}{2}V\right), \\ &= U(\epsilon) + O(\epsilon^2). \end{aligned} \tag{B.32}$$

8. For a more rigorous discussion, the reader is referred to [19].

The advantage of considering $W(\epsilon)$ instead of $U(\epsilon)$ is that its matrix elements can be explicitly written:

$$\langle x|W(\epsilon)|x_0\rangle = \left(\frac{m}{2\pi i\epsilon}\right)^{n/2} \exp\left(i\sum_{i=1}^{n}\left[\frac{m}{2\epsilon}(x_i - x_i')^2 - \frac{\epsilon}{2}\left[V(x) + V(x_0)\right]\right]\right). \tag{B.33}$$

The time interval $[t_0, t]$ can be divided into N small elements so that

$$\epsilon = \frac{t - t_0}{N}. \tag{B.34}$$

Using the result (rigorously proved using the Lie-Kato-Trotter product formula)

$$\exp\{-i(H_0 + V)t\} = \lim_{N\to\infty} W^N(\epsilon), \tag{B.35}$$

and inserting $N - 1$ complete sets of position eigenstates, we get

$$\begin{aligned}\left\langle x|e^{-iHt}|x_0\right\rangle &= \lim_{N\to\infty}\int dx_1\cdots dx_{N-1}\,\langle x|W(\epsilon)|x_1\rangle\cdots\langle x_{N-1}|W(\epsilon)|x_0\rangle\,, \qquad \text{(B.36)}\\ &= \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{nN/2}\int dx_1\cdots dx_{N-1}\exp\left(i\frac{m}{2\epsilon}\sum_{r=0}^{N-1}(x_i(t_r) - x_i(t_{r+1}))^2 - i\frac{\epsilon}{2}\sum_{r=0}^{N}V(x)\right).\end{aligned}$$

Therefore, in the continuum limit

$$\left\langle x|e^{-iHt}|x_0\right\rangle = \int_{x(t_0)=x_0}^{x(t)=x}[\mathscr{D}x(t)]e^{iS(t_0,t)}, \tag{B.37}$$

where the action S is defined as

$$\begin{aligned}S(t_0, t) &= \int_{t_0}^{t} dt\left[\frac{m}{2}\sum_{i=1}^{n}\left(\frac{dx_i}{dt}\right)^2 - V(x)\right], \qquad \text{(B.38)}\\ &= \int_{t_0}^{t} dt L(x, \dot{x}),\end{aligned}$$

and the path integral measure $[\mathscr{D}x(t)]$ is given by

$$[\mathscr{D}x(t)] = \lim_{N\to\infty}\left(\frac{m}{2\pi i\epsilon}\right)^{nN/2} dx_1\cdots dx_{N-1}. \tag{B.39}$$

Thus, we have replaced the quantum mechanical operators with an infinite-dimensional integral which is a path integral. Also, it follows that

$$\int_{t_j}^{t_{j+1}} dtL \approx \epsilon\left[\frac{m}{2}\sum_{i=1}^{n}\left(\frac{x_i(t_{j+1}) - x_i(t_j)}{\epsilon}\right)^2 + \frac{1}{2}\left(V(x(t_{j+1})) + V(x(t_j))\right)\right]. \tag{B.40}$$

As the integrand is complex and strongly oscillating, it is not possible to satisfactorily define mathematically the path integral as an integral over a space of functions. However, the path integral formalism has proved to be extraordinarily useful in quantum field theory giving a compact and easy derivation of many results using formal manipulations of functional integrals. A sound basis of path integrals can be achieved by introducing imaginary times. Specifically, if we define $t = -i\tau$, $\tau > 0$ assuming V is bounded, then the time evolution operator becomes the positive bounded operator $e^{-H\tau}$. The variable τ is referred to as Euclidean time. Then, the path integral formula for the corresponding amplitude is

$$\left\langle x \middle| e^{-H\tau} \middle| x_0 \right\rangle = \int_{x_0}^{x} [\mathscr{D}x(t)] e^{-S_E(\tau_0,\tau)}, \tag{B.41}$$

where the 'Euclidean action' $S_E(\tau_0, \tau)$ is given by

$$\begin{aligned} S_E(\tau_0,\tau) &= -iS(it_0, it), \\ &= \int_{\tau_0}^{\tau} d\tau' \left[\frac{m}{2} \sum_{i=1}^{n} \left(\frac{dx_i}{d\tau} \right)^2 + V(x) \right]. \end{aligned} \tag{B.42}$$

This can be viewed as a formal limit of the expression

$$\begin{aligned} \left\langle x \middle| e^{-H\tau} \middle| x_0 \right\rangle = \lim_{N\to\infty} \int \left\{ \prod_{r=1}^{N-1} d^n x(t_r) \right\} \\ \times \exp\left(-\frac{m}{2\epsilon} \sum_{r=1}^{N-1} (x(t_{r-1}) - x(t_r))^2 - \epsilon \left[\frac{1}{2}[V(x) + V(x_0)] + \sum_{r=1}^{N-1} V(x(t_r)) \right] \right). \end{aligned} \tag{B.43}$$

Clearly, the integrand is manifestly real, and is damped for wildly oscillating paths with large Euclidean action, $S_E(\tau_0, \tau)$, allowing such integrals to be well-defined mathematically. When $V = 0$, it is the Wiener measure. It can be observed that the measure is concentrated on paths such that

$$|x(t_{k+1}) - x(t_k)| \sim \sqrt{\epsilon}, \qquad \text{as } \epsilon \to 0. \tag{B.44}$$

Such paths are continuous everywhere, but not necessarily differentiable.

Finally, it can be shown that the Schrödinger equation for the wavefunction

$$i\frac{\partial}{\partial t}\psi(x,t) = H(\psi(x,t)), \tag{B.45}$$

becomes

$$\frac{\partial \psi_E}{\partial \tau}(x,\tau) = -H\psi_E(x,\tau), \tag{B.46}$$

via continuation to Euclidean time. When $H = H_0$ (or $V(x) = 0$), the Euclidean Schrödinger equation is simply the heat equation; hence it is the average over Brownian paths, as in Wiener's process. When $V \neq 0$, the Euclidean Schrödinger equation is not the Fokker-Planck-Kolmogorov equation, for the Hamiltonian is Hermitian, while the FPK operator is not Hermitian.

Finally, the matrix element

$$\langle x, t | T[\hat{x}(t_1) \cdots \hat{x}(t_n)] | x_0, t_0 \rangle = \int_{x(t_0)=x_0}^{x(t)=x} x(t_1) \cdots x(t_n) e^{iS(t_0,t)}, \tag{B.47}$$

where T stands for time ordering (i.e., the operators are ordered so that $t_1 > \cdots > t_n$) is found to be useful.

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References

[1] Risken, H. (1999), The Fokker-Planck Equation: Methods of Solution and applications, 2^{nd} ed, Springer-Verlag.

[2] Balaji, B. (2008), Estimation of indirectly observable Langevin states: path integral solution using statistical physics methods, *Journal of Statistical Mechanics: Theory and Experiment*, 2008(01), P01014 (17pp).

[3] Jazwinski, A. H. (2007), Stochastic Processes and Filtering Theory, Dover Publications.

[4] Balaji, B. (2009), Universal Nonlinear Filtering Using Path Integrals II: The Continuous-Continuous Model with Additive Noise, *PMC Physics A*, Vol. 3:2.

[5] Yau, S. T. and Yau, S. S.-T (2000), Real Time Solution of Nonlinear Filtering Problem without memory I, *Mathematical Research Letters*, 7, 671–693.

[6] Feynman, R. P. and Hibbs, A. R. (1965), Quantum Mechanics and Path Integrals, McGraw-Hill Book Company.

[7] Yau, S-T. and Yau, S. S-T. (2004), Nonlinear Filtering and Time Varying Schrödinger Equation I, *IEEE Transactions on Aerospace and Electronic Systems*, 40(1), 284–292.

[8] Yau, S.-T. and Yau, S. S.-T (1996), Explicit Solution of a Kolmogorov Equation, *Applied Mathematics and Optimization*, 34(3), 231–266.

[9] Yau, S. S.-T. and Yau, S.-T. (2001), Real Time Algorithm for nonlinear filtering problem, In *Proceedings of the 40th IEEE Conference on Decision and Control*, Orlando, FL, USA.

[10] Yau, S. S-T. and Yau, S-T. (2005), Solution of filtering problem with nonlinear observations, *SIAM Journal of Control and Optimization*, 44(3), 1019–1039.

[11] Yau, S-T. and Yau, S. S.-T. (2008), Real Time Solution of the Nonlinear Filtering Problem without Memory II, *SIAM Journal on Control and Optimization*, 47(1), 163–195.

[12] Balaji, B. (2006), Universal Nonlinear Filtering Using Path Integrals I: The Continuous-Discrete Model with Additive Noise, *submitted to IEEE Transactions on Aerospace and Electronic Systems.*

[13] Zinn-Justin, Jean (2002), Quantum Field Theory and Critical Phenomena, International Series in Monographs on Physics, Oxford University Press.

[14] Moral, P. D. (2004), Feynman-Kăc Formulae, Springer-Verlag.

[15] Bender, C. M., Brody, D. C., and Meister, B. K. (2003), Must a Hamiltonian be Hermitian?, *American Journal of Physics*, 71, 1095–1102.

[16] Montváy, I. and Münster, G. (1997), Quantum Fields on a Lattice, Cambridge Monographs on Mathematical Physics, New York, USA: Cambridge University Press.

[17] Landau, L. D. and Lifshitz, E. M. (1969), Mechanics, 2nd ed, Vol. 1 of *Course of Theoretical Physics*, Pergamon Press.

[18] Dirac, P. A. M. (1982), The Principles of Quantum Mechanics, 4th ed, London, UK: Oxford University Press.

[19] Glimm, J. and Jaffe, A. (1987), Quantum Physics: A Functional Integral Point of View, 2nd ed, New York, USA: Springer-Verlag.

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In this report, it is shown that Euclidean quantum mechanics is closely related to the continuous nonlinear filtering problem. The key is the configuration space Feynman path integral representation of the fundamental solution of a Fokker-Planck type of equation termed the Yau Equation of continuous-continuous filtering. A corollary is the equivalence between nonlinear filtering problem and a time-varying Schrödinger equation previously pointed out by S-T. Yau and Stephen Yau. The path integral formulation is shown to lead to a better conceptual understanding of the origin of this relationship.

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