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Complexity profiles: networks with ring-lattice and small-world relationships

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Abstract

With the recognition of an increasing number of natural and man made systems as complex systems, there is a greater need for characterizing their complexity. One way to describe the complexity of a system is as a measure of the variety of different options it can execute. The complexity profile is a method for characterizing the complexity of a system at each scale or each size of subsets of entities of the system. Thus it shows the variety of effects the system can produce, where these effects require the coordination of different sizes of subsets of entities. This document investigates the complexity profiles for an abstract system consisting of entities with Gaussian state distribution whose relationship to each other is characterized by with ring-lattice and small-world structures. Small-world relationships have been found to be a characteristic of many complex systems. Understanding the relationship of network structure to their complexity profiles can contribute towards methods for engineering complex systems that have desired complexity profiles.

Résumé

Le nombre croissant de systèmes naturels ou construits par des humains qui sont considérés comme systèmes complexes rend plus pressant le besoin de caractériser leur degré de complexité. Cette complexité est un indice de la diversité de tâches qu'un système peut remplir. Le profil de complexité constitue une méthode de caractérisation d'un système pour chaque échelle ou taille de sous-ensembles d'entités d'un système. Il peut ainsi représenter la diversité des effets qui peuvent résulter d'un système, lorsque l'obtention de ces effets exige la coordination de sous-ensembles d'entités de tailles diverses. Le présent document décrit le calcul des profils de complexité d'un système composé d'entités avec une distribution gaussienne dont les relations mutuelles sont caractérisées par leur matrice de covariance avec des relations en treillis en anneau et des relations de type « petit monde ». On a trouvé que les relations de type « petit monde »sont une des caractéristiques d'un bon nombre de systèmes. La compréhension de la relation entre les structures de réseau et leur profil de complexité peut apporter une contribution aux méthodes d'ingénierie de systèmes complexes ayant les profils de complexité voulus.

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Executive summary

Complexity profiles: networks with ring-lattice and small-world relationships

Gerard R. Pieris, Giovanni Fusina; DRDC Ottawa TM 2008-330; Defence R&D Canada – Ottawa; April 2009.

Background: The motivation behind this paper is the Technology Investment Fund project entitled "Methodology For Evaluating Force Capabilities For Complex Threat Environments Using Complexity Profiles" whose purpose is to give DRDC's military clients a better quantitative understanding of the current complex asymmetric battle-space. In this way, the failure of several past and recent military and peace keeping missions (Vietnam, Afghanistan) may be overcome. These failures have been attributed to the lack of understanding of the phenomenon of complex emergent behaviour that characterizes current threat environments and hence the inability to match the resources to the mission to be performed. The characterization, or quantification, of the complex asymmetric battle-space is a crucial step in this project. As an additional motivation, the recognition of an increasing number of natural and man made systems as complex systems entails a greater need for characterizing their complexity.

One can simplistically state that the complexity of a system is a measure of the variety of different options it can execute. The complexity profile is a method for characterizing the complexity of a system at each scale or each size of subsets of entities of the system. Thus it shows the variety of effects the system can produce, where these effects require the coordination of different sizes of subsets of entities. The complexity profile has the property whereby the amount of correlation between multiple entities can be deduced from a covariance matrix representing the pair-wise correlation between the entities. Complexity profiles have been computed for a number of systems such as magnetic spin models and hierarchical enterprise organization structures.

Principal results: In this document we investigate the complexity profile of an abstract system of entities with Gaussian state distribution, whose mutual interactions are given by either a covariance matrix or by coefficients of the bilinear interaction corresponding to the canonical distribution of continuous degrees of freedom (CDCDF). The structure of the entities is modeled by ring-lattice and small-world structures. This abstract system is meant to represent a "real-world" system in a mathematically tractable manner so that preliminary observations can be made about the computation and utility of the complexity profile. Recent investigations into the relationships of biological and social systems have revealed that many of them exhibit the small-world property.

In all models, for small values of covariance, the maximum complexity occurs at the smallest scale, and the magnitude of the complexity reduces sharply with scale. The ring-lattice model of covariance shows oscillations in the complexity profile which increases with the value of covariance. The oscillations also increase with the amount of re-wiring when the ring-lattice is rewired to a small-world model. Oscillations are attributable to the phenomenon of frustration occurring when the system is globally constrained. For the ring-lattice model of bilinear interactions there are no oscillations in the complexity profile. The value of complexity drops sharply from scale 1 to scale 2 and reduces gradually thereafter. Rewiring to a small-world network has negligible effect on the complexity profile.

Significance of results: Characterization of complexity profiles for these models can contribute towards methods for engineering complex systems to achieve desired complexity profiles. The computation of the complexity profile is a viable method to characterize complexity if the mutual interactions follow a CDCDF model.

Future work: We are currently investigating ways of exploiting symmetries in the matrices for more efficient computation of the complexity profiles to extend the study to larger (100 nodes or greater) networks.

Sommaire

Complexity profiles: networks with ring-lattice and small-world relationships

Gerard R. Pieris, Giovanni Fusina ; DRDC Ottawa TM 2008-330 ; R & D pour la défense Canada – Ottawa ; avril 2009.

Considérations générales : : Cet article est centré sur le projet du Fonds d'investissement technologique intitulé « Méthodologie pour l'évaluation des capacités des Forces dans les environnements de menace complexes en utilisant des profils de complexité », dont l'objectif est d'aider les clients militaires de RDDC à avoir une meilleure compréhension quantitative de la notion actuelle de l'espace de bataille asymétrique complexe. Ainsi, pourra t on éviter à l'avenir les revers qu'ont connus plusieurs missions militaires et missions de maintien de la paix récentes ou plus anciennes (Vietnam, Afghanistan). On impute ces cas d'insuccès à une mauvaise compréhension du phénomène du comportement émergent complexe qui caractérise les environnements de menace actuels et, du même coup, l'incapacité des autorités à faire correspondre les ressources à la mission à exécuter. La définition, ou quantification, de l'espace de bataille asymétrique complexe est une étape cruciale de ce projet. En outre, le fait de considérer un nombre croissant de systèmes naturels ou construits par des humains comme des systèmes complexes rend plus pressant le besoin de caractériser leur degré de complexité. On peut affirmer de fa con simpliste que la complexité d'un système est un indice de la diversité des tâches qu'il peut exécuter. Le profil de complexité constitue une méthode de caractérisation d'un système pour chaque échelle ou taille de sous-ensembles d'entités du système. Il peut ainsi représenter la diversité des effets qui peuvent résulter du système, lorsque la production de ces effets exige la coordination des sous-ensembles d'entités de tailles diverses. Une des propriétés du profil de complexité est que l'on peut déduire le degré de corrélation entre de multiples entités à partir d'une matrice de covariance représentant la corrélation par paires entre les entités. On a calculé des profils de complexité pour divers systèmes comme les modèles de spin magnétique et les structures organisationnelles hiérarchiques d'entreprises.

Principaux résultats : : Dans ce document, nous analysons le profil de complexité d'un système abstrait d'entités ayant chacune une distribution gaussienne et dont les interactions mutuelles sont représentées soit par une matrice de covariance ou par les coefficients de l'interaction bilinéaire correspondant à la distribution canonique des degrés de liberté continus (CDCDF « canonical distribution of continuous degrees of freedom »). La structure des entités est modélisée selon des structures en treillis en anneau et de type « petit monde ». Ce système abstrait vise à représenter un système « de situation réelle »par une solution mathématique, de sorte que l'on puisse faire des observations préliminaires sur le calcul et l'utilité du profil de complexité. Des études récentes sur les relations entre les

systèmes biologiques et sociaux ont révélé qu'un grand nombre d'entre eux présentent la propriété du « petit monde ».

Dans tous les modèles, si les valeurs de covariance sont faibles, la complexité est maximale à l'échelle la plus basse, et l'ordre de grandeur de la complexité diminue rapidement en fonction de l'échelle. Le modèle de covariance « treillis en anneau »révèle des oscillations dans le profil de complexité dont l'amplitude augmente avec la valeur de covariance. Nous observons en outre une augmentation de l'amplitude des oscillations en raison directe du volume de restructuration lorsque le treillis en anneau est restructuré dans un modèle « réseau du petit monde ». Les oscillations sont imputables au phénomène de frustration qui se produit lorsque le système est entièrement soumis à des contraintes. En revanche, le modèle d'interactions bilinéaires fondé sur le treillis en anneau ne révèle pas d'oscillations dans le profil de complexité. La valeur de la complexité passe brusquement de l'échelle 1 à l'échelle 2, et diminue progressivement par la suite. La restructuration dans un « réseau du petit monde »a un effet minime sur le profil de complexité.

Portée des résultats : : La caractérisation des profils de complexité peut aider à améliorer les méthodes qui servent à concevoir des systèmes complexes qui présentent les profils de complexité voulus. Le calcul du profil de complexité est un bon moyen de définir la complexité si les interactions mutuelles reposent sur un modèle CDCDF.

Projets de recherche : : Nous examinons actuellement des moyens d'exploiter les propriétés de symétrie des matrices pour calculer plus efficacement les profils de complexité, afin d'étendre l'analyse à des réseaux plus vastes (100 noeuds ou plus).

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1 Introduction

1.1 Motivation and Context

The CARDS (Capabilities for Asymmetric and Radiological Defence and Simulation) section at Defence R & D Canada – Ottawa has been awarded a Technology Investment Fund project entitled “Methodology For Evaluating Force Capabilities For Complex Threat Environments Using Complexity Profiles.” The underlying motivation behind this project is to give DRDC’s military clients a better understanding of the current complex asymmetric battle-space in order to overcome the failure of several past and recent military and peace keeping missions (Vietnam, Afghanistan), which has been attributed to the lack of understanding of the phenomenon of complex emergent behaviour that characterizes current threat environments and hence the inability to match the resources to the mission to be performed. [2], [3], [4]

Recently, the possibility of applying Complex Systems analysis techniques for the solution of modern military problems such as asymmetric threats has begun to be recognized [2], [4], [5]. Complex Systems behaviour characterized by global emergence has been recognized to occur across a range of fields such as physics, biology and sociology. Each field had originally developed their own approaches to the study of these behaviours. The field of Complex Systems arose from the recognition of the underlying common characteristics across these domains and the need for and possibility of developing solutions that can have broad application across such fields.

Complex emergent behaviour occurs when a group of entities interacting with each other exhibits behaviour that cannot be readily predicted from the behaviour of the constituents, and if subsets of the entities when separated from the group will not have the same behaviour as they would have within the group [1]: thus it is directly applicable to the modern battle-space. While traditional warfare has emergent behaviors, the manifold dependences between components in modern warfare require a new level of attention to this phenomenon. Asymmetric threat environments of networked enemy combatants mingled with civilians in complex terrain or urban environments self-organizing to achieve a particular adversarial mission demonstrate this situation. Examples of this phenomenon were seen earlier in Vietnam [2] and are currently demonstrated in Afghanistan and Iraq. It is also characteristic of the broader phenomenon of terrorism with groups widely distributed geographically, mingled among civilians operating in collaboration but with a degree of autonomy.

Complex emergent behaviour can be characterized by the complexity profile [6], [7], [8]. The complexity of a system is a measure of its possible number of states. Thus the complexity of an adversarial force is a measure of the number of options available to the adversary. The complexity profile is the variation of the complexity with the scale at which the system is being considered, e.g. whether at the level of individuals or larger combat units. According to a modified version of Ashby's Law of Requisuite Variety [9], [7], a force designed to identify, track and overcome an adversary while minimizing civilian damage needs to have a degree of complexity equal to or greater than the complexity of the adversary at all scales of force [2], [3]. This generalized principle can equally be applied to the ability of any system to achieve its objectives in a complex environment.

The purpose of the present document is to compute the complexity profile for an abstract system of entities having a Gaussian state distribution [10] that are related (correlated) to each other in either a simple ring-lattice network structure, or a small-world network structure. These terms will be defined below. This abstracted system of entities makes the computation of its complexity profile tractable. This can lead to insights into complexity profiles of network models that represent an organizational structure, even a military one, such as a command and control network.

1.2 Characterization of Complexity

With the recognition of an increasing number of natural and man made systems as complex systems, there is a greater need for characterizing their complexity.

Several approaches, deterministic and probabilistic, have been proposed for characterization of the complexity of a system. Kolmogorov complexity [11] gives a measure of deterministic complexity of a system. Here, the complexity of a system is the minimum length description of the system. Kolmogorov complexity does not provide a useful characterization of systems for our goals as it does not provide a means to determine if a system is suitable for a particular task. Information theory and Statistical Mechanics provide tools that can be developed for a statistical characterization of complexity. In information theory, entropy is an indicator of the uncertainty of the state of the system. The entropy of a group of entities is highest when they display the least amount of coordination and thus the most amount of randomness. When all the entities are coordinated the system has the lowest possible entropy. The entropy of the entire system alone does not provide information on the structural properties of the system: how subsets of entities are related to each other. Mutual information captures the correlation between two sets of entities. Several approaches for extension to correlation of multiple entities have been proposed [12] [13]. The "complexity profile" as defined in [14] is such an extension. It gives the entropy for each size of subset of entities of the system. It has the property whereby the amount of correlation between multiple entities can be deduced from a covariance matrix representing the pair-wise correlation between the entities. The complexity profile together with the law of requisite variety [9] as discussed below has the potential of being able to determine if a system satisfies minimum requirements for performing a particular task.

Complexity profiles have been computed for a number of systems (examples include models of magnetic spins [15], [16] and hierarchical enterprise organization structures [17]). In this document we investigate the complexity profile of an abstract system of entities, each having a Gaussian state distribution whose mutual interactions exhibit different types of relationships. This abstract system of entities is meant to represent a “real-world” system: characterizing a “real-world” system in terms of states and correlation is a very arduous task, but it will be the subject of future documents relating to this project. For now we consider the abstracted systems in order to draw preliminary conclusions about the computation and applicability of the complexity profile.

The Gaussian distribution has been widely used in the analysis of biological and social systems [10], [18]. Recent investigations into the relationships of biological and social systems has revealed that many of them also exhibit the small-world property [19]. In this document we consider four types of relationships between entities. The first and second models investigate the cases where the covariance matrices for the entities exhibit ring-lattice and small world properties respectively. The third and fourth models are based on the canonical distribution for continuous degrees of freedom (CDCDF) with bilinear interactions studied in [15]. In the third case, the bilinear interactions are modeled by a ring-lattice relationship. This case has been studied in the context of magnetic spins [15] and we include this case for reference. In the fourth model the bilinear interactions are represented by a small-world relationship. In the first model we observe oscillations in the complexity profile that increase in amplitude with increasing values of covariance. Also when the ring-lattice of the first model is rewired to give the small-world network of the second model, these oscillations increase with the amount of rewiring. For the third model there are no oscillations in the complexity profile. Rewiring to the small-world network of the fourth model has negligible effect on the complexity profile.

The Law of Requisite Variety [9], has been generalized in [7] to multiple scales that account for the coordination of subsets of entities within a system. According to the Multiscale Law of Requisite Variety, a system designed to perform a task should have a variety (or complexity) equal to or greater than the variety of the task at each scale. Understanding how the complexity profile varies for various complex network structures can assist in the design of systems for such complex tasks.

We describe the models in Section 2. Results of our analysis are presented in Section 3 and conclusions are given in Section 4.

2 Model

The system we consider is based on the model in [15] and consists of n entities, the state of each entity being described by a Gaussian random variable x_i, with mean 0 and variance σ_i^2. The correlations between the random variables are given by the covariance matrix $\mathbf{R}$. The joint probability distribution is then expressed as

$$P(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det \mathbf{R}}} \exp\left(-\frac{1}{2}\mathbf{x}^T\mathbf{R}^{-1}\mathbf{x}\right). \tag{1}$$

From Shannon's information theory [20], the amount of information contained in the system, or its entropy is given by

$$H = \int \ln[P(\mathbf{x})]P(\mathbf{x})d\mathbf{x}, \tag{2}$$

which for the distribution in (1), yields (see pp. 230 in [21])

$$H = \frac{n}{2}[\ln(2\pi)+1] + \frac{1}{2}\ln[\det \mathbf{R}]. \tag{3}$$

For the case of n i.i.d. uncorrelated gaussians, $\mathbf{R}$ reduces to a diagonal matrix with the diagonal elements equal to σ^2. Hence, $\ln(\det \mathbf{R}) = 2n\ln\sigma$ and the entropy is n times that of a single variable.

The multiscale formalism of [14] [15] extends the concept of entropy to capture the relationship among components of the system. It is noted there that the multiscale complexity of an n-component system at scale k,

1. For the smallest scale it should equal the entropy of all degrees of freedom,

2. For a system consisting of distinct subsystems of up to l entities that are coupled within the subsystems but not coupled to other subsystems, $C_n(k)$ should be 0 for $k > l$ and non-zero otherwise,

3. The multiscale complexity of an independent set of entities should be equal to the sum of their complexities.

The following definition of $C_n(k)$ has been shown [14] [15] to uniquely satisfy these conditions:

$$C_n(k) = \sum_{j=0}^{k-1} (-1)^{k-j-1} \binom{n-j-1}{k-j-1} Q(n,j), \tag{4}$$

where

$$Q(n,k) = -\sum_{j \in \{j_1,\ldots,j_l\}} \int P(\mathbf{x} - \{\mathbf{x}_j\}) \ln[P(\mathbf{x} - \{\mathbf{x}_j\})] d\{\mathbf{x} - \{\mathbf{x}_j\}\}. \tag{5}$$

Here, $j_1, \ldots., j_l$ each denotes a $(n-k)$-element subset from the n elements $1, \ldots.n$, where $l = \binom{n}{n-k}$. The term $\mathbf{x} - \{\mathbf{x}_j\}$ denotes a k-row vector obtained by removing the rows with indices corresponding to the numbers in set j from the n-row vector $\mathbf{x}$. Then, for the joint probability density function of (1), equation (5) simplifies to

$$Q(n,k) = -\sum_{j \in \{j_1,\ldots,j_l\}} \frac{n}{2}[\ln(2\pi) + 1] + \frac{1}{2}\ln[\det \mathbf{R}_j]. \tag{6}$$

where $\mathbf{R}_j$ is obtained from $\mathbf{R}$ by deleting the set of $(n-k)$ rows and columns coresponding to the row (and column) indices in set j.

The complexity profile has been computed in [15] using expressions (4) and (6) for various covariance matrices corresponding to a variety of magnetic spin relationships. The authors in [15] observe the close similarity of expression 1 to that for the canonical distribution for continuous degrees of freedom (CDCDF)

$$P(\mathbf{x}) = \frac{\exp(\frac{\beta}{2}\sum_{ij} J_{ij} x_i x_j)}{Z}, \tag{7}$$

with bilinear interactions $H = -\frac{1}{2}\sum_{ij} J_{ij} x_i x_j$, where the partition function Z is the integral of the numerator over state space. Comparing the expressions they have the correspondence $\mathbf{R} = (-\beta\mathbf{J})^{-1}$, where $\mathbf{J}$ is the matrix of elements J_{ij}. Varying β is equivalent to rescaling the variables x_i and x_j and therefore they set $\beta = 1$ and $J_{ii} = -1$ in their analysis.

Our goal in this document is to investigate the complexity profile of small-world type covariance relationships among entities. For comparison, we also model ring-lattice type covariance relationships and small-world type matrices representing the J_{ij} coefficients modeled in [15]. In summary we investigate the following four models:

1. Model 1: The covariance Matrix $\mathbf{R}$ has the structure of a ring lattice.

2. Model 2: The covariance Matrix $\mathbf{R}$ has the structure of a small-world network.

3. Model 3: Bilinear interaction matrix **J**, where the bilinear interactions form a ring lattice. This is studied in [15] but is reproduced here as a reference for the other models.

4. Model 4: Bilinear interaction matrix **J**, where the bilinear interactions form a small world network.

As noted in the introduction, small-world relationships have been found to be common among a variety of man-made and natural systems [1], [22], [19], and the Gaussian distribution has been used to characterize a range of biological and social systems [10] [18].

The model we use for the small-world network is based on that originally proposed by Watts and Strogatz [1]. They consider a sparse regular network on N nodes with k links per node connected in a lattice structure as shown in Figure 1 (a) for $k = 4$. Starting with this structure, the ring is traversed clockwise, rewiring each link with probability p to terminate at a different node. The node to which it is rewired is selected uniformly at random among the remaining $N-2$ nodes. The case where $p = 1$ yields the random network shown in Figure 1 (c). Figure 1 (b) shows a case $0 < p < 1$ where two links have been rewired. Figure 2 shows a plot of the normalized (with respect to the value at $p = 0$) clustering coefficient and normalized characteristic path length as a function of p for $N = 1000$ and $k = 10$. It is evident that the regular lattice has a large clustering coefficient and large characteristic path length. For a random network both the characteristic path length and clustering coefficient are small. A range of values of p yields the small-world property of a large clustering coefficient and small characteristic path length. The small-world property has been found in a range of natural and man-made systems [1], [22], [19]. Note that the clustering coefficient is a measure of the modularity of the network and the characteristic path length is a measure of the average distance between a pair of nodes. For the small size of network and small values of k $(k = 2)$ that we consider, the above method can readily lead to disjointed networks. We only consider the cases where the network remains connected.

Figure 3 shows a matrix of bilinear interactions for a ring-lattice with $k = 2$, where $J_{ij} = a$ for nearest neighbours , $J_{ii} = -1$ for self interactions, and $J_{ij} = 0$ otherwise. It also shows a corresponding matrix for a small-world model after rewiring. Models 1 and 2 studied in this document have covariance matrices with respectively a ring-lattice and a small-world structure. A covariance matrix $\mathbf{R}$ is non-negative definite [23], i.e. $\mathbf{x}^T R\mathbf{x} \geq 0$, for any vector $\mathbf{x}$, thus giving $\det \mathbf{R}_j \geq 0$ in equation (6). Figure 4 shows a covariance matrix when $k = 4$ for a ring lattice and a corresponding matrix for a small-world model after rewiring. We vary ρ such that $\mathbf{R}$ remains non-negative definite. The value of ρ_{max} denotes the largest value of ρ, $0 < \rho < 1$ for which we were able to satisfy this condition in our numerical computation (Note that in this document our focus is not in investigating the nature of the variation of this condition as a function of ρ). Finally, bilinear interactions corresponding to models 1 and 2 can be obtained from $\mathbf{J} = -\frac{1}{\beta}\mathbf{R}^{-1}$.

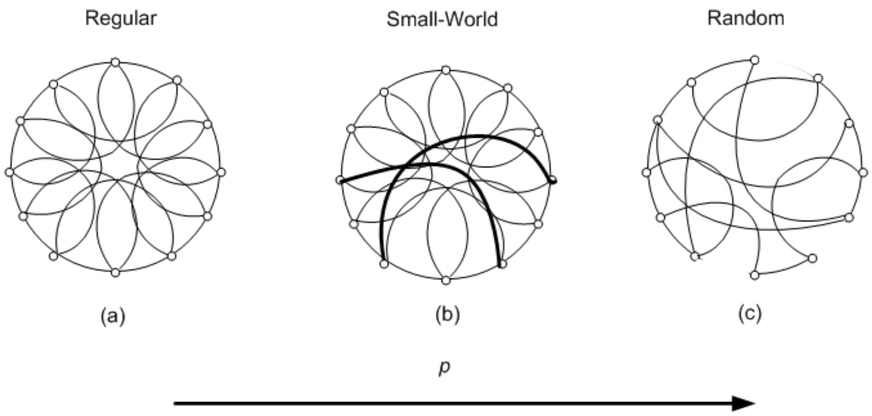


Figure 1: *Construction of a Small-World Network.*

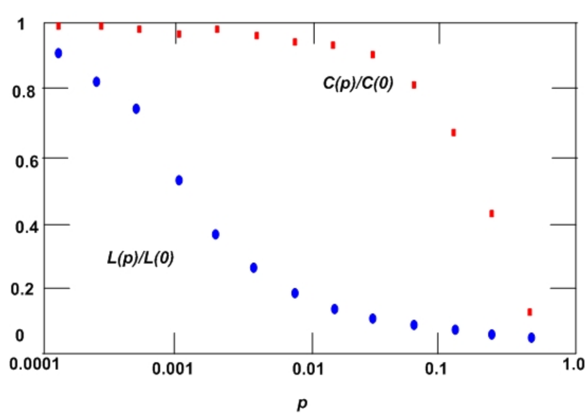


Figure 2: *Variation of Characteristic Path Length and Clustering Coefficient with p for $N = 1000$ and $k = 10$.The figure is adapted from [1].*

$$
\begin{pmatrix}
-1 & a & 0 & 0 & 0 & 0 & 0 & a \\
a & -1 & a & 0 & 0 & 0 & 0 & 0 \\
0 & a & -1 & a & 0 & 0 & 0 & 0 \\
0 & 0 & a & -1 & a & 0 & 0 & 0 \\
0 & 0 & 0 & a & -1 & a & 0 & 0 \\
0 & 0 & 0 & 0 & a & -1 & a & 0 \\
0 & 0 & 0 & 0 & 0 & a & -1 & a \\
a & 0 & 0 & 0 & 0 & 0 & a & -1
\end{pmatrix}
\qquad
\begin{pmatrix}
-1 & 0 & 0 & 0 & \mathbf{a} & 0 & 0 & a \\
0 & -1 & a & 0 & 0 & 0 & 0 & 0 \\
0 & a & -1 & 0 & 0 & \mathbf{a} & 0 & 0 \\
0 & 0 & 0 & -1 & a & 0 & 0 & 0 \\
\mathbf{a} & 0 & 0 & a & -1 & a & 0 & 0 \\
0 & 0 & \mathbf{a} & 0 & a & -1 & a & 0 \\
0 & 0 & 0 & 0 & 0 & a & 1 & a \\
a & 0 & 0 & 0 & 0 & 0 & a & -1
\end{pmatrix}
$$

Figure 3: *Bilinear interaction coefficients with $k = 2$ for a 8-node ring lattice and its rewired small-world network.*

$$
\begin{pmatrix}
1 & \rho & \rho & 0 & 0 & 0 & 0 & 0 & \rho & \rho \\
\rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 & 0 & \rho \\
\rho & \rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 & 0 \\
0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 \\
0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 & 0 \\
0 & 0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 \\
0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho & \rho \\
\rho & 0 & 0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho \\
\rho & \rho & 0 & 0 & 0 & 0 & 0 & \rho & \rho & 1
\end{pmatrix}
\qquad
\begin{pmatrix}
1 & \rho & 0 & 0 & 0 & \rho & 0 & 0 & \rho & \rho \\
\rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 & 0 & \rho \\
0 & \rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 & 0 \\
0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 & 0 & 0 \\
0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 & 0 \\
\rho & 0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 & 0 \\
0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho & \rho & 0 \\
0 & 0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho & \rho \\
\rho & 0 & 0 & 0 & 0 & 0 & \rho & \rho & 1 & \rho \\
\rho & \rho & 0 & 0 & 0 & 0 & 0 & \rho & \rho & 1
\end{pmatrix}
$$

Figure 4: *Covariance matrix with $k = 4$ for a 10-node ring lattice and its rewired small-world network.*

3 Results and Analysis

3.1 Models 1 and 2

Table 1 summarizes the results for the ring-lattice and small-world network models of the covariances for $k = 2$. The value ρ_{max} denotes the largest value of ρ for which $\mathbf{R}$ remained non-negative definite.

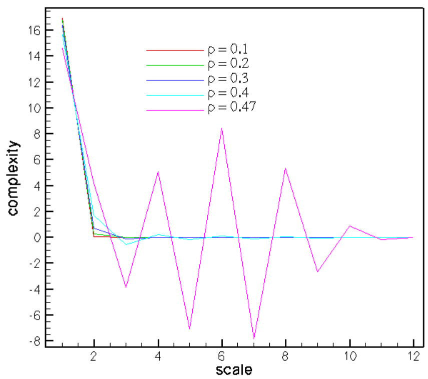
Number of Nodes	p	ρ_{max}	Number of rewired nodes
12	0/12 = 0.0	0.49	0
12	1/12 = 0.083	0.45	1
12	3/12 = 0.250	0.49	2
12	5/12 = 0.416	0.43	6
12	7/12 = 0.583	0.40	6
14	0/14 = 0.0	0.49	0
14	1/14 = 0.071	0.45	1
14	3/14 = 0.214	0.45	3
14	5/14 = 0.357	0.45	6
14	7/14 = 0.5	0.47	6
18	0.0	0.49	0
18	0.1	0.47	1
18	0.3	0.43	6
18	0.5	0.43	9
22	0.0	0.49	0
22	0.1	0.47	1
22	0.3	0.48	4
22	0.5	0.46	7

Table 1: *Summary of results for ring-lattice and small-world models of the covariances.*

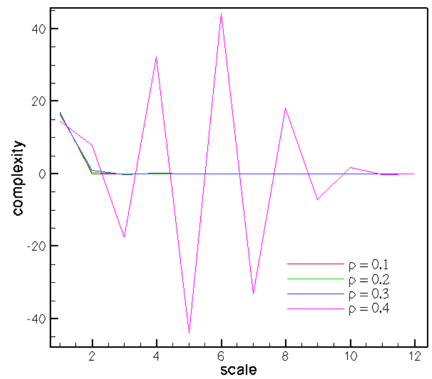
Figure 5, shows the complexity profiles for a 12-node network. Figure 5(a) shows the complexity profile when the covariance matrix is tri-diagonal and ρ is varied. For small values of ρ the complexity is large at scale 1 and drops sharply at larger scales. The profile oscillates with small amplitude, alternatively taking positive and negative values at each increment in scale. The oscillations increase with ρ and is largest at $\rho = 0.47$. At $\rho = 0.5$, $\mathbf{R}$ ceases to be positive definite. The oscillations in the complexity profile can be attributed to global constraints[8], [14]; $\sum x_i = 0$, thus making one of the variables redundant.

In Figure 5(c) the network is re-wired for varying rewiring probabilities p, with $\rho = 0.1$. For this case, there is negligible change in the complexity profile with rewiring. At higher values of ρ in figures 5(b) and 5(d) there is increased oscillation with rewiring. Also, from table 1, the value of ρ_{max}, the largest value of ρ for which the $\mathbf{R}$ remains non-negative definite, decreases with rewiring.

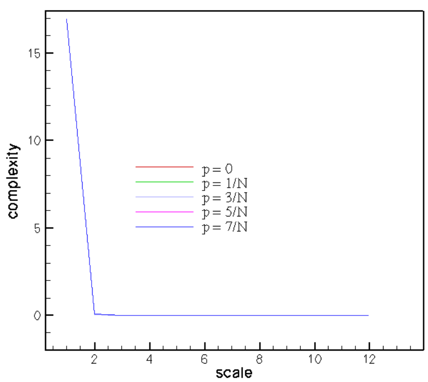
Figures 6, 7 and 8 show the complexity profiles for 14-node, 18-node and 22-node networks respectively. Similar trends are observed for these networks. Also, the complexity at scale 1, and the magnitude of oscillation for large ρ increases with the size of the network.



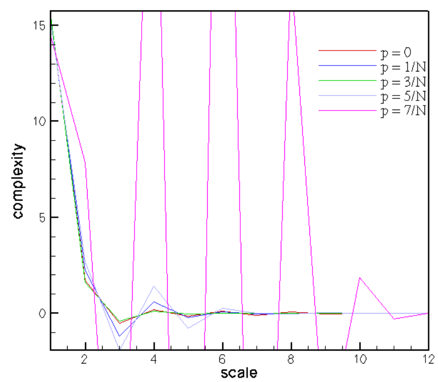
(a) without rewiring



(b) different values of ρ with rewiring probability $p = 7/12$

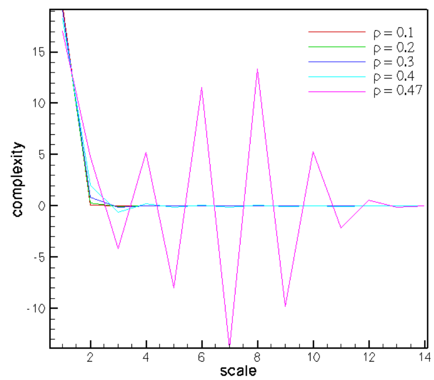


(c) different values of rewiring probabilities p, with $\rho = 0.1$. N is the number of nodes in the network.

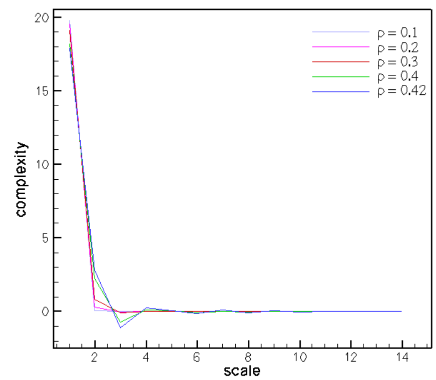


(d) different values of rewiring probabilities p, with $\rho = 0.4$.

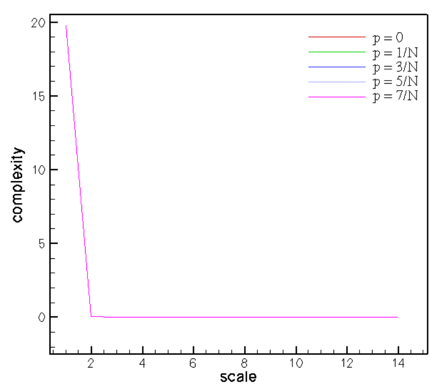
Figure 5: *Complexity profile for a 12 x 12 tridiagonal covariance matrix*



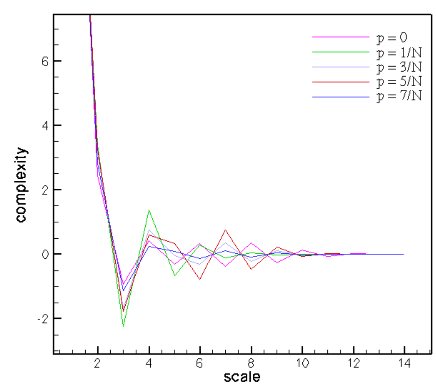
(a) without rewiring



(b) different values of ρ with rewiring probability $p = 7/14$

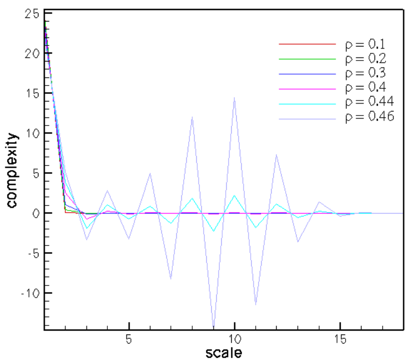


(c) different values of rewiring probabilities p, with $\rho = 0.1$.

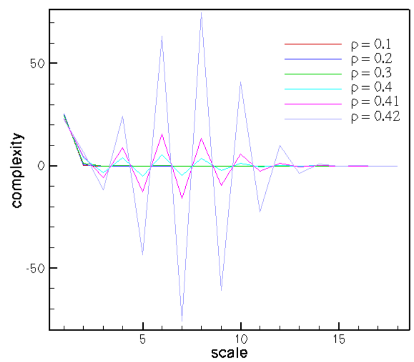


(d) different values of rewiring probabilities p, with $\rho = 0.42$.

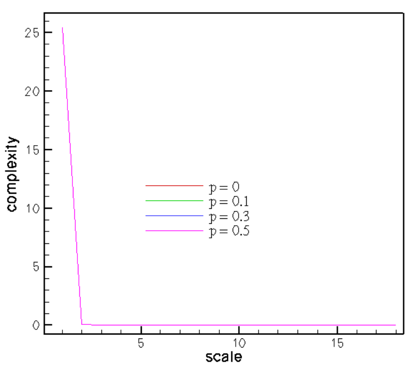
Figure 6: *Complexity profile for a 14 x 14 tridiagonal covariance matrix*



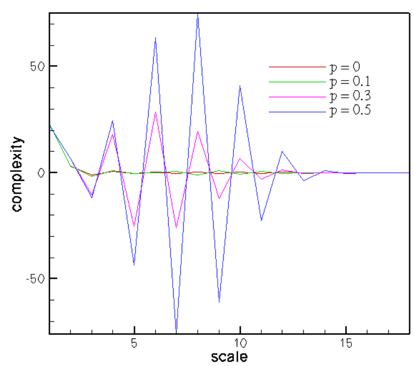
(a) without rewiring



(b) different values of ρ with rewiring probability $p = 0.5$

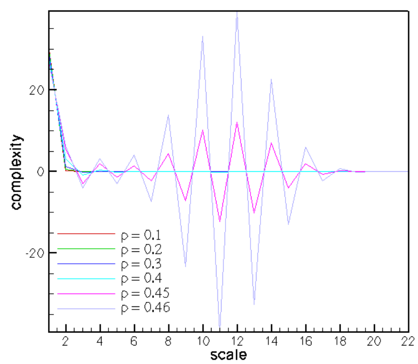


(c) different values of rewiring probabilities p, with $\rho = 0.1$.

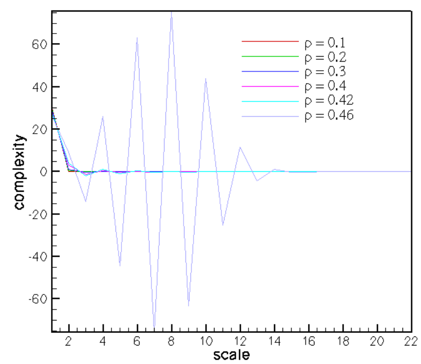


(d) different values of rewiring probabilities p, with $\rho = 0.42$.

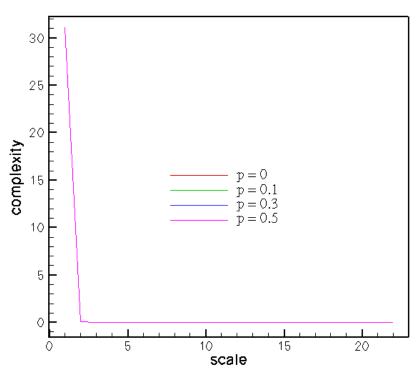
Figure 7: *Complexity profile for a 18 x 18 tridiagonal covariance matrix*



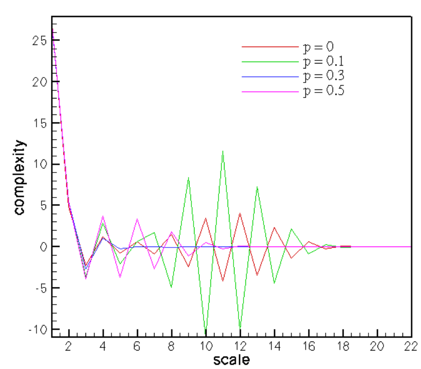
(a) without rewiring



(b) different values of ρ with rewiring probability $p = 0.5$



(c) different values of rewiring probabilities p, with $\rho = 0.1$.



(d) different values of rewiring probabilities p, with $\rho = 0.44$.

Figure 8: *Complexity profile for a 22 x 22 tridiagonal covariance matrix*

3.2 Models 3 and 4

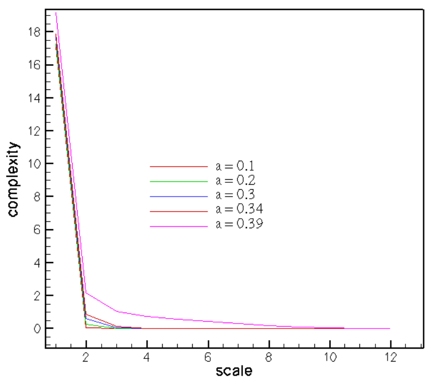
Table 2 summarizes the results for the ring-lattice with $k = 2$ and the small-world network obtained from rewiring the links. The ring-lattice model is used in [15] and is reproduced here for reference. Here, a_{max} denotes the largest value of a for which the matrix **R** remained non-negative definite.

Number of nodes	p	a_{max}	Number of rewired nodes
12	0/12 = 0.0	0.49	0
12	1/12 = 0.083	0.45	1
12	3/12 = 0.25	0.41	2
12	5/12 = 0.416	0.41	6
12	7/12 = 0.583	0.40	6
14	0/12 = 0.0	0.49	0
14	1/12 = 0.083	0.45	1
14	3/12 = 0.25	0.43	3
14	5/12 = 0.416	0.42	6
14	7/12 = 0.583	0.44	6
18	0.0	0.49	0
18	0.1	0.47	1
18	0.3	0.43	6
18	0.5	0.43	9
22	0.0	0.49	0
22	0.1	0.47	1
22	0.3	0.44	4
22	0.5	0.41	7

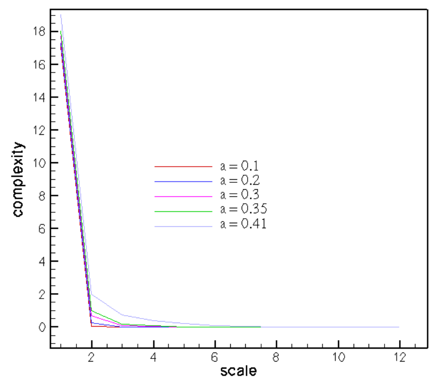
Table 2: *Summary of results for Models 3 and 4.*

Figure 9 shows a plot of the complexity profile for a ring lattice of bilinear interactions on 12 nodes with $k = 2$. The complexity is highest at a scale of 1 and it rapidly and steadily decreases with scale. It is seen that complexity at small scales steadily increases with a. Rewiring the network has negligible effect on the complexity profile. These same characteristics were also observed for 14, 18 and 22 node networks. The 22-node case is shown in Figure 12.

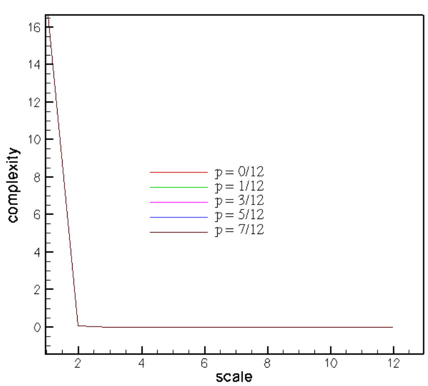
In summary, there is little difference in the complexity profile whether the bilinear interactions are represented by a simple ($k = 2$) small ring lattice network or a simple small small-world type network. When the covariance between the entities is represented by simple small ring lattice network oscillations in the complexity profile occurs with increasing covariance. When the network is rewired to a simple small small-world network of covariances, these oscillations increase with the amount of rewiring.



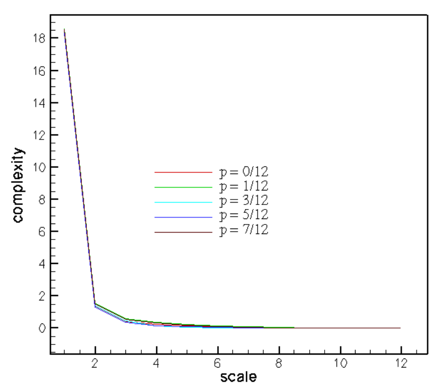
(a) without rewiring



(b) different values of a with rewiring probability $p = 7/12$

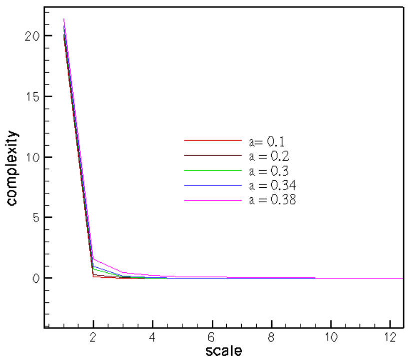


(c) different values of rewiring probabilities p, with $a = 0.1$.

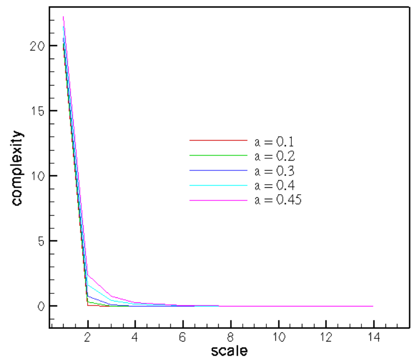


(d) different values of rewiring probabilities p, with $a = 0.38$.

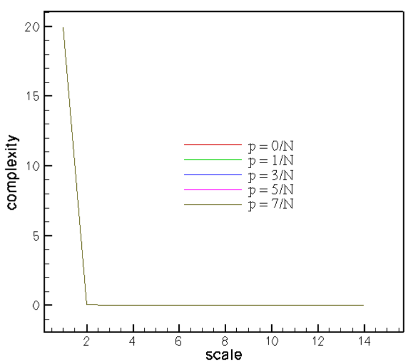
Figure 9: *Complexity profile for a 12-node ring lattice.*



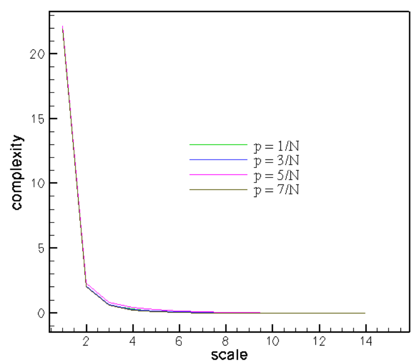
(a) without rewiring



(b) different values of a with rewiring probability $p = 7/12$

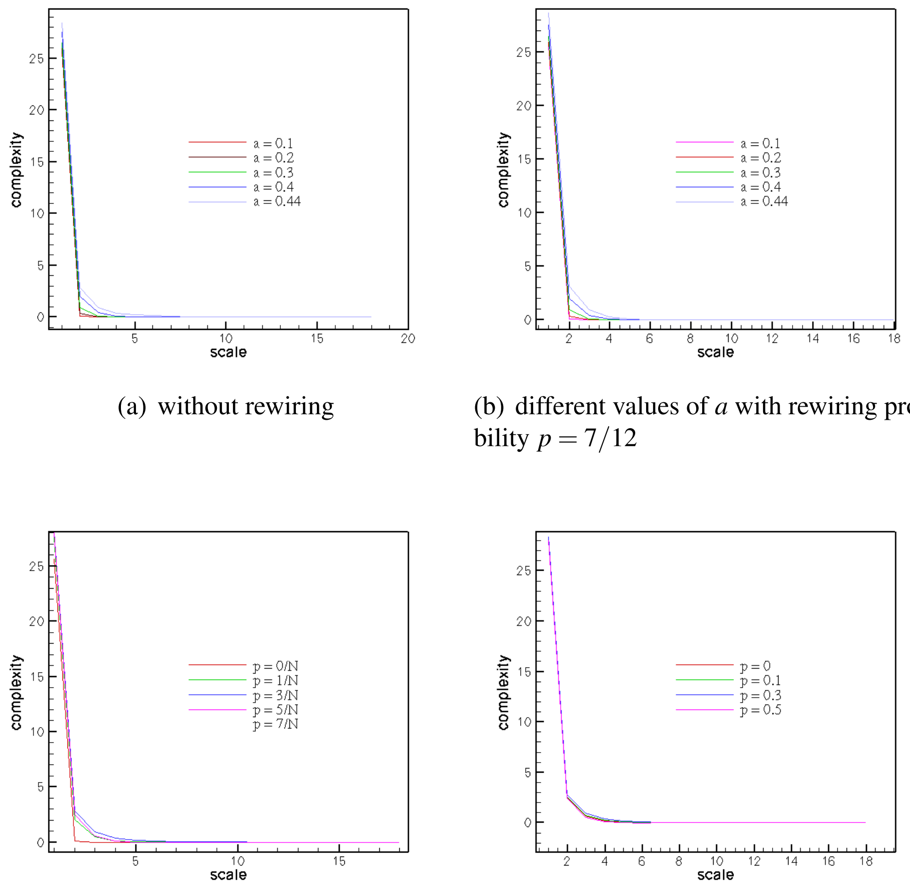


(c) different values of rewiring probabilities p, with $a = 0.1$.



(d) different values of rewiring probabilities p, with $a = 0.43$.

Figure 10: *Complexity profile for a 14-node ring lattice.*



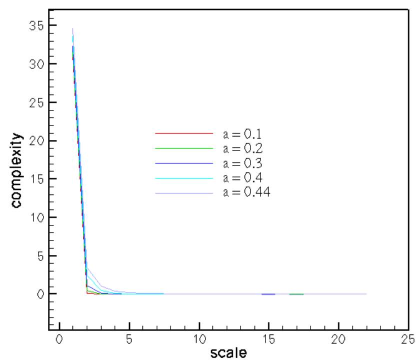
(a) without rewiring

(b) different values of a with rewiring probability $p = 7/12$

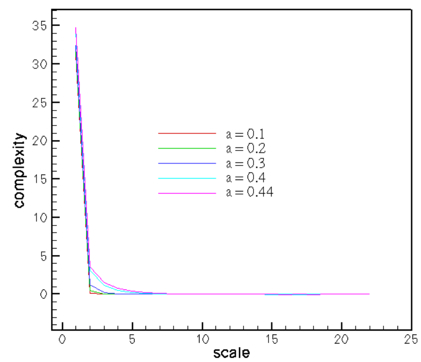
(c) different values of rewiring probabilities p, with $a = 0.1$.

(d) different values of rewiring probabilities p, with $a = 0.43$.

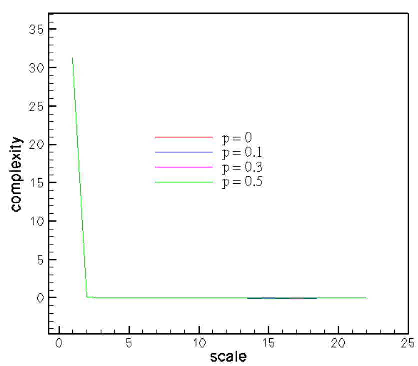
Figure 11: *Complexity profile for a 18-node ring lattice*



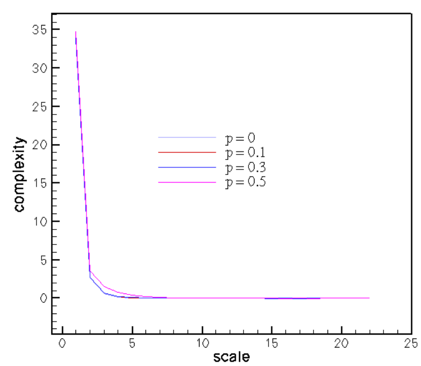
(a) without rewiring



(b) different values of a with rewiring probability $p = 0.5$



(c) different values of rewiring probabilities p, with $a = 0.1$.



(d) different values of rewiring probabilities p, with $a = 0.43$.

Figure 12: *Complexity profile for a 22-node ring lattice.*

4 Conclusions and Future Work

The complexity profile is a way to characterize the complexity of complex systems, whereby the clustering of interactions of entities in the system can be represented. The small-world property has been observed in many complex systems. The main goal of this paper was to observe how the small-world property might affect the complexity profile. We considered a system where entities have a Gaussian state distribution, and where the entities have ring-lattice and simple small world type relationships. These relationship models were investigated for their covariance matrices and for the coefficients of the bilinear interaction corresponding to the canonical distribution of continuous degrees of freedom studied in [15].

The bilinear interaction models (models 3 and 4), show a high complexity at the smallest scale and small complexity for higher scales. There is negligible change in the complexity profile with rewiring to a simple small world type network.

For small values of covariance, the ring lattice relationship of covariance also showed a high value of complexity at scale 1 and a sharp decrease for larger scales. However there were oscillations of small magnitude. Oscillations are attributable to the presence of global constrains on the system thereby making some variables redundant [14]. Rewiring to a simple small-world type network caused a small increase in the oscillations. For large values of covariance, approaching 0.5, the ring lattice relationship showed significant oscillations in the complexity profile. Rewiring to simple small-world models further increased the oscillations in the complexity profile.

It would be interesting to see whether these trends extend to general ring-lattice and small-world networks of larger size and higher values of k. We are currently investigating algorithms for more efficient computation of the complexity profiles to explore how these properties extend to such networks.

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With the recognition of an increasing number of natural and man made systems as complex systems, there is a greater need for characterizing their complexity. One way to describe the complexity of a system is as a measure of the variety of different options it can execute. The complexity profile is a method for characterizing the complexity of a system at each scale or each size of subsets of entities of the system. Thus it shows the variety of effects the system can produce, where these effects require the coordination of different sizes of subsets of entities. This document investigates the complexity profiles for an abstract system consisting of entities with Gaussian state distribution whose relationship to each other is characterized by with ring-lattice and small-world structures. Small-world relationships have been found to be a characteristic of many complex systems. Understanding the relationship of network structure to their complexity profiles can contribute towards methods for engineering complex systems that have desired complexity profiles.

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