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Eigenfunctions of the Fourier transformation over a circle — I

Approximation of Sturmian eigenvalues

Richard F. Boivin

Defence R&D Canada – Ottawa
Technical Memorandum
DRDC Ottawa TM 2008-342
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Canada

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Abstract

We clarify and expand upon Slepian's perturbation scheme for approximating the eigenvalues of the Sturm-Liouville equation characterizing the eigenfunctions of the finite Fourier transformation over a circle. The eigenvalues are approximated as power series in terms of c^2 or $1/c$, where c represents the adjustable scaling constant that controls mapping of the frequency domain of the transformation to the field domain. Analytical expressions of the series coefficients are worked out up to fifth order. An algorithm is also provided for numerical determination of higher-order coefficients. The accuracy of the series is investigated; prospects for extensive computations of the eigenvalue spectrum are discussed; and some applications of the eigenfunctions are outlined.

Résumé

Nous clarifions et développons le procédé de perturbation de Slepian pour l'approximation des valeurs propres de l'équation de Sturm-Liouville qui caractérise les fonctions propres de la transformation finie de Fourier sur un cercle. L'approximation des valeurs propres se fait par des séries en puissances de c^2 ou $1/c$, où c représente la constante d'échelle ajustable qui contrôle la projection du domaine de fréquence de la transformation sur le domaine du champ. Nous élaborons jusqu'à l'ordre cinq des expressions analytiques des coefficients des séries. Nous fournissons aussi un algorithme pour la détermination numérique des coefficients d'ordre plus élevé. Nous étudions la justesse de ces séries ; nous discutons les perspectives de calculs étendus du spectre des valeurs propres ; et nous esquissons quelques applications des fonctions propres.

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Executive summary

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Richard F. Boivin; DRDC Ottawa TM 2008-342; Defence R&D Canada – Ottawa; March 2009.

Background: This work is concerned with certain functions of two real variables which, upon being Fourier-transformed over a circular domain, *regenerate*—up to a multiplicative constant—over another circle c times larger or smaller, where c is a freely adjustable scaling constant. These eigenfunctions of the particular Fourier transformation involved have completeness and orthogonality properties that make them attractive for dealing with various field problems. The standard method of calculating such mathematical entities consists of solving a Sturm-Liouville differential equation which they happen to satisfy. This requires accurate knowledge of the spectrum of the equation's eigenvalue, χ, which depends on scaling constant c. A first step is to compute a reasonable approximation of each eigenvalue. For this purpose D. Slepian [1] proposed a perturbation scheme that expresses χ as a series in powers of c^2 or $1/c$, according as c may be considered "small" or "large" in some sense. The series coefficients are obtained through a little-understood, recursive process.

Main results: We first review some known properties of the eigenfunctions in the limits $c \to 0$ and $c \to \infty$, which are prerequisite to application of Slepian's perturbation scheme. Next we derive the latter, *ab initio*, in its original general form, and describe its recursive character. From this process we work out explicit, non-recursive expressions of the perturbation-series coefficients up to fifth order. We then specialize these expressions to the two specific cases in hand, where the perturbation parameter is either c^2 or $1/c$. Our analysis corrects unreported misprints in the papers by Slepian [1] and Heurtley [2]. We provide several concrete examples of the resulting fifth-order series, as well as illustrations of their employment to approximate eigenvalue χ in various instances. We also examine the accuracy of these series and confirm the existence of an interval of values of c where both perform poorly. Finally, we convert Slepian's recursive process to a *non*-recursive algorithm that allows computation of the perturbation-series coefficients up to any desired order, thus increasing the accuracy potential of the approximation scheme.

Significance of results: Slepian did not explain the internal mechanics of his recursive process for calculating the coefficients of the perturbation series. Together with the fact that some of his final formulæ contained undocumented misprints, this may account for the absence of reported use of his eigenvalue approximation method. The clarifications supplied here should correct this situation.

Our calculation of the fourth- and fifth-order terms in the power series for approximating eigenvalue χ appears to be an entirely novel contribution to the field, as the few such formulæ found in the literature (some of which are in error) reach only to third order.

All of our expressions consist of analytical combinations of certain integers that serve to select χ; they amount, in the end, to exact ratios of integers, so that the coefficients of the perturbation series are, in effect, known with infinite accuracy. These analytical formulæ will contribute to a better understanding of the series' convergence properties and may suggest a general analytical formalism free from recursion.

However, the most significant gain resulting from this work may be our non-recursive algorithm for computing successive perturbation-series coefficients up to any order. The evidence we present suggests that the additional terms accessible with this algorithm would permit χ to be usefully approximated across the interval of c values where the fifth-order series prove inadequate. Hence, the way is now open for computing a reasonable approximation of the entire eigenvalue spectrum.

Future work: Once an approximation of eigenvalue χ is available, it can be refined to any degree of accuracy by adapting the scheme Bouwkamp [3] devised for computing the prolate spheroidal wave functions, which are the one-dimensional analog of the eigenfunctions considered here. This complementary scheme is based on the three-term recurrence satisfied by the coefficients of the series expansion representing the wave functions. We plan to perform the necessary adaptation (thus filling another gap in the literature), and then to use the resulting algorithm to compute a table of highly accurate eigenvalues covering a suitably wide range of the parameters that define them. Because Bouwkamp's scheme is also optimal for computing the coefficients of the expansion it is based upon, the opportunity should be seized to construct a table of expansion coefficients for the eigenfunctions themselves. At that point the latter will have become available for application to problems of interest to the field analyst or designer. The Conclusion outlines specific applications to instrumental optics, optical data processing, antenna design, and the mitigation of diffraction.

Sommaire

Eigenfunctions of the Fourier transformation over a circle — I: Approximation of Sturmian eigenvalues

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Contexte : Ce travail porte sur certaines fonctions de deux variables réelles qui, en subissant la transformation de Fourier sur un domaine circulaire, *se reproduisent*—à une constante multiplicative près—sur un autre cerle c fois plus grand ou plus petit, où c est une constante d'échelle que l'on peut ajuster à volonté. Ces fonctions propres de la transformation de Fourier particulière en cause possèdent des propriétés d'intégralité et d'orthogonalité qui les rendent attrayantes pour traiter divers problèmes de champ. La façon usuelle de calculer les objets mathématiques de ce genre consiste à résoudre une équation différentielle de Sturm-Liouville qu'ils s'avèrent satisfaire. Cela demande une connaissance précise du spectre de la valeur propre de l'équation, χ, laquelle est fonction de la constante d'échelle c. Comme première étape on calcule une approximation acceptable de chaque valeur propre. À cette fin D. Slepian [1] a proposé un procédé de perturbation qui exprime χ en série de puissances de c^2 ou $1/c$, selon que c peut être considéré “petit” ou “grand” dans un certain sens. Les coefficients de la série sont obtenus au moyen d'un mécanisme récursif peu compris.

Principaux résultats : Nous commençons par passer en revue certaines propriétés connues des fonctions propres dans les limites $c \to 0$ et $c \to \infty$, préparatoires à l'application du procédé de perturbation de Slepian. Ensuite nous démontrons ce dernier du tout début, dans sa forme originale générale, et nous décrivons son caractère récursif. À partir de ce mécanisme nous élaborons jusqu'à l'ordre cinq des expressions explicites, non récursives, des coefficients de la série de perturbation. Puis nous restreignons ces expressions aux deux cas précis visés, où le paramètre de perturbation est soit c^2, soit $1/c$. Notre analyse corrige des coquilles apparaissant dans les articles de Slepian [1] et Heurtley [2] et qui n'ont jamais été signalées. Nous fournissons plusieurs exemples concrets des séries de cinquième ordre résultantes et illustrons leur emploi dans l'approximation de la valeur propre χ sous diverses conditions. De plus, nous étudions la justesse de ces séries et confirmons l'existence d'un intervalle de valeurs de c sur lequel l'une comme l'autre donne des résultats médiocres. Enfin, nous convertissons le mécanisme récursif de Slepian en un algorithme *non* récursif qui permet de calculer les coefficients de la série de perturbation jusqu'à tout ordre souhaité et qui, ce faisant, augmente le potentiel de précision du procédé de perturbation.

Portée des résultats : Slepian n'a pas expliqué le fonctionnement interne de son mécanisme récursif pour le calcul des coefficients de la série de perturbation. Joint au fait que certaines de ses formules finales contenaient des fautes typographiques jamais consignées, cela peut faire comprendre qu'on ne rapporte pas d'utilisation de sa méthode d'approximation des valeurs propres. Les éclaircissements apportés ici devraient corriger cette situation.

Il semble que notre calcul des termes de quatrième et cinquième ordres des séries de puissances pour l'approximation de la valeur propre χ soit une contribution totalement originale à ce domaine, car les quelques formules de ce genre que l'on trouve dans la documentation (erronées dans certains cas) ne se rendent qu'au troisième ordre. Toutes nos expressions se composent de combinaisons analytiques de certains entiers qui servent à choisir χ ; elles se ramènent, au bout du compte, à des rapports exacts d'entiers, de telle sorte que les coefficients des séries de perturbation sont, de fait, connus avec une précision infinie. Ces formules analytiques vont contribuer à une meilleure compréhension des propriétés de convergence des séries et peuvent suggérer une formulation analytique générale libre de toute récurrence.

Cependant, il se peut que le gain le plus considérable résultant de ce travail soit notre algorithme non récursif pour le calcul des coefficients successifs de la série de perturbation jusqu'à n'importe quel ordre. Les données que nous présentons suggèrent que les termes additionnels accessibles au moyen de cet algorithme pourraient permettre une approximation utile de χ à travers l'intervalle de valeurs de c où les séries de cinquième ordre se démontrent insuffisantes. Ainsi, la voie est maintenant ouverte pour le calcul d'une approximation acceptable du spectre entier de valeurs propres.

Recherches futures : Une fois que l'on dispose d'une approximation de la valeur propre χ, on peut l'affiner jusqu'à n'importe quel niveau de précision en adaptant le procédé inventé par Bouwkamp [3] pour le calcul des fonctions d'onde du sphéroïde allongé, qui constituent l'analogue unidimensionnel des fonctions propres considérées ici. Ce procédé complémentaire est fondé sur la récurrence à trois termes que satisfont les coefficients du développement en série représentant les fonctions d'onde. Nous nous proposons d'effectuer l'adaptation requise (comblant ainsi une autre lacune dans la documentation), puis d'employer l'algorithme résultant pour calculer une table de valeurs propres très précises, embrassant une gamme suffisamment large des paramètres qui les définissent. Parce que le procédé de Bouwkamp est aussi la façon optimale de calculer les coefficients du développement sur lequel il est fondé, on devrait profiter de l'occasion pour construire une table de coefficients de développement des fonctions propres elles-mêmes. Ces dernières seront, à ce stade, devenues disponibles pour en faire l'application à des problèmes intéressant l'analyste ou concepteur de champ. Notre conclusion esquisse des applications particulières à l'optique instrumentale, au traitement des données optiques, à la conception des antennes, et à la mitigation de la diffraction.

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Acknowledgements

Jean-François Rivest's support of this research is gratefully acknowledged.

1 Introduction

Many physical systems are governed by an *integral transformation*; this relates, through an integral, the state at any one point of some domain of the system to the state at *every point* of *another* domain. If the domain over which the transformation is performed has limited extent—as the need for physical realism often demands—one must deal with a so-called *finite* transformation. Finite or not, the transformation mechanism suggests the existence of its *eigenfunctions*, which describe states that the transformation merely *reproduces*. Physically, states like these correspond to what are known as the *modes* of a system.

One well known integral transformation is that associated with the name of the French mathematical physicist Jean-Baptiste Joseph Fourier. Essentially, the Fourier transformation links the complex amplitude at some field point to the complex amplitude over a frequency domain. It is central to numerous areas of field assessment or design—for example, the analysis of image formation and the optimization of radiative structures, to single out but two major applications rooted in electromagnetism. One can also point to communications theory, signal processing, and even quantum physics. If the system is two-dimensional (2-d), cannot be assumed infinite, and has rotational symmetry, the transformation is performed over a circular domain. This leads to the concept of eigenfunctions of the Fourier transformation over a circle. The present work seeks to make these mathematical entities readily available to the field analyst or designer.

From a quite general mathematical standpoint, one may consider eigenfunctions of the *multi-dimensional* finite Fourier transformation. By definition such functions, upon being Fourier-transformed over some multi-dimensional domain of finite extent, will regenerate—up to a multiplicative constant—over a scaled version of the transformation domain. The simplest instance of these eigenfunctions occurs in the case of the usual, one-dimensional (1-d) Fourier transformation, where they are defined by

$$\int_{-1}^{+1} \psi(x')\, e^{iux'} dx' = \alpha\, \psi(u/c), \qquad |u| \le c. \tag{1–1}$$

Here the transformation domain is the real line segment $-1 \le x' \le 1$, and the behaviour of eigenfunction ψ therein gets mapped, through the transformation, onto the real line segment $-c \le u \le c$, with c a positive scaling constant. Allowance for multiplicative loss or gain is made through constant α. By writing $u = cx$ we obtain a more symmetric form of Eq. (1–1), viz.

$$\alpha\, \psi(x) = \int_{-1}^{+1} \psi(x')\, e^{icxx'} dx', \qquad |x| \le 1. \tag{1–2}$$

More generally, denote points in Euclidean space of D dimensions, E_D, by vectors $\mathbf{x} = (x_1, x_2, \ldots, x_D)$, and let dx stand for $\prod dx_i$, the infinitesimal element of volume in said space. The multi-dimensional equivalent of Eq. (1–1) is

$$\int_R \psi(\mathbf{x}')\, e^{i\mathbf{u}\cdot\mathbf{x}'} dx' = \alpha\, \psi(\mathbf{u}/c), \tag{1–3}$$

where R is some bounded region of E_D and $\mathbf{u} \in cR$. By analogy with Eq. (1–2) we may rewrite this as

$$\alpha\,\psi(\mathbf{x}) = \int_R \psi(\mathbf{x}')\,e^{ic\mathbf{x}\cdot\mathbf{x}'}dx', \qquad \mathbf{x}\in R. \tag{1–4}$$

A theory of such multi-dimensional *finite Fourier self-transforms* (FFSTs) was developed by D. Slepian [1], under the assumption that R has a certain kind of symmetry, namely: $\mathbf{x}\in R$ implies $-\mathbf{x}\in R$. Slepian showed that, under this sole and relatively weak restriction, the solutions to Fredholm integral equation (1–4) enjoy the following properties:

- they can be chosen real and either even or odd, α being real in the former instance and imaginary in the latter;
- they are orthogonal in R and complete in the class of functions square-integrable in R;
- when extended to all of Euclidean space by demanding that Eq. (1–4) hold for all $\mathbf{x}\in E_D$, they are found to be orthogonal in E_D as well as complete in the class of cR-limited functions—that is, functions whose inverse Fourier transform vanishes outside region cR of the frequency domain.

It is the orthogonality and completeness of the FFSTs over dual domains that makes them so relevant to physics and engineering problems in which finite Fourier transforms arise.[1] In any case, Slepian then limited his analysis to the special case where R is a D-sphere. He was able to reduce this to two dimensions—R a circle, as we are concerned with. But what is more, scaling arguments show that it is sufficient to consider, as Slepian did, a *unit* circle. By switching to polar coordinates and appealing to Fourier-Bessel theory, Slepian exhibited the corresponding FFSTs as follows:

$$\begin{gathered}\psi_{N,n}(r,\theta) = \phi_{N,n}(r)\cos(N\theta), \qquad \alpha_{N,n} = 2\pi i^N \beta_{N,n},\\ N,n = 0,1,2,\ldots,\end{gathered} \tag{1–5}$$

where

$$\beta_{N,n}\,\phi_{N,n}(r) = \int_0^1 \phi_{N,n}(r')J_N(crr')r'\,dr', \qquad 0\le r\le 1. \tag{1–6}$$

Each FFST appears as the product of the Fourier harmonic of order N, which provides the azimuthal dependence, with one of a denumerable infinity of finite *Hankel* self-transforms (FHSTs), denoted $\phi_{N,n}$, which provide the radial dependence.[2] Enumeration is via rank n, which, like order N, ranges up to infinity. This doubly-infinite set of eigenfunctions $\psi_{N,n}$ and eigenvalues $\alpha_{N,n}$ depends on scaling constant c, a fact which can be rendered explicit by modifying the notation to $\psi_{N,n}(c,r,\theta)$, $\phi_{N,n}(c,r)$, $\alpha_{N,n}(c)$ and $\beta_{N,n}(c)$. The scaling constant may be shown to allow complete freedom in selecting the radii of the frequency domain and the conjugate time or space domain. In fact c must be taken as their product, which explains why it is often referred to as the *time-* (or *space-*) *bandwidth product*.

[1] B.R. Frieden [4] provides an early but highly suggestive review of applications to signal evaluation, design and extrapolation.

[2] We recall that, given a function f of the real variable x, its *Hankel transform of order* ν is defined as

$$g(z) = \int_0^\infty f(x)J_\nu(zx)x\,dx,$$

where z is likewise real and J_ν is the Bessel function of the first kind of real order ν.

For our purposes, the basic piece of information emerging from Slepian's seminal monograph is that calculation of the eigenfunctions of the Fourier transformation over a circle amounts to solving the integral equation

$$\beta\,\phi(r) = \int_0^1 \phi(r')J_N(crr')r'\,dr', \qquad 0 \le r \le 1. \tag{1–7}$$

When Slepian was approaching this problem, it had been known for some time that in the equivalent 1-d case the eigenfunctions—that is, the solutions to Eq. (1–2), which, without risk of confusion, we may denote $\psi_n(c,x)$—were identical with the prolate spheroidal wave functions (PSWFs) of order zero. These constitute one family of solutions to the wave equation in prolate spheroidal coordinates and were known, in this connection, to satisfy the following Sturm-Liouville equation:

$$\frac{d}{dx}\left[(1-x^2)\,\frac{d\psi_n}{dx}\right] + (\chi_n - c^2x^2)\,\psi_n = 0. \tag{1–8}$$

Here $\chi_n(c)$ is the (real, positive) Sturmian eigenvalue associated with $\alpha_n(c)$, one of the eigenvalues of Fredholm integral equation (1–2). The above differential equation provided a convenient—indeed, the standard—means of computing functions ψ_n. Therefore, it was a natural step for Slepian to seek a similar differential equation for functions $\phi_{N,n}$. For this purpose he introduced two auxiliary variables, $\varphi(r) = \sqrt{r}\,\phi(r)$ and $\gamma = \sqrt{c}\,\beta$, which enabled him to symmetrize the kernel of Eq. (1–7):

$$\gamma\,\varphi(r) = \int_0^1 \varphi(r')J_N(crr')\sqrt{crr'}\,dr', \qquad 0 \le r \le 1. \tag{1–9}$$

Slepian then managed to find a second-order, self-adjoint differential operator that *commuted* with the symmetric integral operator, thus establishing auxiliary self-transform φ as also the solution to a Sturm-Liouville equation. Specifically, every $\varphi_{N,n}$ must satisfy

$$\frac{d}{dr}\left[(1-r^2)\,\frac{d\varphi_{N,n}}{dr}\right] + \left(\chi_{N,n} - c^2r^2 + \frac{\frac{1}{4}-N^2}{r^2}\right)\varphi_{N,n} = 0, \tag{1–10}$$

where $\chi_{N,n}(c)$ is the (real, positive) eigenvalue associated with $\gamma_{N,n}(c)$, one of the eigenvalues of Fredholm integral equation (1–9). Note that by making $N = \pm\frac{1}{2}$ in Eq. (1–10) we retrieve Eq. (1–8), which is why Slepian referred to the $\varphi_{N,n}$ as *generalized prolate spheroidal functions* (GPSFs). It should be borne in mind, however, that these differ by a factor $\sqrt{r}$ from the FHSTs, which constitute the radial part of the FFSTs:

$$\varphi_{N,n}(c,r) = \sqrt{r}\,\phi_{N,n}(c,r), \qquad \gamma_{N,n}(c) = \sqrt{c}\,\beta_{N,n}(c). \tag{1–11}$$

In fact, converting back from φ to ϕ we find that every $\phi_{N,n}$ satisfies[3]

$$(1-r^2)\,\frac{d^2\phi_{N,n}}{dr^2} + \left(\frac{1}{r} - 3r\right)\frac{d\phi_{N,n}}{dr} + \left(\chi_{N,n} - \frac{3}{4} - c^2r^2 - \frac{N^2}{r^2}\right)\phi_{N,n} = 0. \tag{1–12}$$

[3] Eq. (1–12) was independently established by J.C. Heurtley [2], who also obtained, regarding eigenvalues $\chi_{N,n}$, several results equivalent to Slepian's.

Here the eigenvalues are the same as in Eq. (1–10). They are typically ranked so that

$$0 < \chi_{N,0} < \chi_{N,1} < \cdots < \infty. \tag{1–13}$$

As a result it is found that the quantities $\lambda_{N,n} = (c\beta_{N,n})^2$ settle into the following sequence:

$$1 > \lambda_{N,0} > \lambda_{N,1} > \cdots > 0. \tag{1–14}$$

For computing his GPSFs, Slepian devised or advocated methods patterned upon those used for computing the PSWFs. Essentially, his approach consists of solving differential equation (1–10) by means of an expansion in terms of orthogonal polynomials. Of course such a procedure requires accurate knowledge of the eigenvalue spectrum—a goal achievable in two steps. First, having identified the eigenfunctions in the twin limits $c \to 0$ and $c \to \infty$, Slepian applies perturbation methods to generate, from the differential equation, series expansions of $\chi_{N,n}(c)$ in powers of either c^2 or $1/c$, with coefficients depending only on N and n. For fixed N and n, such expansions are applicable to small and large values of c, respectively. Conversely, for fixed c the series in powers of c^2 is applicable to large combinations of N and n, whereas the series in powers of $1/c$ is applicable to small ones, with the threshold between "large" and "small" defined by some relationship between N and n on one hand and c on the other. In any case, these series are meant to supply only a starting approximation of $\chi_{N,n}$. This can then be refined to any degree of accuracy through an ingenious scheme devised by C.J. Bouwkamp [3] for computing the PSWFs, and based on the three-term recurrence satisfied by the coefficients of the orthogonal-polynomial expansion representing these eigenfunctions. The recurrence may be recast as a continued fraction for ratios of successive coefficients, which leads to a transcendental equation valid for any χ_n. From this equation an expression can be derived for correcting some such approximation of χ_n as power series like those discussed above will supply, and the process can be iterated until the correction vanishes.

Slepian's methods are, to say the least, not straightforward, and over the years other techniques were proposed for computing the GPSFs. For the most part these amount to formulating a matrix eigenvalue problem equivalent to Eq. (1–9) and solving it numerically by means of standard algorithms. But in this context matrix methods are generally regarded as adequate only for computing low-rank eigenfunctions—i.e., those associated with small eigenvalues $\chi_{N,n}$—since accuracy requirements, for large eigenvalues, translate into ever-increasing demands on matrix size. An alternative is to integrate numerically differential equation (1–10) or some substitute. But discretization techniques have built-in, irreducible sources of error, and small step sizes—when feasible—rapidly translate into prohibitive demands on computer resources.

Thus it would appear that the classical computation methods inaugurated by Slepian and his predecessors still have their usefulness. However, when setting out to use these methods, one is soon confronted with two substantial lacunæ. Firstly, Slepian merely outlined his perturbation scheme for approximating eigenvalues $\chi_{N,n}$ and, unfortunately, fatal, unreported misprints crept into his final formulæ. Secondly, he gave no indication on how to adapt Bouwkamp's critical, but esoteric, scheme to refinement of the eigenvalue approximations in the case of the GPSFs. We intend to fill these lacunæ by clarifying and, where possible, extending the classical methods of generating accurate eigenvalues $\chi_{N,n}$. Furthermore, because computing the eigenvalues remains a delicate undertaking, we propose to create an extensive table of same, covering a suitably wide array of values of scaling constant c, order N and rank n. Given such a table of eigenvalues, the calculation of expansion coefficients for the eigenfunctions themselves should present no major difficulty.

This first report addresses Slepian's perturbation scheme for approximating Sturmian eigenvalues $\chi_{N,n}(c)$. Section 2 recapitulates some known properties of self-transforms $\phi_{N,n}(c,r)$ in the limits $c \to 0$ and $c \to \infty$, which subsequent developments depend upon. Section 3 derives the perturbation scheme in its original, general form, and describes its recursive character. Section 4 provides explicit, non-recursive expressions of the perturbation-series coefficients in the particular case relevant here, up to fifth order. Sections 5 and 6 further specialize the analysis, by performing the exact determination, in terms of N and n, of these coefficients for eigenvalues $\chi_{N,n}(c)$ when c can be considered large or small. The end results are fifth-order series expansions of $\chi_{N,n}$ in powers of $1/c$ or c^2, respectively, which constitute the centrepiece of the report. In Section 7 we veer from analytical methods to numerical ones, by transforming Slepian's recursive process into a *non*-recursive algorithm suitable for computing, given N and n, the values of successive coefficients of either series up to any order. Finally, Section 8 concludes the report by summarizing the results obtained here, pointing the way forward, and suggesting avenues of further research on this topic.

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2 Limit behaviour of eigenfunctions

In his diffraction theory of aberrations, F. Zernike [5] introduced a family of two-index polynomials, defined over the unit radial interval $0 \le r \le 1$ and usually denoted $R_q^p(r)$, which satisfy the following differential equation:

$$(1-r^2)\,\frac{d^2 R_q^p}{dr^2} + \left(\frac{1}{r} - 3r\right)\frac{dR_q^p}{dr} + \left[q(q+2) - \frac{p^2}{r^2}\right] R_q^p = 0. \tag{2–1}$$

By making $p = N$ and $q = N + 2n$ we find that $R_{N+2n}^N(r)$ obeys

$$(1-r^2)\,\frac{d^2 R_{N+2n}^N}{dr^2} + \left(\frac{1}{r} - 3r\right)\frac{dR_{N+2n}^N}{dr} + \left[\left(N+2n+\frac{1}{2}\right)\left(N+2n+\frac{3}{2}\right) - \frac{3}{4} - \frac{N^2}{r^2}\right] R_{N+2n}^N = 0. \tag{2–2}$$

Comparison with Eq. (1–12) leads to the conclusion that, in the limit where $c = 0$, self-transforms $\phi_{N,n}(c, r)$ become identical—up to some arbitrary normalization factor—with the Zernike polynomials:

$$\phi_{N,n}(0, r) \propto R_{N+2n}^N(r), \qquad \chi_{N,n}(0) = \left(N+2n+\frac{1}{2}\right)\left(N+2n+\frac{3}{2}\right). \tag{2–3}$$

Thus the behaviour of the Zernike polynomials is indicative of that of the FHSTs when c is small in comparison with the Sturmian eigenvalue. Among their more important properties in the present context we may point out that:

- They are expressible in the guise of the familiar Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$:

  $$R_{N+2n}^N(r) = r^N P_n^{(0,N)}(2r^2 - 1). \tag{2–4}$$

  This makes it easy to establish that

  $$\left.\frac{R_{N+2n}^N(r)}{r^N}\right|_{r=0} = (-1)^n \binom{N+n}{n}, \quad R_{N+2n}^N(0) = (-1)^n \delta_{N0}, \quad R_{N+2n}^N(1) = 1, \tag{2–5}$$

  and

  $$\left|R_{N+2n}^N(r)\right| \le 1, \qquad 0 \le r \le 1. \tag{2–6}$$

- They are orthogonal over the unit radial interval:

  $$\int_0^1 R_{N+2m}^N(r) R_{N+2n}^N(r) r\,dr = \frac{\delta_{mn}}{2(N+2n+1)}. \tag{2–7}$$

- In combination with the Fourier harmonics, they are complete over the unit circle within the class of functions square-integrable there.

- Their finite Hankel transform of order N generates Bessel functions:

  $$\int_0^1 R_{N+2n}^N(r) J_N(zr) r\,dr = (-1)^n \frac{J_{N+2n+1}(z)}{z}. \tag{2–8}$$

- They may readily be computed by means of the following three-term recurrence:

$$r^2 R_{N+2n}^N(r) = a_{N,n}^1 R_{N+2(n+1)}^N(r) + a_{N,n}^0 R_{N+2n}^N(r) + a_{N,n}^{-1} R_{N+2(n-1)}^N(r), \quad n = 0,1,2,\ldots, \tag{2–9}$$

  where

$$R_j^N(r) \stackrel{\text{def}}{\equiv} 0 \quad \text{if} \quad j < N, \qquad R_N^N(r) = r^N, \qquad R_{N+2}^N(r) = r^N \left[(N+2)\,r^2 - (N+1)\right], \tag{2–10}$$

  and where coefficients $a_{N,j}^m$ are defined as follows:[4]

$$a_{N,j}^1 = \begin{cases} \dfrac{(j+1)(N+j+1)}{(N+2j+1)(N+2j+2)} & \text{if } j \geq 0, \\ 0 & \text{if } j < 0; \end{cases} \tag{2–11}$$

$$a_{N,j}^0 = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{N^2}{(N+2j)(N+2j+2)}\right] & \text{if } j > 0, \\ \dfrac{N+1}{N+2} & \text{if } j = 0, \\ 0 & \text{if } j < 0; \end{cases} \tag{2–12}$$

$$a_{N,j}^{-1} = \begin{cases} \dfrac{j(N+j)}{(N+2j)(N+2j+1)} & \text{if } j \geq 1, \\ 0 & \text{if } j < 1. \end{cases} \tag{2–13}$$

It will later prove convenient that some of the above facts be expressed in the language of operators and Slepian's notation. Let

$$\mathbf{L} \stackrel{\text{def}}{\equiv} (1-r^2)\frac{d^2}{dr^2} - 2r\frac{d}{dr} + \frac{\frac{1}{4} - N^2}{r^2}, \qquad \mathbf{M} \stackrel{\text{def}}{\equiv} r^2, \tag{2–14}$$

in terms of which operators Eq. (1–10) may be written as

$$(\mathbf{L} - c^2\mathbf{M})\,\varphi_{N,n} + \chi_{N,n}\,\varphi_{N,n} = 0. \tag{2–15}$$

Also, following Slepian, let

$$T_{N,n}(r) \stackrel{\text{def}}{\equiv} r^{N+\frac{1}{2}}\,\mathscr{R}_{N,n}(r), \qquad 0 \leq r \leq 1, \tag{2–16}$$

where

$$\mathscr{R}_{N,n}(r) \stackrel{\text{def}}{\equiv} \binom{N+n}{n}^{-1} P_n^{(N,0)}(1-2r^2) = (-1)^n \binom{N+n}{n}^{-1} \frac{R_{N+2n}^N(r)}{r^N}, \tag{2–17}$$

such that

$$\mathscr{R}_{N,n}(0) = 1, \qquad \mathscr{R}_{N,n}(1) = (-1)^n \binom{N+n}{n}^{-1}, \qquad \left|\mathscr{R}_{N,n}(r)\right| \leq 1. \tag{2–18}$$

[4] Note that in $a_{N,j}^m$ superscript m is merely a label, not an exponent nor an index. The same convention applies to coefficients $\gamma_{N,j}^m$ and $\mu_{N,j}^m$ defined further down.

In view of result (2–3) we can state that every $T_{N,n}$ satisfies the simpler equation

$$\mathbf{L}T_{N,n} + \tau_{N,n}T_{N,n} = 0, \qquad \tau_{N,n} = \left(N+2n+\tfrac{1}{2}\right)\left(N+2n+\tfrac{3}{2}\right). \tag{2–19}$$

Furthermore, recurrence relation (2–9) is equivalent to

$$\begin{gathered}\mathbf{M}T_{N,n}(r) = \gamma^{1}_{N,n}T_{N,n+1}(r) + \gamma^{0}_{N,n}T_{N,n}(r) + \gamma^{-1}_{N,n}T_{N,n-1}(r),\\ n = 0,1,2,\ldots,\end{gathered} \tag{2–20}$$

where $T_{N,j}(r) \overset{\text{def}}{\equiv} 0$ if $j < 0$ and, by virtue of the relationship between $T_{N,n}$ and R^{N}_{N+2n} expressed in Eqs. (2–16) and (2–17),

$$\gamma^{1}_{N,j} = \begin{cases} -\dfrac{(N+j+1)^2}{(N+2j+1)(N+2j+2)} & \text{if } j \geq 0,\\ 0 & \text{if } j < 0;\end{cases} \tag{2–21}$$

$$\gamma^{0}_{N,j} = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{N^2}{(N+2j)(N+2j+2)}\right] & \text{if } j > 0,\\ \dfrac{N+1}{N+2} & \text{if } j = 0,\\ 0 & \text{if } j < 0;\end{cases} \tag{2–22}$$

$$\gamma^{-1}_{N,j} = \begin{cases} -\dfrac{j^2}{(N+2j)(N+2j+1)} & \text{if } j \geq 1,\\ 0 & \text{if } j < 1.\end{cases} \tag{2–23}$$

The properties of the Zernike polynomials make them an ideal basis for representing self-transforms $\phi_{N,n}$ when $c > 0$. Slepian indirectly acknowledged this by proposing the following expansion over the unit radial interval:

$$\varphi_{N,n}(c,r) = \sum_{j=0}^{\infty} d^{N,n}_{j}(c)\,T_{N,j}(r), \qquad 0 \leq r \leq 1. \tag{2–24}$$

Substitution into differential equation (1–10), plus appeal to properties (2–19) and (2–20) of polynomials $T_{N,n}$, leads to a three-term recurrence for the coefficients:

$$-c^2\gamma^{-1}_{N,j+1}d^{N,n}_{j+1} = \left[\chi_{N,j}(0) - \chi_{N,n}(c) + c^2\gamma^{0}_{N,j}\right]d^{N,n}_{j} + c^2\gamma^{1}_{N,j-1}d^{N,n}_{j-1}, \qquad j \geq 0, \tag{2–25}$$

where $d^{N,n}_{j} \overset{\text{def}}{\equiv} 0$ if $j < 0$. Representation (2–24) is, of course, entirely equivalent to

$$\phi_{N,n}(c,r) = \sum_{j=0}^{\infty}(-1)^j\binom{N+j}{j}^{-1}d^{N,n}_{j}(c)\,R^{N}_{N+2j}(r), \qquad 0 \leq r \leq 1. \tag{2–26}$$

This may easily be extended beyond $r = 1$ through defining integral equation (1–6) and property (2–8) of the Zernike polynomials:

$$\phi_{N,n}(c,r) = \beta^{-1}_{N,n}(c)\sum_{j=0}^{\infty}\binom{N+j}{j}^{-1}d^{N,n}_{j}(c)\,\frac{J_{N+2j+1}(cr)}{cr}, \qquad r \geq 0. \tag{2–27}$$

An expression for eigenvalues $\beta_{N,n}$ can be obtained by dividing both sides of the last equation by r^N and taking the limit as $r \to 0$:

$$\beta_{N,n}(c) = \frac{c^N}{2^{N+1}(N+1)!} \frac{d_0^{N,n}(c)}{\sum_{j=0}^{\infty} d_j^{N,n}(c)} \tag{2–28}$$

Thus, as noted in the Introduction, once accurate eigenvalues $\chi_{N,n}(c)$ are known, computation of eigenfunctions $\psi_{N,n}$ poses no difficulty of principle.

Having considered the limit $c \to 0$, we proceed to the other extreme—the limit $c \to \infty$. Following Slepian, we start by making the change of variable $t = \sqrt{c}\,r$. It is straightforward to show that this turns Eq. (1–10) into

$$[\mathbf{L} - (1/c)\mathbf{M}]\,\varphi_{N,n} + (\chi_{N,n}/c)\varphi_{N,n} = 0, \tag{2–29}$$

where

$$\mathbf{L} \overset{\text{def}}{\equiv} \frac{d^2}{dt^2} - t^2 + \frac{\frac{1}{4} - N^2}{t^2}, \qquad \mathbf{M} \overset{\text{def}}{\equiv} t^2 \frac{d^2}{dt^2} + 2t\frac{d}{dt}. \tag{2–30}$$

Let us consider functions

$$U_{N,n}(t) = \sqrt{t}\,S_{N,n}(t), \tag{2–31}$$

where $S_{N,n}$ denotes the *associated Gauss-Laguerre functions* (AGLFs):

$$S_{N,n}(t) \overset{\text{def}}{\equiv} e^{-t^2/2}\,t^N L_n^{(N)}(t^2), \qquad t \geq 0. \tag{2–32}$$

It is known that every $U_{N,n}$ satisfies

$$\mathbf{L}\,U_{N,n} + \sigma_{N,n} U_{N,n} = 0, \qquad \sigma_{N,n} = 2\,(N + 2n + 1). \tag{2–33}$$

(See, for instance, [6], p.781, #22.6.18.) It can also be shown that

$$\mathbf{M}\,U_{N,n}(t) = \mu^1_{N,n} U_{N,n+2}(t) + \mu^0_{N,n} U_{N,n}(t) + \mu^{-1}_{N,n} U_{N,n-2}(t), \qquad n = 0, 1, 2, \ldots, \tag{2–34}$$

where $U_{N,j}(t) \overset{\text{def}}{\equiv} 0$ if $j < 0$ and

$$\mu^1_{N,j} = (j+1)(j+2), \tag{2–35}$$

$$\mu^0_{N,j} = -\left[(2j+1)\left(N + j + \tfrac{1}{2}\right) + \tfrac{3}{4}\right], \tag{2–36}$$

$$\mu^{-1}_{N,j} = (N+j)(N+j-1). \tag{2–37}$$

Now, as $c \to \infty$ in Eq. (2–29), operator $(1/c)\mathbf{M}$ becomes negligible with respect to operator $\mathbf{L}$. Comparing with result (2–33), we are led to conclude that, in the limit $c \to \infty$, $\varphi_{N,n}(c, r)$ becomes identical—up to some arbitrary normalization factor—with $U_{N,n}(\sqrt{c}\,r)$. An immediate corollary is that self-transforms $\phi_{N,n}(c, r)$ tend toward the AGLFs:

$$\lim_{c\to\infty} \phi_{N,n}(c, r) \propto S_{N,n}(\sqrt{c}\,r), \qquad \lim_{c\to\infty} \chi_{N,n}(c)/c = 2\,(N + 2n + 1). \tag{2–38}$$

Thus the behaviour of the AGLFs is indicative of that of the FHSTs when c is large in comparison with the Sturmian eigenvalue. Among their most important properties in the present context we may point out that:

- They are orthogonal over the *infinite* radial interval:

$$\int_0^\infty S_{N,m}(r) S_{N,n}(r) r\, dr = \frac{(N+n)!}{2\,n!}\,\delta_{mn}. \tag{2–39}$$

- In combination with the Fourier harmonics, they are complete in E_2 within the class of square-integrable functions.

- They are Hankel self-transforms of order N:

$$\int_0^\infty S_{N,n}(r) J_N(zr) r\, dr = (-1)^n S_{N,n}(z). \tag{2–40}$$

This last property asumes great significance in laser theory because, viewed as an integral equation, it describes the amplitude distribution over the mirrors of a gas laser oscillating in its fundamental modes.

As we shall see in the next section, Eqs. (2–15), (2–19) and (2–20) on one hand, and Eqs. (2–29), (2–33) and (2–34) on the other, form the basis of Slepian's perturbation scheme for approximating eigenvalues $\chi_{N,n}(c)$ when c is small or large, respectively. Furthermore, Eq. (2–25) is the starting point of Bouwkamp's scheme for refining them.

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3 Perturbation scheme

We now derive the central equations of Slepian's perturbation scheme for approximating eigenvalues $\chi_{N,n}(c)$.

Consider some linear operator $\mathbf{L}$ whose eigenfunctions u_n and eigenvalues λ_n are known:

$$\mathbf{L}\,u_n + \lambda_n u_n = 0, \qquad n = 0, 1, 2, \ldots. \tag{3–1}$$

We seek solutions ψ_n, χ_n of the perturbed equation

$$(\mathbf{L} - \epsilon\,\mathbf{M})\,\psi_n + \chi_n \psi_n = 0. \tag{3–2}$$

Let us expand these unknowns in powers of perturbation parameter ϵ:

$$\psi_n = u_n + \sum_{j=1}^{\infty} \epsilon^j Q_j, \tag{3–3}$$

$$\chi_n = \lambda_n + \sum_{j=1}^{\infty} \epsilon^j a_j. \tag{3–4}$$

Here coefficients a_j and Q_j will depend on index n, and the Q_j will be functions of the same independent variable that the u_n depend upon. After introducing the above two expansions into Eq. (3–2), taking Eq. (3–1) into account, and grouping terms, we find

$$\sum_{j=1}^{\infty} \epsilon^j (\mathbf{L}\,Q_j + \lambda_n Q_j) - \epsilon\,\mathbf{M}\,u_n - \sum_{j=1}^{\infty} \epsilon^{j+1}\,\mathbf{M}\,Q_j \\ + \sum_{j=1}^{\infty} \epsilon^j a_j u_n + \sum_{j=1}^{\infty} \epsilon^j a_j \sum_{i=1}^{\infty} \epsilon^i Q_i = 0. \tag{3–5}$$

Setting $u_n \stackrel{\text{def}}{\equiv} Q_0$ allows us to write

$$-\epsilon\,\mathbf{M}\,u_n - \sum_{j=1}^{\infty} \epsilon^{j+1}\,\mathbf{M}\,Q_j = -\sum_{j=0}^{\infty} \epsilon^{j+1}\,\mathbf{M}\,Q_j = -\sum_{j=1}^{\infty} \epsilon^{j}\,\mathbf{M}\,Q_{j-1} \tag{3–6}$$

and

$$\sum_{j=1}^{\infty} \epsilon^j a_j u_n + \sum_{j=1}^{\infty} \epsilon^j a_j \sum_{i=1}^{\infty} \epsilon^i Q_i = \sum_{j=1}^{\infty} \epsilon^j a_j \sum_{i=0}^{\infty} \epsilon^i Q_i. \tag{3–7}$$

Furthermore,

$$\begin{aligned} \sum_{j=1}^{\infty} \epsilon^j a_j \sum_{i=0}^{\infty} \epsilon^i Q_i = \quad & \epsilon\,a_1(Q_0 + \epsilon\,Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \cdots) \\ & + \epsilon^2 a_2(Q_0 + \epsilon\,Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \cdots) \\ & + \epsilon^3 a_3(Q_0 + \epsilon\,Q_1 + \epsilon^2 Q_2 + \epsilon^3 Q_3 + \cdots) + \cdots, \\ = {} & \epsilon\,a_1 Q_0 + \epsilon^2(a_1 Q_1 + a_2 Q_0) + \epsilon^3(a_1 Q_2 + a_2 Q_1 + a_3 Q_0) + \cdots, \\ = {} & \sum_{j=1}^{\infty} \epsilon^j (a_1 Q_{j-1} + a_2 Q_{j-2} + \cdots + a_j Q_0) = \sum_{j=1}^{\infty} \epsilon^j \sum_{k=1}^{j} a_k Q_{j-k}. \end{aligned} \tag{3–8}$$

Thus Eq. (3–5) is equivalent to

$$\sum_{j=1}^{\infty} \epsilon^j \left(\mathbf{L} Q_j + \lambda_n Q_j - \mathbf{M} Q_{j-1} + \sum_{k=1}^{j} a_k Q_{j-k} \right) = 0, \tag{3–9}$$

and since this is true for any value of ϵ we must have $\big($[1], Eq. (119)$\big)$

$$\mathbf{L} Q_j + \lambda_n Q_j = \mathbf{M} Q_{j-1} - \sum_{k=1}^{j} a_k Q_{j-k}, \qquad j = 1, 2, 3, \ldots. \tag{3–10}$$

Let us now assume that application of perturbating operator $\mathbf{M}$ upon u_n results in a finite, linear combination of functions u_j with constant coefficients:

$$\mathbf{M} u_n = \sum_{k=-l}^{l} \gamma_n^k \, u_{n+\alpha k}, \qquad n = 0, 1, 2, \ldots. \tag{3–11}$$

Here l sets the *number* of terms on either side of u_n, while α controls the *index jump* between them. We shall be concerned with two instances of this: $l = \alpha = 1$, for which

$$\mathbf{M} u_n = \gamma_n^{-1} u_{n-1} + \gamma_n^0 u_n + \gamma_n^1 u_{n+1}, \tag{3–12}$$

and $l = 1$, $\alpha = 2$, for which

$$\mathbf{M} u_n = \gamma_n^{-1} u_{n-2} + \gamma_n^0 u_n + \gamma_n^1 u_{n+2}. \tag{3–13}$$

Note that in γ_n^k, k is just a label, not an exponent, while n is the same index as that of eigenfunction u_n.

Let us also assume that eigenfunctions u_n are linearly independent and complete in the space of solutions ψ_n of Eq. (3–2). We may then write, in a fashion similar to (3–11),

$$Q_j = \sum_{k=-jl}^{jl} A_k^j(n) \, u_{n+\alpha k}, \qquad j = 0, 1, 2, \ldots, \tag{3–14}$$

so that

$$\psi_n = \sum_{j=0}^{\infty} \epsilon^j Q_j = \sum_{j=0}^{\infty} \epsilon^j \sum_{k=-jl}^{jl} A_k^j \, u_{n+\alpha k}. \tag{3–15}$$

Note that in A_k^j, j is not an exponent but a label, corresponding via Q_j to the power of ϵ in expansion (3–3); k, on the other hand, plays a similar role as in γ_n^k of Eq. (3–11). For $l = \alpha = 1$ we shall have

$$\begin{aligned} \psi_n &= \sum_{j=0}^{\infty} \epsilon^j \sum_{k=-j}^{j} A_k^j \, u_{n+k}, \\ &= A_0^0 u_n + \epsilon \, (A_{-1}^1 u_{n-1} + A_0^1 u_n + A_1^1 u_{n+1}) \\ &\qquad + \epsilon^2 (A_{-2}^2 u_{n-2} + A_{-1}^2 u_{n-1} + A_0^2 u_n + A_1^2 u_{n+1} + A_2^2 u_{n+2}) \\ &\qquad + \epsilon^3 (A_{-3}^3 u_{n-3} + A_{-2}^3 u_{n-2} + A_{-1}^3 u_{n-1} + A_0^3 u_n \\ &\qquad\qquad + A_1^3 u_{n+1} + A_2^3 u_{n+2} + A_3^3 u_{n+3}) + \cdots. \end{aligned} \tag{3–16}$$

Regrouping function-wise, we find

$$\begin{aligned}
\psi_n = \quad & u_n \quad (A_0^0 + A_0^1\epsilon + A_0^2\epsilon^2 + A_0^3\epsilon^3 + \cdots) \\
& + u_{n-1}(A_{-1}^1\epsilon + A_{-1}^2\epsilon^2 + A_{-1}^3\epsilon^3 + \cdots) \\
& + u_{n-2}(A_{-2}^2\epsilon^2 + A_{-2}^3\epsilon^3 + \cdots) \\
& + u_{n-3}(A_{-3}^3\epsilon^3 + \cdots) + \cdots \\
& + u_0 \quad (A_{-n}^n\epsilon^n + A_{-n}^{n+1}\epsilon^{n+1} + \cdots) \\
& + u_{n+1}(A_1^1\epsilon + A_1^2\epsilon^2 + A_1^3\epsilon^3 + \cdots) \\
& + u_{n+2}(A_2^2\epsilon^2 + A_2^3\epsilon^3 + \cdots) \\
& + u_{n+3}(A_3^3\epsilon^3 + \cdots) + \cdots = \sum_{j=-n}^{\infty} u_{n+j} \sum_{k=|j|}^{\infty} A_j^k\,\epsilon^k.
\end{aligned} \tag{3–17}$$

The case $l = 1$, $\alpha = 2$ can be shown, through the same process, to yield

$$\psi_n = \sum_{j=-s}^{\infty} u_{n+2j} \sum_{k=|j|}^{\infty} A_j^k\,\epsilon^k, \quad \text{where} \quad s = \begin{cases} n/2 & \text{for } n = 0, 2, 4, \ldots, \\ (n-1)/2 & \text{for } n = 1, 3, 5, \ldots. \end{cases} \tag{3–18}$$

These expressions clearly establish that representation (3–14) of the functions Q_j which appear in series (3–3) leads to infinite expansions of ψ_n in terms of the complete u_j. In one case the expansion is full, while in the other, half the u_j are missing—those with odd index if n is even, those with even index if n is odd. The discrepancy reflects the difference in the assumed effect of perturbating operator $\mathbf{M}$, as expressed in Eqs. (3–12) and (3–13). An instance of expansion (3–17) would be Eq. (2–24), where $d_{n+j}^{N,n} = \sum_{k=|j|}^{\infty} A_j^k(N,n)\,c^{2k}$, $j = -n, -n+1, \ldots$. Likewise, an instance of (3–18) would be an expansion of $\varphi_{N,n}(c,r)$ in terms of functions $U_{N,j}(\sqrt{c}\,r)$.

Note that, in order for expression (3–14) to agree with our definition of Q_0 as being u_n, we must set $A_0^0 = 1$ and $A_{\pm1}^0 = A_{\pm2}^0 = \cdots = 0$. Neither are contributions in u_n needed from other functions Q_j, therefore we set $A_0^1 = A_0^2 = \cdots = 0$. Altogether, then, coefficients A_k^j must obey the following rule:

$$A_k^0 = A_0^k = \delta_{k0}. \tag{3–19}$$

Furthermore, the limits of the sum in Eq. (3–14) require that

$$A_k^j = 0 \quad \text{for} \quad |\,k\,| > j\,l, \tag{3–20}$$

and in order for no function u_j therein to bear a negative index it is necessary that

$$A_k^j = 0 \quad \text{for} \quad \alpha k < -n. \tag{3–21}$$

We now return to Eq. (3–10). Substituting for Q_j from Eq. (3–14), we obtain

$$\begin{aligned}
&\sum_{m=-jl}^{jl} A_m^j(\mathbf{L}\,u_{n+\alpha m} + \lambda_n u_{n+\alpha m}) = \\
&\qquad \sum_{m=-(j-1)l}^{(j-1)l} A_m^{j-1}\mathbf{M}\,u_{n+\alpha m} - \sum_{k=1}^{j} a_k \sum_{m=-(j-k)l}^{(j-k)l} A_m^{j-k}\,u_{n+\alpha m}, \quad j = 1, 2, \ldots.
\end{aligned} \tag{3–22}$$

But, on account of Eq. (3–1),

$$\mathbf{L}\, u_{n+\alpha m} = -\lambda_{n+\alpha m} u_{n+\alpha m}, \tag{3–23}$$

and, in view of Eq. (3–11),

$$\mathbf{M}\, u_{n+\alpha m} = \sum_{k=-l}^{l} \gamma^k_{n+\alpha m}\, u_{n+\alpha(m+k)}. \tag{3–24}$$

Hence Eq. (3–22) becomes

$$\begin{aligned} &\sum_{m=-jl}^{jl} A^j_m (\lambda_n - \lambda_{n+\alpha m})\, u_{n+\alpha m} = \\ &\quad \sum_{k=-l}^{l} \sum_{m=-(j-1)l}^{(j-1)l} A^{j-1}_m \gamma^k_{n+\alpha m}\, u_{n+\alpha(m+k)} - \sum_{k=1}^{j} \sum_{m=-(j-k)l}^{(j-k)l} a_k\, A^{j-k}_m\, u_{n+\alpha m}, \\ &\qquad j = 1, 2, \ldots. \end{aligned} \tag{3–25}$$

Consider the inner sum of the second term on the right-hand side. Its limits may be taken beyond $\pm(j-k)l$ since, by rule (3–20), any A^{j-k}_m will be zero if $|m| > (j-k)l$. Now, because $l \in (0, 1, 2, \ldots)$ and $k \in (1, 2, \ldots, j)$, where $j \in (1, 2, 3, \ldots)$, kl cannot be negative, whence $jl \geq jl - kl = (j-k)l$ and the sum's limits may be taken, instead, as $\pm jl$. Thus, said second term is equivalent to

$$-\sum_{k=1}^{j} \sum_{m=-jl}^{jl} a_k\, A^{j-k}_m\, u_{n+\alpha m} = -\sum_{m=-jl}^{jl} \sum_{k=1}^{j} a_k\, A^{j-k}_m\, u_{n+\alpha m}. \tag{3–26}$$

Now consider the inner sum of the first term. Its limits may be taken beyond $\pm(j-1)l$ since, by rule (3–20) again, any A^{j-1}_m will be zero if $|m| > (j-1)l$. As the outer sum imposes $k \leq l$, it follows that $-k \geq -l$, which in turn implies that $jl - k \geq jl - l = (j-1)l$. The outer sum also sets $k \geq -l$, or $-k \leq l$, so that $-jl-k \leq -jl+l = -(j-1)l$. Altogether, then, the sum may be extended to run from $m = -jl - k$ up to $m = jl - k$, and said first term is found to be equivalent to

$$\begin{aligned} \sum_{k=-l}^{l} \sum_{m=-jl-k}^{jl-k} A^{j-1}_m \gamma^k_{n+\alpha m}\, u_{n+\alpha(m+k)} &= \sum_{k=-l}^{l} \sum_{m=-jl}^{jl} A^{j-1}_{m-k} \gamma^k_{n+\alpha(m-k)}\, u_{n+\alpha m}, \\ &= \sum_{m=-jl}^{jl} \sum_{k=-l}^{l} A^{j-1}_{m-k} \gamma^k_{n+\alpha(m-k)}\, u_{n+\alpha m}. \end{aligned} \tag{3–27}$$

In view of the foregoing, Eq. (3–25) reduces to

$$\begin{aligned} \sum_{m=-jl}^{jl} \left[A^j_m (\lambda_{n+\alpha m} - \lambda_n) - \sum_{k=1}^{j} a_k\, A^{j-k}_m + \sum_{k=-l}^{l} A^{j-1}_{m-k} \gamma^k_{n+\alpha(m-k)} \right] u_{n+\alpha m} = 0, \\ j = 1, 2, \ldots. \end{aligned} \tag{3–28}$$

Now, since functions u_j have been assumed linearly independent, the above can be true only if the coefficient of $u_{n+\alpha m}$ is zero for every m. It follows that $\big([1]$, Eq. $(123)\big)$

$$(\lambda_{n+\alpha m} - \lambda_n)\, A_m^j = \sum_{k=1}^{j} a_k\, A_m^{j-k} - \sum_{k=-l}^{l} A_{m-k}^{j-1}\, \gamma_{n+\alpha(m-k)}^k, \qquad m = -jl, -jl+1, \ldots, jl-1, jl, \qquad j = 1, 2, \ldots. \tag{3–29}$$

In particular, for $m = 0$ this reduces, by virtue of rule (3–19), to $\big([1]$, Eq. $(122)\big)$

$$a_j = \sum_{k=-l}^{l} A_{-k}^{j-1}\, \gamma_{n-\alpha k}^k, \qquad j = 1, 2, \ldots. \tag{3–30}$$

Eqs. (3–29) and (3–30) form the central apparatus of Slepian's eigenvalue approximation process. Specifically, formula (3–30) allows successive determination of coefficients a_j of expansion (3–4) from knowledge of the γs that define $\mathbf{M}\, u_n$ via Eq. (3–11), and from prior evaluation of the necessary quantities A_{-k}^{j-1}. When these fulfil none of special conditions (3–19)–(3–21) they are calculated by means of formula (3–29). On the face of it the latter seems to assume foreknowledge of the coefficients, but closer inspection reveals that calculation of A_{-k}^{j-1}, for purposes of determining a_j, involves coefficients a_1, a_2, …, a_{j-1} only—which numbers are presumably available. However, starting with $j = 3$, some of the quantities A_q^p on which A_{-k}^{j-1}depends via formula (3–29) are *not* special cases. Hence they, themselves, require application of formula (3–29), thus making the latter recursive.

Application of Slepian's perturbation scheme comes from recognition of Eqs. (2–19), (2–15) and (2–20) on one hand, and Eqs. (2–33), (2–29) and (2–34) on the other, as instances of Eqs. (3–1), (3–2) and (3–11)—with $\epsilon = c^2$ (small c) and $\epsilon = 1/c$ (large c), respectively; with ψ_n replaced by $\varphi_{N,n}$; and with the following additional correspondences:

Small c:

$$u_n \to T_{N,n}, \qquad \chi_n \to \chi_{N,n}, \qquad \lambda_n \to \big(N + 2n + \tfrac{1}{2}\big)\big(N + 2n + \tfrac{3}{2}\big), \qquad l = 1, \qquad \alpha = 1, \qquad \gamma_n^k \to \gamma_{N,n}^k. \tag{3–31}$$

Large c:

$$u_n \to U_{N,n}, \qquad \chi_n \to \chi_{N,n}/c, \qquad \lambda_n \to 2\,(N + 2n + 1), \qquad l = 1, \qquad \alpha = 2, \qquad \gamma_n^k \to \mu_{N,n}^k. \tag{3–32}$$

Note that the $\gamma_{N,n}^k$ and the $\mu_{N,n}^k$ are given by Eqs. (2–21)–(2–23) and Eqs. (2–35)–(2–37), respectively. In the next three sections we shall specialize the scheme to those instances. First we shall avail ourselves of the fact that $l = 1$ in both cases to convert formulæ (3–29) and (3–30) into explicit, non-recursive expressions of coefficients a_j for $j = 0$, 1, …, 5. Next we shall substitute therein the actual definitions of λ_n and γ_n^k that obtain when c is large, then when it is small.[5] This process will result in fifth-order, analytical series expansions of $\chi_{N,n}(c)$ in powers of $1/c$ or c^2, fit for approximating the eigenvalues.

[5] We start with the case of large c because it is rather less complicated than that of small c.

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4 Explicit expressions of perturbation coefficients

Slepian's power series (3–4) for eigenvalue approximation may be recast as

$$\chi_n = \sum_{j=0}^{\infty} a_j \epsilon^j, \tag{4–1}$$

where

$$a_0 = \lambda_n \tag{4–2}$$

and subsequent coefficients are calculated by means of formula (3–30). The quantities A_{-k}^{j-1} which appear therein are gotten from recursive formula (3–29)—insofar as they fulfil none of conditions (3–19)–(3–21). Our task now consists of producing, by use of these formulæ, explicit, *non*-recursive expressions for coefficients a_1, a_2, …, in the particular case $l = 1$ which we are concerned with.

For $l = 1$ formula (3–30) becomes

$$\begin{aligned} a_j &= \sum_{k=-1}^{1} A_{-k}^{j-1} \gamma_{n-\alpha k}^{k}, \\ &= A_1^{j-1}\gamma_{n+\alpha}^{-1} + A_0^{j-1}\gamma_n^0 + A_{-1}^{j-1}\gamma_{n-\alpha}^1, \qquad j = 1, 2, \ldots. \end{aligned} \tag{4–3}$$

In view of the above, more specific expression for a_j, it is instructive to revisit forthwith condition (3–19), which, for the sake of convenience, we repeat below together with its sister conditions (3–20) and (3–21):

$$A_k^0 = A_0^k = \delta_{k0}, \tag{4–4a}$$

$$A_k^j = 0 \quad \text{for} \quad |k| > j, \tag{4–4b}$$

$$A_k^j = 0 \quad \text{for} \quad n < -\alpha k. \tag{4–4c}$$

We have $A_1^0 = A_{-1}^0 = 0$, while $A_0^0 = 1$. Thus Eq. (4–3) yields

$$a_1 = A_1^0 \gamma_{n+\alpha}^{-1} + A_0^0 \gamma_n^0 + A_{-1}^0 \gamma_{n-\alpha}^1 = \gamma_n^0. \tag{4–5}$$

Conversely, we have $A_0^{j-1} = 0$ for all $j \in (2, 3, \ldots)$. Hence, for purposes of determining subsequent coefficients a_2, a_3, …, Eq. (4–3) reduces to

$$a_j = A_1^{j-1}\gamma_{n+\alpha}^{-1} + A_{-1}^{j-1}\gamma_{n-\alpha}^1, \qquad j = 2, 3, \ldots. \tag{4–6}$$

If a_j is being evaluated for χ_0, then $n = 0$ and, by rule (4–4c), A_{-1}^{j-1} is zero (recall that α is either 1 or 2). Otherwise, both A_1^{j-1} and A_{-1}^{j-1} must be calculated through formula (3–29), which, for $l = 1$, becomes

$$\begin{gathered} (\lambda_{n+\alpha m} - \lambda_n)\, A_m^j = \sum_{k=1}^{j} a_k A_m^{j-k} - \sum_{k=-1}^{1} A_{m-k}^{j-1} \gamma_{n+\alpha(m-k)}^{k}, \\ m = -j, -j+1, \ldots, j-1, j, \qquad j = 1, 2, \ldots, \end{gathered} \tag{4–7}$$

or, more explicitly,

$$A_m^j = h_{\alpha m}\Big[a_1 A_m^{j-1} + a_2 A_m^{j-2} + \cdots + a_j A_m^0 \\ - A_{m+1}^{j-1}\gamma_{n+\alpha(m+1)}^{-1} - A_m^{j-1}\gamma_{n+\alpha m}^0 - A_{m-1}^{j-1}\gamma_{n+\alpha(m-1)}^1\Big], \qquad (4\text{–}8)$$
$$m = -j, -j+1, \ldots, j-1, j, \qquad j = 1, 2, \ldots .$$

Here, following Slepian, we have introduced the handy factor

$$h_j \stackrel{\text{def}}{\equiv} (\lambda_{n+j} - \lambda_n)^{-1}. \qquad (4\text{–}9)$$

We shall proceed to apply Eqs. (4–6) and (4–8)—which, together with conditions (4–4), constitute our working formulæ—to derive the desired expressions for a_2, a_3, a_4 and a_5.

When $j = 2$, Eq. (4–6) gives

$$a_2 = A_1^1\gamma_{n+\alpha}^{-1} + A_{-1}^1\gamma_{n-\alpha}^1. \qquad (4\text{–}10)$$

On one hand, using Eq. (4–8) and applying rule (4–4a), we obtain

$$A_1^1 = h_\alpha\big(a_1A_1^0 - A_2^0\gamma_{n+2\alpha}^{-1} - A_1^0\gamma_{n+\alpha}^0 - A_0^0\gamma_n^1\big) = -h_\alpha\gamma_n^1. \qquad (4\text{–}11)$$

On the other hand, by rule (4–4c), $A_{-1}^1 = 0$ if $n < \alpha$; otherwise, using Eq. (4–8) we find that

$$A_{-1}^1 = h_{-\alpha}\big(a_1A_{-1}^0 - A_0^0\gamma_n^{-1} - A_{-1}^0\gamma_{n-\alpha}^0 - A_{-2}^0\gamma_{n-2\alpha}^1\big) = -h_{-\alpha}\gamma_n^{-1}. \qquad (4\text{–}12)$$

Altogether, then, we have

$$a_2 = -\,h_\alpha\gamma_n^1\gamma_{n+\alpha}^{-1} - \phi_\alpha h_{-\alpha}\gamma_{n-\alpha}^1\gamma_n^{-1}, \qquad (4\text{–}13)$$

where ϕ_m has the role of an *inclusion switch*: in any expression a term that bears ϕ_m is to be included if and only if $n \geq m$.

When $j = 3$, Eq. (4–6) gives

$$a_3 = A_1^2\gamma_{n+\alpha}^{-1} + A_{-1}^2\gamma_{n-\alpha}^1. \qquad (4\text{–}14)$$

By using Eq. (4–8) together with rules (4–4a) and (4–4b) we find that

$$A_1^2 = h_\alpha\big(a_1A_1^1 + a_2A_1^0 - A_2^1\gamma_{n+2\alpha}^{-1} - A_1^1\gamma_{n+\alpha}^0 - A_0^1\gamma_n^1\big) = h_\alpha A_1^1\big(a_1 - \gamma_{n+\alpha}^0\big). \qquad (4\text{–}15)$$

The recursive character of formula (4–8) [or (3–29)] now becomes apparent. Of course, we can avoid recursion because Eq. (4–11) provides A_1^1. However, in automatic calculations matters would not be so simple: the order in which formula (4–8) is applied would matter a great deal—an issue we shall revisit in Section 7. As for a_1, we refer back to Eq. (4–5). Thus,

$$A_1^2 = -\big(h_\alpha\big)^2\gamma_n^1\big(\gamma_n^0 - \gamma_{n+\alpha}^0\big). \qquad (4\text{–}16)$$

Turning to A^2_{-1} now, by rule (4–4c) this is zero if $n < \alpha$. Otherwise Eq. (4–8) gives

$$\begin{aligned} A^2_{-1} &= h_{-\alpha}\big(a_1 A^1_{-1} + a_2 A^0_{-1} - A^1_0 \gamma^{-1}_n - A^1_{-1}\gamma^0_{n-\alpha} - A^1_{-2}\gamma^1_{n-2\alpha}\big), \\ &= h_{-\alpha} A^1_{-1}\big(a_1 - \gamma^0_{n-\alpha}\big). \end{aligned} \tag{4–17}$$

We make use of result (4–5) again, and we bring in expression (4–12) for A^1_{-1}, to find

$$A^2_{-1} = -\,\phi_\alpha \big(h_{-\alpha}\big)^2 \gamma^{-1}_n \big(\gamma^0_n - \gamma^0_{n-\alpha}\big). \tag{4–18}$$

Finally, substituting for A^2_1 and A^2_{-1} into Eq. (4–14), we obtain

$$a_3 = -\,\big(h_\alpha\big)^2 \gamma^1_n \gamma^{-1}_{n+\alpha}\big(\gamma^0_n - \gamma^0_{n+\alpha}\big) - \phi_\alpha \big(h_{-\alpha}\big)^2 \gamma^1_{n-\alpha}\gamma^{-1}_n \big(\gamma^0_n - \gamma^0_{n-\alpha}\big). \tag{4–19}$$

When $j = 4$, Eq. (4–6) gives

$$a_4 = A^3_1 \gamma^{-1}_{n+\alpha} + A^3_{-1}\gamma^1_{n-\alpha}. \tag{4–20}$$

Using Eq. (4–8) we find successively that

$$\begin{aligned} A^3_1 &= h_\alpha\big(a_1 A^2_1 + a_2 A^1_1 + a_3 A^0_1 - A^2_2 \gamma^{-1}_{n+2\alpha} - A^2_1 \gamma^0_{n+\alpha} - A^2_0 \gamma^1_n\big), \\ &= h_\alpha\big[A^2_1\big(a_1 - \gamma^0_{n+\alpha}\big) + A^1_1 a_2 - A^2_2 \gamma^{-1}_{n+2\alpha}\big], \end{aligned} \tag{4–21}$$

and

$$\begin{aligned} A^2_2 &= h_{2\alpha}\big(a_1 A^1_2 + a_2 A^0_2 - A^1_3 \gamma^{-1}_{n+3\alpha} - A^1_2 \gamma^0_{n+2\alpha} - A^1_1 \gamma^1_{n+\alpha}\big), \\ &= -h_{2\alpha} A^1_1 \gamma^1_{n+\alpha} = h_\alpha h_{2\alpha} \gamma^1_n \gamma^1_{n+\alpha}. \end{aligned} \tag{4–22}$$

Substituting for A^2_2 and, from Eq. (4–15), for A^2_1 into Eq. (4–21), we obtain

$$\begin{aligned} A^3_1 &= h_\alpha\big[h_\alpha A^1_1 \big(a_1 - \gamma^0_{n+\alpha}\big)^2 + A^1_1 a_2 + h_{2\alpha} A^1_1 \gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big], \\ &= h_\alpha A^1_1\big[a_2 + h_\alpha\big(a_1 - \gamma^0_{n+\alpha}\big)^2 + h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big]. \end{aligned} \tag{4–23}$$

Finally, in view of results (4–5) and (4–11), we get

$$A^3_1 = -\big(h_\alpha\big)^2 \gamma^1_n \big[a_2 + h_\alpha\big(\gamma^0_n - \gamma^0_{n+\alpha}\big)^2 + h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big]. \tag{4–24}$$

Turning to A^3_{-1} now, we follow the same route, paying attention, however, to instances of n that will cause A^j_{-m} to vanish due to rule (4–4c). In fact, $A^3_{-1} = 0$ if $n < \alpha$; otherwise Eq. (4–8) yields

$$\begin{aligned} A^3_{-1} &= h_{-\alpha}\big(a_1 A^2_{-1} + a_2 A^1_{-1} + a_3 A^0_{-1} - A^2_0 \gamma^{-1}_n - A^2_{-1}\gamma^0_{n-\alpha} - A^2_{-2}\gamma^1_{n-2\alpha}\big), \\ &= h_{-\alpha}\big[A^2_{-1}\big(a_1 - \gamma^0_{n-\alpha}\big) + A^1_{-1} a_2 - A^2_{-2}\gamma^1_{n-2\alpha}\big]. \end{aligned} \tag{4–25}$$

Similarly, $A^2_{-2} = 0$ if $n < 2\alpha$; otherwise

$$\begin{aligned} A^2_{-2} &= h_{-2\alpha}\big(a_1 A^1_{-2} + a_2 A^0_{-2} - A^1_{-1}\gamma^{-1}_{n-\alpha} - A^1_{-2}\gamma^0_{n-2\alpha} - A^1_{-3}\gamma^1_{n-3\alpha}\big), \\ &= -h_{-2\alpha} A^1_{-1}\gamma^{-1}_{n-\alpha} = h_{-\alpha} h_{-2\alpha}\gamma^{-1}_n \gamma^{-1}_{n-\alpha}. \end{aligned} \tag{4–26}$$

Substituting for A^2_{-2} and, from Eq. (4–17), for A^2_{-1} into Eq. (4–25), and introducing the required switches ϕ_m, we obtain

$$
\begin{aligned}
A^3_{-1} &= \phi_\alpha h_{-\alpha}\big[h_{-\alpha}A^1_{-1}(a_1-\gamma^0_{n-\alpha})^2 + A^1_{-1}a_2 + \phi_{2\alpha}h_{-2\alpha}A^1_{-1}\gamma^1_{n-2\alpha}\gamma^{-1}_{n-\alpha}\big],\\
&= \phi_\alpha h_{-\alpha}A^1_{-1}\big[a_2 + h_{-\alpha}(a_1-\gamma^0_{n-\alpha})^2 + \phi_{2\alpha}h_{-2\alpha}\gamma^1_{n-2\alpha}\gamma^{-1}_{n-\alpha}\big]. \qquad (4–27)
\end{aligned}
$$

Finally, in view of results (4–5) and (4–12), we get

$$
A^3_{-1} = -\phi_\alpha(h_{-\alpha})^2\gamma^{-1}_n\big[a_2 + h_{-\alpha}(\gamma^0_n-\gamma^0_{n-\alpha})^2 + \phi_{2\alpha}h_{-2\alpha}\gamma^1_{n-2\alpha}\gamma^{-1}_{n-\alpha}\big]. \qquad (4–28)
$$

We may now substitute for A^3_1 and A^3_{-1} into Eq. (4–20) to arrive at the desired expression for a_4:

$$
\begin{aligned}
a_4 = &-(h_\alpha)^2\gamma^1_n\gamma^{-1}_{n+\alpha}\big[a_2 + h_\alpha(\gamma^0_n-\gamma^0_{n+\alpha})^2 + h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big]\\
&-\phi_\alpha(h_{-\alpha})^2\gamma^1_{n-\alpha}\gamma^{-1}_n\big[a_2 + h_{-\alpha}(\gamma^0_n-\gamma^0_{n-\alpha})^2 + \phi_{2\alpha}h_{-2\alpha}\gamma^1_{n-2\alpha}\gamma^{-1}_{n-\alpha}\big]. \qquad (4–29)
\end{aligned}
$$

When $j = 5$, Eq. (4–6) gives

$$
a_5 = A^4_1\gamma^{-1}_{n+\alpha} + A^4_{-1}\gamma^1_{n-\alpha}. \qquad (4–30)
$$

Using Eq. (4–8) we find successively that

$$
\begin{aligned}
A^4_1 &= h_\alpha\big(a_1A^3_1 + a_2A^2_1 + a_3A^1_1 + a_4A^0_1 - A^3_2\gamma^{-1}_{n+2\alpha} - A^3_1\gamma^0_{n+\alpha} - A^3_0\gamma^1_n\big),\\
&= h_\alpha\big[A^3_1(a_1-\gamma^0_{n+\alpha}) + A^2_1a_2 + A^1_1a_3 - A^3_2\gamma^{-1}_{n+2\alpha}\big], \qquad (4–31)
\end{aligned}
$$

and that

$$
\begin{aligned}
A^3_2 &= h_{2\alpha}\big(a_1A^2_2 + a_2A^1_2 + a_3A^0_2 - A^2_3\gamma^{-1}_{n+3\alpha} - A^2_2\gamma^0_{n+2\alpha} - A^2_1\gamma^1_{n+\alpha}\big),\\
&= h_{2\alpha}\big[A^2_2(a_1-\gamma^0_{n+2\alpha}) - A^2_1\gamma^1_{n+\alpha}\big]. \qquad (4–32)
\end{aligned}
$$

Substituting for A^3_2 and, from Eq. (4–21), for A^3_1 into Eq. (4–31), we obtain

$$
\begin{aligned}
A^4_1 = h_\alpha\big\{&h_\alpha\big[A^2_1(a_1-\gamma^0_{n+\alpha}) + A^1_1a_2 - A^2_2\gamma^{-1}_{n+2\alpha}\big](a_1-\gamma^0_{n+\alpha})\\
&+ A^2_1a_2 + A^1_1a_3 - h_{2\alpha}\big[A^2_2(a_1-\gamma^0_{n+2\alpha}) - A^2_1\gamma^1_{n+\alpha}\big]\gamma^{-1}_{n+2\alpha}\big\},\\
= h_\alpha\big\{&A^1_1\big[h_\alpha a_2(a_1-\gamma^0_{n+\alpha}) + a_3\big]\\
&+ A^2_1\big[h_\alpha(a_1-\gamma^0_{n+\alpha})^2 + a_2 + h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big]\\
&- A^2_2\gamma^{-1}_{n+2\alpha}\big[h_\alpha(a_1-\gamma^0_{n+\alpha}) + h_{2\alpha}(a_1-\gamma^0_{n+2\alpha})\big]\big\}. \qquad (4–33)
\end{aligned}
$$

Substituting into this from Eqs. (4–15) and (4–22) for A^2_1 and A^2_2, respectively, we obtain

$$
\begin{aligned}
A^4_1 = h_\alpha\big\{&A^1_1\big[h_\alpha a_2(a_1-\gamma^0_{n+\alpha}) + a_3\big]\\
&+ A^1_1h_\alpha(a_1-\gamma^0_{n+\alpha})\big[h_\alpha(a_1-\gamma^0_{n+\alpha})^2 + a_2 + h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big]\\
&+ A^1_1h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big[h_\alpha(a_1-\gamma^0_{n+\alpha}) + h_{2\alpha}(a_1-\gamma^0_{n+2\alpha})\big]\big\},\\
= h_\alpha A^1_1\big\{&a_3 + h_\alpha(a_1-\gamma^0_{n+\alpha})\big[2a_2 + h_\alpha(a_1-\gamma^0_{n+\alpha})^2\big]\\
&+ h_{2\alpha}\gamma^1_{n+\alpha}\gamma^{-1}_{n+2\alpha}\big[2h_\alpha(a_1-\gamma^0_{n+\alpha}) + h_{2\alpha}(a_1-\gamma^0_{n+2\alpha})\big]\big\}. \qquad (4–34)
\end{aligned}
$$

Finally, in view of results (4–5) and (4–11), we get

$$A_1^4 = -\,(h_\alpha)^2\gamma_n^1\{a_3 + h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0)[2a_2 + h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0)^2] + h_{2\alpha}\gamma_{n+\alpha}^1\gamma_{n+2\alpha}^{-1}[2h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0) + h_{2\alpha}(\gamma_n^0 - \gamma_{n+2\alpha}^0)]\}. \qquad (4\text{–}35)$$

Turning to A_{-1}^4 now, we proceed along the same lines, with due regard for cases of A_{-m}^j vanishing on account of rule (4–4c). In fact, $A_{-1}^4 = 0$ if $n < \alpha$; otherwise Eq. (4–8) yields

$$\begin{aligned} A_{-1}^4 &= h_{-\alpha}(a_1A_{-1}^3 + a_2A_{-1}^2 + a_3A_{-1}^1 + a_4A_{-1}^0 - A_0^3\gamma_n^{-1} - A_{-1}^3\gamma_{n-\alpha}^0 - A_{-2}^3\gamma_{n-2\alpha}^1), \\ &= h_{-\alpha}[A_{-1}^3(a_1 - \gamma_{n-\alpha}^0) + A_{-1}^2a_2 + A_{-1}^1a_3 - A_{-2}^3\gamma_{n-2\alpha}^1]. \qquad (4\text{–}36) \end{aligned}$$

Similarly, $A_{-2}^3 = 0$ if $n < 2\alpha$; otherwise

$$\begin{aligned} A_{-2}^3 &= h_{-2\alpha}(a_1A_{-2}^2 + a_2A_{-2}^1 + a_3A_{-2}^0 - A_{-1}^2\gamma_{n-\alpha}^{-1} - A_{-2}^2\gamma_{n-2\alpha}^0 - A_{-3}^2\gamma_{n-3\alpha}^1), \\ &= h_{-2\alpha}[A_{-2}^2(a_1 - \gamma_{n-2\alpha}^0) - A_{-1}^2\gamma_{n-\alpha}^{-1}]. \qquad (4\text{–}37) \end{aligned}$$

Substituting for A_{-2}^3 and, from Eq. (4–25), for A_{-1}^3 into Eq. (4–36), we obtain

$$\begin{aligned} A_{-1}^4 &= \phi_\alpha h_{-\alpha}\{h_{-\alpha}[A_{-1}^2(a_1 - \gamma_{n-\alpha}^0) + A_{-1}^1a_2 - \phi_{2\alpha}A_{-2}^2\gamma_{n-2\alpha}^1](a_1 - \gamma_{n-\alpha}^0) \\ &\qquad + A_{-1}^2a_2 + A_{-1}^1a_3 - \phi_{2\alpha}h_{-2\alpha}[A_{-2}^2(a_1 - \gamma_{n-2\alpha}^0) - A_{-1}^2\gamma_{n-\alpha}^{-1}]\gamma_{n-2\alpha}^1\}, \\ &= \phi_\alpha h_{-\alpha}\{A_{-1}^1[h_{-\alpha}a_2(a_1 - \gamma_{n-\alpha}^0) + a_3] \\ &\qquad + A_{-1}^2[h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0)^2 + a_2 + \phi_{2\alpha}h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}] \\ &\qquad - \phi_{2\alpha}A_{-2}^2\gamma_{n-2\alpha}^1[h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0) + h_{-2\alpha}(a_1 - \gamma_{n-2\alpha}^0)]\}. \qquad (4\text{–}38) \end{aligned}$$

Substituting into this from Eqs. (4–17) and (4–26) for A_{-1}^2 and A_{-2}^2, respectively, we obtain

$$\begin{aligned} A_{-1}^4 &= \phi_\alpha h_{-\alpha}\{A_{-1}^1[h_{-\alpha}a_2(a_1 - \gamma_{n-\alpha}^0) + a_3] \\ &\qquad + A_{-1}^1h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0)[h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0)^2 + a_2 + \phi_{2\alpha}h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}] \\ &\qquad + \phi_{2\alpha}A_{-1}^1h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}[h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0) + h_{-2\alpha}(a_1 - \gamma_{n-2\alpha}^0)]\}, \\ &= \phi_\alpha h_{-\alpha}A_{-1}^1\{a_3 + h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0)[2a_2 + h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0)^2] \\ &\qquad + \phi_{2\alpha}h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}[2h_{-\alpha}(a_1 - \gamma_{n-\alpha}^0) + h_{-2\alpha}(a_1 - \gamma_{n-2\alpha}^0)]\}. \qquad (4\text{–}39) \end{aligned}$$

Finally, in view of results (4–5) and (4–12), we get

$$A_{-1}^4 = -\,\phi_\alpha(h_{-\alpha})^2\gamma_n^{-1}\{a_3 + h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0)[2a_2 + h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0)^2] + \phi_{2\alpha}h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}[2h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0) + h_{-2\alpha}(\gamma_n^0 - \gamma_{n-2\alpha}^0)]\}. \qquad (4\text{–}40)$$

We may now substitute for A_1^4 and A_{-1}^4 into Eq. (4–30) to arrive at the desired expression for a_5:

$$\begin{aligned} a_5 &= -\,(h_\alpha)^2\gamma_n^1\gamma_{n+\alpha}^{-1}\{a_3 + h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0)[2a_2 + h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0)^2] \\ &\qquad + h_{2\alpha}\gamma_{n+\alpha}^1\gamma_{n+2\alpha}^{-1}[2h_\alpha(\gamma_n^0 - \gamma_{n+\alpha}^0) + h_{2\alpha}(\gamma_n^0 - \gamma_{n+2\alpha}^0)]\} \\ &\quad - \phi_\alpha(h_{-\alpha})^2\gamma_{n-\alpha}^1\gamma_n^{-1}\{a_3 + h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0)[2a_2 + h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0)^2] \\ &\qquad + \phi_{2\alpha}h_{-2\alpha}\gamma_{n-2\alpha}^1\gamma_{n-\alpha}^{-1}[2h_{-\alpha}(\gamma_n^0 - \gamma_{n-\alpha}^0) + h_{-2\alpha}(\gamma_n^0 - \gamma_{n-2\alpha}^0)]\}. \qquad (4\text{–}41) \end{aligned}$$

The process of deriving $a_2, \ldots, a_5$, being relatively straightforward, could no doubt be extended to further coefficients. However, it may have occurred to the reader that, as the order j of the coefficient a_j being determined increases by one unit, the length of the final expression for a_j more or less doubles. Thus, for a_6 one can expect eight lines such as the four on the right-hand side of Eq. (4–41), for a_7 sixteen, and so on; eventually the algebra is bound to become unmanageable. Of course, the number of coefficients a_j required to approximate $\chi_{N,n}$ depends on the accuracy sought for the approximation, as well as on the convergence of the approximating series itself, Eq. (4–1). Thus, before committing to the substantial effort of generating expressions for one or, possibly, two more coefficients, it would be prudent to investigate the behaviour of the approximating series up to order 5—that is, with the six coefficients $a_0, \ldots, a_5$ now available. We shall look into this matter at the end of the next two sections. If more coefficients turn out to be needed, it will be easy to generate them numerically by means of the algorithm described in Section 7—which, essentially, performs in the right sequence the operations prescribed by Eqs. (4–6), (4–4) and (4–8). Nor should we lose sight of our goal, namely, formulæ that will provide mere *approximations* of $\chi_{N,n}$: one will be able to refine these to any degree of accuracy by means of Bouwkamp's scheme.

Comparison between this section and Slepian's paper reveals some misprints in the latter, which we correct below:

[1], Eq. (59):

$$\mu^k_{N,n+2}(m-k) \quad \rightarrow \quad \mu^k_{N,n+2(m-k)} \qquad \text{[see our Eq. (4–7)]}$$

[1], p. 3024, line before last:

$$k < -n \quad \rightarrow \quad 2k < -n \qquad \text{[see our Eq. (3–21)]}$$

[1], p. 3053:

$$a_3 = (h_\alpha)^2 \gamma^1_n \gamma^{-1}_{n+\alpha}\left(-\gamma^0_n + \gamma^0_{n+1}\right) + \ldots$$

$$\rightarrow \quad a_3 = (h_\alpha)^2 \gamma^1_n \gamma^{-1}_{n+\alpha}\left(-\gamma^0_n + \gamma^0_{\underset{\uparrow}{n+\alpha}}\right) + \ldots \qquad \text{[see our Eq. (4–19)]}$$

$$A^3_{-1} = h_{-\alpha}\left[A^2_{-\alpha}\left(\gamma^0_n - \gamma^0_{n-\alpha}\right) + a_2 A^2_{-1} - \ldots\right.$$

$$\rightarrow \quad A^3_{-1} = h_{-\alpha}\left[A^2_{\underset{\uparrow}{-1}}\left(\gamma^0_n - \gamma^0_{n-\alpha}\right) + a_2 A^{\overset{}{1}}_{\underset{\uparrow}{-1}} - \ldots\right. \qquad \text{[see our Eq. (4–25)]}$$

5 Fifth-order approximation of eigenvalues for large c

We now specialize the general expressions obtained in the preceding section for $a_0, \ldots, a_5$ to the case $\alpha = 2$. As per Eqs. (3–32) and (4–1), $\chi_{N,n}(c)$ is then approximated through the following sum of inverse powers of c:

$$\chi_{N,n}(c) = a_0 c + a_1 + a_2/c + a_3/c^2 + a_4/c^3 + a_5/c^4 + O(c^{-5}). \tag{5–1}$$

In accordance with Eq. (4–2) we have

$$a_0 = \lambda_n = 2N + 4n + 2 = 2z, \tag{5–2}$$

where

$$z \stackrel{\text{def}}{\equiv} N + 2n + 1. \tag{5–3}$$

This is one of two "canonical variables" we shall be using, the other being

$$x \stackrel{\text{def}}{\equiv} n(N + n). \tag{5–4}$$

Clearly $\lambda_{n+j} = \lambda_n + 4j$, therefore, in accordance with Eq. (4–9),

$$\left(h_j\right)^{-1} \stackrel{\text{def}}{\equiv} \lambda_{n+j} - \lambda_n = 4j. \tag{5–5}$$

Thus

$$h_2 = 1/8, \qquad h_{-2} = -1/8, \qquad h_4 = 1/16, \qquad h_{-4} = -1/16. \tag{5–6}$$

For this case we also have, from Eqs. (2–35)–(2–37),

$$\gamma_j^1 = (j+1)(j+2), \tag{5–7}$$

$$\gamma_j^0 = -\left[(2j+1)\left(N + j + \tfrac{1}{2}\right) + \tfrac{3}{4}\right] = -\left[2j\,(N + j + 1) + N + \tfrac{5}{4}\right], \tag{5–8}$$

and

$$\gamma_j^{-1} = (N + j)(N + j - 1). \tag{5–9}$$

Finally, the general expressions for coefficients $a_1 \ldots a_5$—Eqs. (4–5), (4–13), (4–19), (4–29) and (4–41)—appear as the following ones in terms of the above quantities h_j and γ_j^k:

$$a_1 = \gamma_n^0, \tag{5–10}$$

$$a_2 = -h_2 \gamma_n^1 \gamma_{n+2}^{-1} - \phi_2 h_{-2} \gamma_{n-2}^1 \gamma_n^{-1}, \tag{5–11}$$

$$a_3 = -\left(h_2\right)^2 \gamma_n^1 \gamma_{n+2}^{-1} \left(\gamma_n^0 - \gamma_{n+2}^0\right) - \phi_2 \left(h_{-2}\right)^2 \gamma_{n-2}^1 \gamma_n^{-1} \left(\gamma_n^0 - \gamma_{n-2}^0\right), \tag{5–12}$$

$$\begin{aligned} a_4 = &-\left(h_2\right)^2 \gamma_n^1 \gamma_{n+2}^{-1} \left[a_2 + h_2\left(\gamma_n^0 - \gamma_{n+2}^0\right)^2 + h_4 \gamma_{n+2}^1 \gamma_{n+4}^{-1}\right] \\ &- \phi_2 \left(h_{-2}\right)^2 \gamma_{n-2}^1 \gamma_n^{-1} \left[a_2 + h_{-2}\left(\gamma_n^0 - \gamma_{n-2}^0\right)^2 + \phi_4 h_{-4} \gamma_{n-4}^1 \gamma_{n-2}^{-1}\right], \end{aligned} \tag{5–13}$$

$$\begin{aligned} a_5 = &-\left(h_2\right)^2 \gamma_n^1 \gamma_{n+2}^{-1} \Big\{a_3 + h_2\left(\gamma_n^0 - \gamma_{n+2}^0\right)\left[2a_2 + h_2\left(\gamma_n^0 - \gamma_{n+2}^0\right)^2\right] \\ &\qquad + h_4 \gamma_{n+2}^1 \gamma_{n+4}^{-1}\left[2h_2\left(\gamma_n^0 - \gamma_{n+2}^0\right) + h_4\left(\gamma_n^0 - \gamma_{n+4}^0\right)\right]\Big\} \\ &- \phi_2\left(h_{-2}\right)^2 \gamma_{n-2}^1 \gamma_n^{-1} \Big\{a_3 + h_{-2}\left(\gamma_n^0 - \gamma_{n-2}^0\right)\left[2a_2 + h_{-2}\left(\gamma_n^0 - \gamma_{n-2}^0\right)^2\right] \\ &\qquad + \phi_4 h_{-4} \gamma_{n-4}^1 \gamma_{n-2}^{-1}\left[2h_{-2}\left(\gamma_n^0 - \gamma_{n-2}^0\right) + h_{-4}\left(\gamma_n^0 - \gamma_{n-4}^0\right)\right]\Big\}. \end{aligned} \tag{5–14}$$

We recall the role of ϕ_m as an *inclusion switch*: a term bearing ϕ_m is to be included if and only if $n \geq m$, where n is the rank of the eigenvalue being approximated.

We start by developing convenient expressions for the combinations of quantities γ_j^k that occur above. First we note that

$$2n(N+n+1)+N+1 \;=\; N+2n+1+2n(N+n) \;=\; z+2x. \tag{5–15}$$

As a corollary, from Eqs. (5–10) and (5–8) we find that

$$-4a_1 = 4(z+2x)+1. \tag{5–16}$$

We also have from Eq. (5–8) that

$$\gamma_n^0 - \gamma_{n+j}^0 \;=\; -\,2n(N+n+1)+2(n+j)(N+n+j+1) \;=\; 2j(z+j). \tag{5–17}$$

Thus

$$\left.\begin{aligned} \gamma_n^0 - \gamma_{n+2}^0 &= 4(z+2), & \gamma_n^0 - \gamma_{n+4}^0 &= 8(z+4),\\ \gamma_n^0 - \gamma_{n-2}^0 &= -4(z-2), & \gamma_n^0 - \gamma_{n-4}^0 &= -8(z-4). \end{aligned}\right\} \tag{5–18}$$

Furthermore, from Eqs. (5–7) and (5–9) we find that

$$\gamma_{n+j}^1\gamma_{n+j+2}^{-1} \;=\; (n+j+1)(N+n+j+1)(n+j+2)(N+n+j+2). \tag{5–19}$$

But

$$\begin{aligned}(n+j)(N+n+j) &= n(N+n)+j(N+2n)+j^2,\\ &= n(N+n)+j(N+2n+1)-j+j^2,\\ &= x+j(j-1)+jz.\end{aligned} \tag{5–20}$$

Thus

$$\gamma_{n+j}^1\gamma_{n+j+2}^{-1} \;=\; [x+j(j+1)+(j+1)z][x+(j+1)(j+2)+(j+2)z]. \tag{5–21}$$

In particular,

$$\begin{aligned}\gamma_{n-2}^1\gamma_n^{-1} &= (n-1)(N+n-1)n(N+n),\\ &= x(x+2-z) \stackrel{\text{def}}{\equiv} \zeta,\end{aligned} \tag{5–22a}$$

$$\begin{aligned}\gamma_n^1\gamma_{n+2}^{-1} &= (n+1)(N+n+1)(n+2)(N+n+2),\\ &= (x+z)(x+2+2z) \stackrel{\text{def}}{\equiv} \eta,\end{aligned} \tag{5–22b}$$

$$\begin{aligned}\gamma_{n-4}^1\gamma_{n-2}^{-1} &= (n-3)(N+n-3)(n-2)(N+n-2),\\ &= (x+6-2z)(x+12-3z) \stackrel{\text{def}}{\equiv} \mu,\end{aligned} \tag{5–22c}$$

$$\begin{aligned}\gamma_{n+2}^1\gamma_{n+4}^{-1} &= (n+3)(N+n+3)(n+4)(N+n+4),\\ &= (x+6+3z)(x+12+4z) \stackrel{\text{def}}{\equiv} \nu.\end{aligned} \tag{5–22d}$$

Finally, for later purposes we define

$$P(z,x) \stackrel{\text{def}}{\equiv} z(z+2x+1) \tag{5–23}$$

and

$$Q(z,x) \stackrel{\text{def}}{\equiv} z(z+x+1)+x(x+2), \tag{5–24}$$

such that

$$\eta - \zeta = 2P, \tag{5–25}$$

$$\eta + \zeta = 2Q, \tag{5–26}$$

$$\nu - \mu \,=\, 6z(z+2x+17) \,=\, 6(P+16z) \,=\, 3(\eta-\zeta)+96z, \tag{5–27}$$

and

$$\begin{aligned} \nu + \mu &= 2\,[z(9z+x+9)+(x+6)(x+12)], \\ &= 2\,[Q+8(z^2+z+2x+9)], \\ &= \eta+\zeta+16(z^2+z+2x+9). \end{aligned} \tag{5–28}$$

We may now substitute from Eqs. (5–6), (5–18) and (5–22) into expressions (5–11)–(5–14) for coefficients a_2, a_3, a_4 and a_5. First we must obtain a_2 and a_3, as they occur in the expressions for a_4 and a_5. We note that ϕ_2 may safely be ignored in both instances, since the presence of factors n and $n-1$ in $\gamma^1_{n-2}\gamma^{-1}_n$ [see Eq. (5–22a)] guarantees that the associated term will vanish when $n=0$ or $n=1$. Thus

$$a_2 \,=\, -\,(1/8)\eta + (1/8)\zeta, \tag{5–29}$$

or

$$-8\,a_2 \,=\, \eta - \zeta, \tag{5–30}$$

that is, in view of result (5–25),

$$-4\,a_2 \,=\, P \,=\, z(z+2x+1). \tag{5–31}$$

Next,

$$a_3 \,=\, -\,(1/64)\eta\cdot 4(z+2)+(1/64)\zeta\cdot 4(z-2), \tag{5–32}$$

or,

$$-16\,a_3 \,=\, z(\eta-\zeta)+2\,(\eta+\zeta), \tag{5–33}$$

that is,

$$-8\,a_3 \,=\, zP+2Q \,=\, z^2(z+2x+1)+2\,[z(z+x+1)+x(x+2)]. \tag{5–34}$$

Armed with expressions (5–30) and (5–33) for a_2 and a_3, we can now tackle a_4 and a_5. Again we note that ϕ_2 and ϕ_4 may be disregarded, since $\gamma^1_{n-2}\gamma^{-1}_n = 0$ for $n=0$ and $n=1$, while $\gamma^1_{n-4}\gamma^{-1}_{n-2} = 0$ for $n=2$ and $n=3$. Thus, from Eq. (5–13) we obtain

$$\begin{aligned} a_4 \,=\, &-\,(1/64)\,\eta\,\left[a_2+(1/8)\cdot 16(z+2)^2+(1/16)\nu\right] \\ &-\,(1/64)\,\zeta\,\left[a_2-(1/8)\cdot 16(z-2)^2-(1/16)\mu\right], \end{aligned} \tag{5–35}$$

or,

$$-64\,a_4 = 2\,(z^2+4)(\eta-\zeta) + (a_2+8z)(\eta+\zeta) + (1/16)(\eta\nu-\zeta\mu). \qquad (5\text{–}36)$$

But we may write

$$(\eta\nu-\zeta\mu) = (1/2)(\nu+\mu)(\eta-\zeta) + (1/2)(\nu-\mu)(\eta+\zeta), \qquad (5\text{–}37)$$

so that, multiplying Eq. (5–36) through by 32, we find

$$-2048\,a_4 = A(\eta-\zeta) + B(\eta+\zeta), \qquad (5\text{–}38)$$

where, in view of Eqs. (5–27), (5–28) and (5–30),

$$A \stackrel{\text{def}}{\equiv} 64\,(z^2+4) + (\nu+\mu) = (\eta+\zeta) + 16\,(5z^2+z+2x+25) \qquad (5\text{–}39)$$

and

$$B \stackrel{\text{def}}{\equiv} 32\,a_2 + 256\,z + (\nu-\mu) = -\,(\eta-\zeta) + 352\,z. \qquad (5\text{–}40)$$

Substituting for A and B back into Eq. (5–38), and taking advantage of the cancellation of the two terms in $(\eta+\zeta)(\eta-\zeta)$, we are led to the desired result:

$$-128\,a_4 = (5z^2+z+2x+25)(\eta-\zeta) + 22\,z(\eta+\zeta), \qquad (5\text{–}41)$$

that is,

$$\begin{aligned} -64\,a_4 &= (5z^2+z+2x+25)P + 22\,zQ, \\ &= z\left\{(z+2x+1)(5z^2+z+2x+25) + 22\,[z(z+x+1)+x(x+2)]\right\}. \end{aligned} \qquad (5\text{–}42)$$

From Eq. (5–14), finally, we obtain

$$\begin{aligned} a_5 &= -\,(1/64)\,\eta\left\{a_3 + (1/8)\cdot 4(z+2)\left[2\,a_2 + (1/8)\cdot 16(z+2)^2\right]\right. \\ &\qquad\qquad \left. +\,(1/16)\,\nu\left[(1/4)\cdot 4(z+2) + (1/16)\cdot 8(z+4)\right]\right\} \\ &\quad -\,(1/64)\,\zeta\left\{a_3 + (1/8)\cdot 4(z-2)\left[2\,a_2 - (1/8)\cdot 16(z-2)^2\right]\right. \\ &\qquad\qquad \left. -\,(1/16)\,\mu\left[(1/4)\cdot 4(z-2) + (1/16)\cdot 8(z-4)\right]\right\}, \\ &= -\,(1/64)\,\eta\left\{a_3 + (z+2)\left[a_2 + (z+2)^2\right] + (1/32)\,\nu\,(3z+8)\right\} \\ &\quad -\,(1/64)\,\zeta\left\{a_3 + (z-2)\left[a_2 - (z-2)^2\right] - (1/32)\,\mu\,(3z-8)\right\}, \end{aligned} \qquad (5\text{–}43)$$

or,

$$\begin{aligned} -64\,a_5 = {} & \left[2\,a_2 + z(z^2+12)\right](\eta-\zeta) + \left[a_3 + z a_2 + 2\,(3z^2+4)\right](\eta+\zeta) \\ & + (3z/32)(\eta\nu-\zeta\mu) + (1/4)(\eta\nu+\zeta\mu). \end{aligned} \qquad (5\text{–}44)$$

Using once again expression (5–37) for $(\eta\nu-\zeta\mu)$, together with

$$(\eta\nu+\zeta\mu) = (1/2)(\nu-\mu)(\eta-\zeta) + (1/2)(\nu+\mu)(\eta+\zeta), \qquad (5\text{–}45)$$

and multiplying Eq. (5–44) through by 64, we find

$$-4096\,a_5 = C(\eta-\zeta) + D(\eta+\zeta), \tag{5–46}$$

where, in view of Eqs. (5–25), (5–27), (5–28), (5–30) and (5–33),

$$\begin{aligned} C &\overset{\text{def}}{\equiv} 128\,a_2 + 64z(z^2+12) + 3z(\nu+\mu) + 8(\nu-\mu), \\ &= -16(\eta-\zeta) + 64z(z^2+12) + 3z(\eta+\zeta) \\ &\qquad + 48z(z^2+z+2x+9) + 24(\eta-\zeta) + 768z, \\ &= 3z(\eta+\zeta) + 8(\eta-\zeta) + 16z(7z^2+3z+6x+123), \\ &= 3z(\eta+\zeta) + 16z\big[7z^2+4(z+2x+31)\big], \end{aligned} \tag{5–47}$$

and

$$\begin{aligned} D &\overset{\text{def}}{\equiv} 64\,a_3 + 64z\,a_2 + 128(3z^2+4) + 3z(\nu-\mu) + 8(\nu+\mu), \\ &= -4z(\eta-\zeta) - 8(\eta+\zeta) - 8z(\eta-\zeta) + 128(3z^2+4) \\ &\qquad + 8(\eta+\zeta) + 9z(\eta-\zeta) + 128(z^2+z+2x+9) + 288z^2, \\ &= -3z(\eta-\zeta) + 32\big[25z^2+4(z+2x+13)\big]. \end{aligned} \tag{5–48}$$

Substituting for C and D back into Eq. (5–46), and taking advantage of the cancellation of the two terms in $z(\eta+\zeta)(\eta-\zeta)$, we are led to the desired result:

$$-256\,a_5 = z\big[7z^2+4(z+2x+31)\big](\eta-\zeta) + 2\big[25z^2+4(z+2x+13)\big](\eta+\zeta), \tag{5–49}$$

that is,

$$\begin{aligned} -128\,a_5 &= z\big[7z^2+4(z+2x+31)\big]P + 2\big[25z^2+4(z+2x+13)\big]Q, \\ &= z^2(z+2x+1)\big[7z^2+4(z+2x+31)\big] \\ &\qquad + 2\big[z(z+x+1)+x(x+2)\big]\big[25z^2+4(z+2x+13)\big]. \end{aligned} \tag{5–50}$$

For the sake of convenience we gather below our formulæ so far, in the form best suited to computations:

$$\left.\begin{aligned} a_0 &= 2z, \\ a_1 &= -(1/4)\big[4(z+2x)+1\big], \\ a_2 &= -(1/4)P, \\ a_3 &= -(1/8)(zP+2Q), \\ a_4 &= -(1/64)\big[(5z^2+z+2x+25)P+22\,zQ\big], \\ a_5 &= -(1/128)\big\{z\big[7z^2+4(z+2x+31)\big]P + 2\big[25z^2+4(z+2x+13)\big]Q\big\}. \end{aligned}\right\} \tag{5–51}$$

Here P and Q are as defined in Eqs. (5–23) and (5–24). We note that a_1 and a_2 are functions of $z+2x$ and $z+2x+1$. This suggests that, more generally, a_{m+1} might be expressible in terms of the form $(z+2x+j)$, with $(z+2x+m)$ the lead term and secondary terms arising beyond a threshold value of m. Starting from Eq. (5–34) it is straightforward to express a_3 as such a function of $(z+2x+2)$:

$$\begin{aligned} -8\,a_3 &= 2\,[z(z+x+1)+x(x+2)] + z^2(z+2x+1), \\ &= z(z+z+2x+2) + z^2(z+2x+2-1) + 2x(x+2), \\ &= z(z+1)(z+2x+2) + 2x(x+2). \end{aligned} \tag{5–52}$$

We have obtained similar representations of a_4 and a_5, under the twin requirements that 1) every term in $(z+2x+j)$ otherwise depend only on z—as found for $a_1, \ldots, a_3$—and 2) no negative quantity nor subtraction occur. Omitting all details we display our best results below, together with explicit expressions of a_0, a_1, a_2 and a_3 in order to show the evolution of the suggested representation from one coefficient to the next:

$$\left.\begin{aligned} a_0 &= 2z, \\ -4\,a_1 &= 4(z+2x)+1, \\ -4\,a_2 &= z(z+2x+1), \\ -8\,a_3 &= z(z+1)(z+2x+2) + 2x(x+2), \\ -64\,a_4 &= z\big[(z+1)(5z+7)(z+2x+3) + (z+4)(z+2x+1) \\ &\qquad\qquad + 22x(x+3) + 4x(x+2)\big], \\ -128\,a_5 &= z(z+1)\big[(z+2)(7z+10)(z+2x+4) + 2(z+6)(z+2x+2)\big] \\ &\qquad + 66z^2x(x+4) + 24zx(x+3) + 8(2x+13)x(x+2). \end{aligned}\right\} \tag{5–53}$$

The above leads to the speculation that any further coefficient a_{m+1} might be written as a combination of terms $(z+2x+m)$, $(z+2x+m-2)$, $\ldots$, $(z+2x+u)$, where $u=2$ if m is even and $u=1$ if m is odd, joined with terms $x(x+m)$, $x(x+m-1)$, $\ldots$, $x(x+2)$. However, the existence of such a representation must eventually become a moot point, as it is bound to grow lengthier than the original expression in terms of P and Q: compare, for instance, a_5 as given above and in Eq. (5–50).

Slepian has provided explicit expressions in terms of N and n for a_0, a_1 and a_2 $\big($[1], Eq. (60)$\big)$:

$$\left.\begin{aligned} a_0 &= \quad 4n+2N+2\,, \\ a_1 &= -\,\big[(2n+1)\big(n+N+\tfrac{1}{2}\big)+\tfrac{3}{4}\big], \\ a_2 &= -\,\tfrac{1}{4}(N+2n+1)\big[2n^2+2n(N+1)+N+2\big]. \end{aligned}\right\} \tag{5–54}$$

These results are easily reconciled with our own formulæ [see Eqs. (5–2); (5–10) and (5–8); (5–31) and (5–15)]. Heurtley, on the other hand, provided expressions for the same three coefficients plus a_3, with n substituted for our N and m substituted for our $N+2n$ $\big($[2], Eqs. (24) and (26a)–(28b)$\big)$:

$$\left.\begin{aligned}
a_0 &= 2(m+1),\\
a_1 &= -\tfrac{1}{2}\big[(m-n)(m+n+2)+2n+\tfrac{5}{2}\big],\\
a_2 &= \tfrac{1}{128}\big[\quad (m-n-2)(m-n)(m+n-2)(m+n)\\
&\qquad -(m-n+2)(m-n+4)(m+n+2)(m+n+4)\big],\\
a_3 &= \tfrac{1}{256}\big[\quad (m-1)(m-n-2)(m-n)(m+n-2)(m+n)\\
&\qquad -(m+3)(m-n+2)(m-n+4)(m+n+2)(m+n+4)\big].
\end{aligned}\right\} \tag{5–55}$$

Conversion to our notation shows these expressions to be equivalent to ours [see Eqs. (5–2); (5–10) and (5–8); (5–29), (5–22a) and (5–22b); (5–32), (5–22a) and (5–22b)]. Our expressions (5–42) for a_4 and (5–50) for a_5, however, appear to be entirely new; indeed, no one seems to have calculated these two coefficients in analytical form before.

To provide some idea of how, in this case of $\alpha = 2$, coefficients $a_j(N, n)$ change with N and n, we list below several concrete instances of Eq. (5–1):

$$\chi_{0,0}(c) = 2c - \tfrac{5}{4} - \tfrac{1}{2}c^{-1} - \tfrac{3}{4}c^{-2} - \tfrac{53}{32}c^{-3} - \tfrac{297}{64}c^{-4} + O(c^{-5}). \tag{5–56}$$

$$\chi_{0,1}(c) = 6c - \tfrac{21}{4} - \tfrac{9}{2}c^{-1} - \tfrac{45}{4}c^{-2} - \tfrac{1{,}269}{32}c^{-3} - \tfrac{10{,}935}{64}c^{-4} + O(c^{-5}). \tag{5–57}$$

$$\chi_{0,2}(c) = 10c - \tfrac{53}{4} - \tfrac{35}{2}c^{-1} - \tfrac{249}{4}c^{-2} - \tfrac{9{,}775}{32}c^{-3} - \tfrac{115{,}371}{64}c^{-4} + O(c^{-5}). \tag{5–58}$$

$$\chi_{0,3}(c) = 14c - \tfrac{101}{4} - \tfrac{91}{2}c^{-1} - \tfrac{855}{4}c^{-2} - \tfrac{43{,}631}{32}c^{-3} - \tfrac{661{,}365}{64}c^{-4} + O(c^{-5}). \tag{5–59}$$

$$\chi_{0,4}(c) = 18c - \tfrac{165}{4} - \tfrac{189}{2}c^{-1} - \tfrac{2{,}223}{4}c^{-2} - \tfrac{140{,}697}{32}c^{-3} - \tfrac{2{,}624{,}157}{64}c^{-4} + O(c^{-5}). \tag{5–60}$$

$$\chi_{0,5}(c) = 22c - \tfrac{245}{4} - \tfrac{341}{2}c^{-1} - \tfrac{4{,}833}{4}c^{-2} - \tfrac{366{,}553}{32}c^{-3} - \tfrac{8{,}150{,}787}{64}c^{-4} + O(c^{-5}). \tag{5–61}$$

$$\chi_{0,6}(c) = 26c - \tfrac{341}{4} - \tfrac{559}{2}c^{-1} - \tfrac{9{,}285}{4}c^{-2} - \tfrac{822{,}419}{32}c^{-3} - \tfrac{21{,}285{,}855}{64}c^{-4} + O(c^{-5}). \tag{5–62}$$

$$\chi_{0,7}(c) = 30c - \tfrac{453}{4} - \tfrac{855}{2}c^{-1} - \tfrac{16{,}299}{4}c^{-2} - \tfrac{1{,}653{,}075}{32}c^{-3} - \tfrac{48{,}878{,}721}{64}c^{-4} + O(c^{-5}). \tag{5–63}$$

$$\chi_{1,0}(c) = 4c - \tfrac{9}{4} - \tfrac{3}{2}c^{-1} - 3c^{-2} - \tfrac{273}{32}c^{-3} - 30c^{-4} + O(c^{-5}). \tag{5–64}$$

$$\chi_{1,1}(c) = 8c - \tfrac{33}{4} - 9c^{-1} - 27c^{-2} - \tfrac{1{,}809}{16}c^{-3} - \tfrac{2{,}295}{4}c^{-4} + O(c^{-5}). \tag{5–65}$$

$$\chi_{1,2}(c) = 12c - \tfrac{73}{4} - \tfrac{57}{2}c^{-1} - 117c^{-2} - \tfrac{21{,}027}{32}c^{-3} - 4{,}410c^{-4} + O(c^{-5}). \tag{5–66}$$

$$\chi_{1,3}(c) = 16c - \tfrac{129}{4} - 66c^{-1} - 348c^{-2} - \tfrac{19{,}833}{8}c^{-3} - 20{,}895c^{-4} + O(c^{-5}). \tag{5–67}$$

$$\chi_{1,4}(c) = 20c - \tfrac{201}{4} - \tfrac{255}{2}c^{-1} - 825c^{-2} - \tfrac{229{,}125}{32}c^{-3} - 73{,}050c^{-4} + O(c^{-5}). \tag{5–68}$$

$$\chi_{1,5}(c) = 24c - \tfrac{289}{4} - 219c^{-1} - 1{,}683c^{-2} - \tfrac{276{,}339}{16}c^{-3} - \tfrac{829{,}935}{4}c^{-4} + O(c^{-5}). \tag{5–69}$$

$$\chi_{1,6}(c) = 28c - \tfrac{393}{4} - \tfrac{693}{2}c^{-1} - 3{,}087c^{-2} - \tfrac{1{,}171{,}863}{32}c^{-3} - 507{,}150c^{-4} + O(c^{-5}). \tag{5–70}$$

$$\chi_{1,7}(c) = 32c - \tfrac{513}{4} - 516c^{-1} - 5{,}232c^{-2} - \tfrac{282{,}009}{4}c^{-3} - 1{,}106{,}940c^{-4} + O(c^{-5}). \tag{5–71}$$

$$\chi_{2,0}(c) = 6c - \tfrac{13}{4} - 3c^{-1} - \tfrac{15}{2}c^{-2} - \tfrac{417}{16}c^{-3} - \tfrac{3{,}525}{32}c^{-4} + O(c^{-5}). \tag{5–72}$$

$$\chi_{2,1}(c) = 10c - \tfrac{45}{4} - 15c^{-1} - \tfrac{105}{2}c^{-2} - \tfrac{4{,}065}{16}c^{-3} - \tfrac{47{,}355}{32}c^{-4} + O(c^{-5}). \tag{5–73}$$

$$\chi_{2,2}(c) = 14c - \tfrac{93}{4} - 42c^{-1} - 195c^{-2} - \tfrac{9{,}849}{8}c^{-3} - \tfrac{147{,}885}{16}c^{-4} + O(c^{-5}). \tag{5–74}$$

$$\chi_{2,3}(c) = 18c - \tfrac{157}{4} - 90c^{-1} - 525c^{-2} - \tfrac{32{,}985}{8}c^{-3} - \tfrac{610{,}995}{16}c^{-4} + O(c^{-5}). \tag{5–75}$$

$$\chi_{3,0}(c) = 8c - \tfrac{17}{4} - 5c^{-1} - 15c^{-2} - \tfrac{985}{16}c^{-3} - \tfrac{1{,}215}{4}c^{-4} + O(c^{-5}). \tag{5–76}$$

$$\chi_{3,1}(c) = 12c - \tfrac{57}{4} - \tfrac{45}{2}c^{-1} - 90c^{-2} - \tfrac{15{,}795}{32}c^{-3} - 3{,}240c^{-4} + O(c^{-5}). \tag{5–77}$$

$$\chi_{3,2}(c) = 16c - \tfrac{113}{4} - 58c^{-1} - 300c^{-2} - \tfrac{16{,}801}{8}c^{-3} - 17{,}415c^{-4} + O(c^{-5}). \tag{5–78}$$

$$\chi_{3,3}(c) = 20c - \tfrac{185}{4} - \tfrac{235}{2}c^{-1} - 750c^{-2} - \tfrac{205{,}685}{32}c^{-3} - 64{,}800c^{-4} + O(c^{-5}). \tag{5–79}$$

$$\chi_{4,4}(c) = 26c - \tfrac{309}{4} - \tfrac{507}{2}c^{-1} - \tfrac{8{,}277}{4}c^{-2} - \tfrac{721{,}227}{32}c^{-3} - \tfrac{18{,}374{,}775}{64}c^{-4} + O(c^{-5}). \tag{5–80}$$

$$\chi_{5,5}(c) = 32c - \tfrac{465}{4} - 468c^{-1} - 4{,}662c^{-2} - \tfrac{247{,}041}{4}c^{-3} - \tfrac{1{,}907{,}451}{2}c^{-4} + O(c^{-5}). \tag{5–81}$$

$$\chi_{6,6}(c) = 38c - \tfrac{653}{4} - 779c^{-1} - \tfrac{18{,}339}{2}c^{-2} - \tfrac{2{,}291{,}989}{16}c^{-3} - \tfrac{83{,}308{,}905}{32}c^{-4} + O(c^{-5}). \tag{5–82}$$

$$\chi_{7,7}(c) = 44c - \tfrac{873}{4} - \tfrac{2{,}409}{2}c^{-1} - 16{,}365c^{-2} - \tfrac{9{,}430{,}971}{32}c^{-3} - 6{,}166{,}380c^{-4} + O(c^{-5}). \tag{5–83}$$

Finally, we consider approximations of $\chi_{N,n}$ of successive orders p, that is,

$$\chi_{N,n}(c) \approx \sum_{j=0}^{p} a_j(N,n)\, c^{-j+1}, \qquad p = 0, 1, \ldots, 5. \tag{5–84}$$

Table 1 lists such numbers for $N = n = 0$, $c = 100$ and $c = 10$. For comparison we provide values exact to fifteen significant digits, as computed by means of Bouwkamp's scheme. With the larger value of c the fifth-order approximation is accurate up to the eleventh significant digit, while with the smaller value it is accurate only up to the fifth significant digit. As a general rule the accuracy of the approximation is found to rise/drop according as c increases/decreases. Thus, there exists a lower threshold of c below which all accuracy is lost. Conversely, a higher threshold can always be found above which a specified accuracy will obtain. *Both these thresholds increase with N and n*, as Table 2 illustrates for $N = 2$ and $n = 3$: now, with $c = 100$ the fifth-order approximation produces eight digits of accuracy: with $c = 50$, three to four.

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Table 1. Progression of successive approximations of $\chi_{0,0}(c)$ toward exact value — $c = 100, 10$.

Approx.	$\chi_{0,0}(100)$	$\chi_{0,0}(10)$
$p = 0$	$2.00000000000000 \times 10^2$	$2.00000000000000 \times 10^1$
$p = 1$	$1.98750000000000 \times 10^2$	$1.87500000000000 \times 10^1$
$p = 2$	$1.98745000000000 \times 10^2$	$1.87000000000000 \times 10^1$
$p = 3$	$1.98744925000000 \times 10^2$	$1.86925000000000 \times 10^1$
$p = 4$	$1.98744923343750 \times 10^2$	$1.86908437500000 \times 10^1$
$p = 5$	$1.98744923297344 \times 10^2$	$1.86903796875000 \times 10^1$
exact	$1.98744923295734 \times 10^2$	$1.86901099396909 \times 10^1$

Table 2. Progression of successive approximations of $\chi_{2,3}(c)$ toward exact value — $c = 100, 50$.

Approx.	$\chi_{2,3}(100)$	$\chi_{2,3}(50)$
$p = 0$	$1.80000000000000 \times 10^3$	$9.00000000000000 \times 10^2$
$p = 1$	$1.76075000000000 \times 10^3$	$8.60750000000000 \times 10^2$
$p = 2$	$1.75985000000000 \times 10^3$	$8.58950000000000 \times 10^2$
$p = 3$	$1.75979750000000 \times 10^3$	$8.58740000000000 \times 10^2$
$p = 4$	$1.75979337687500 \times 10^3$	$8.58707015000000 \times 10^2$
$p = 5$	$1.75979299500313 \times 10^3$	$8.58700905050000 \times 10^2$
exact	$1.75979295052608 \times 10^3$	$8.58699269327762 \times 10^2$

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6 Fifth-order approximation of eigenvalues for small c

We now specialize the general expressions developed in Section 4 for $a_0, \ldots, a_5$ to the case $\alpha = 1$. As per Eqs. (3–31) and (4–1), $\chi_{N,n}(c)$ is then approximated through the following sum of powers of c^2:

$$\chi_{N,n}(c) = a_0 + a_1 c^2 + a_2 c^4 + a_3 c^6 + a_4 c^8 + a_5 c^{10} + O(c^{12}). \tag{6–1}$$

This case is more complicated than the case where c is large ($\alpha = 2$), which was dealt with in the preceding section. Difficulties arise because the formulæ for calculating the coefficients in expansion (6–1) turn out to involve differences between small fractions—a notorious source of precision loss. In a calculation scheme such as the present one, enough error is incurred due to the approximation mechanism itself without more being added by the expansion coefficients—especially if convergence is to be examined. Therefore, one seeks to represent the coefficients as exact ratios of two integers—which, in the present case, complicates the algebra substantially. Nevertheless, with the proper approach, the manipulations become straightforward and the results prove to be surprisingly concise.

In accordance with Eq. (4–2) we have

$$a_0 = \lambda_n = \left(N + 2n + \tfrac{1}{2}\right)\left(N + 2n + \tfrac{3}{2}\right) = \left(z - \tfrac{1}{2}\right)\left(z + \tfrac{1}{2}\right) = z^2 - \tfrac{1}{4}, \tag{6–2}$$

where we recognize canonical variable $z \stackrel{\text{def}}{\equiv} N + 2n + 1$ from Section 5. Thus,

$$4a_0 = 4z^2 - 1 = (2z-1)(2z+1) = (2N + 4n + 1)(2N + 4n + 3). \tag{6–3}$$

Furthermore, in accordance with Eq. (4–9),

$$\left(h_j\right)^{-1} \stackrel{\text{def}}{\equiv} \lambda_{n+j} - \lambda_n = (z + 2j)^2 - z^2 = 4j(z + j), \tag{6–4}$$

whence

$$h_1 = \frac{1}{4(z+1)}, \qquad h_{-1} = -\frac{1}{4(z-1)}, \qquad h_2 = \frac{1}{8(z+2)}, \qquad h_{-2} = -\frac{1}{8(z-2)}. \tag{6–5}$$

For this case we also have, from Eqs. (2–21)–(2–23),

$$\gamma_j^1 = \begin{cases} -\dfrac{(N+j+1)^2}{(N+2j+1)(N+2j+2)} & \text{if } j \geq 0, \\ 0 & \text{if } j < 0; \end{cases} \tag{6–6}$$

$$\gamma_j^0 = \begin{cases} \dfrac{1}{2}\left[1 + \dfrac{N^2}{(N+2j)(N+2j+2)}\right] = \dfrac{(N+j)(N+j+1) + j(j+1)}{(N+2j)(N+2j+2)} & \text{if } j > 0, \\ \dfrac{N+1}{N+2} & \text{if } j = 0, \\ 0 & \text{if } j < 0; \end{cases} \tag{6–7}$$

and

$$\gamma_j^{-1} = \begin{cases} -\dfrac{j^2}{(N+2j)(N+2j+1)} & \text{if } j \geq 1, \\ 0 & \text{if } j < 1. \end{cases} \tag{6–8}$$

Finally, the general expressions for coefficients $a_1 \ldots a_5$—Eqs. (4–5), (4–13), (4–19), (4–29) and (4–41)—appear as the following ones in terms of the above quantities h_j and γ_j^k:

$$a_1 = \gamma_n^0, \tag{6–9}$$

$$a_2 = -h_1\gamma_n^1\gamma_{n+1}^{-1} - \phi_1 h_{-1}\gamma_{n-1}^1\gamma_n^{-1}, \tag{6–10}$$

$$a_3 = -(h_1)^2\gamma_n^1\gamma_{n+1}^{-1}(\gamma_n^0 - \gamma_{n+1}^0) - \phi_1(h_{-1})^2\gamma_{n-1}^1\gamma_n^{-1}(\gamma_n^0 - \gamma_{n-1}^0), \tag{6–11}$$

$$\begin{aligned} a_4 = &-(h_1)^2\gamma_n^1\gamma_{n+1}^{-1}\big[a_2 + h_1(\gamma_n^0 - \gamma_{n+1}^0)^2 + h_2\gamma_{n+1}^1\gamma_{n+2}^{-1}\big] \\ &-\phi_1(h_{-1})^2\gamma_{n-1}^1\gamma_n^{-1}\big[a_2 + h_{-1}(\gamma_n^0 - \gamma_{n-1}^0)^2 + \phi_2 h_{-2}\gamma_{n-2}^1\gamma_{n-1}^{-1}\big], \end{aligned} \tag{6–12}$$

$$\begin{aligned} a_5 = &-(h_1)^2\gamma_n^1\gamma_{n+1}^{-1}\big\{a_3 + h_1(\gamma_n^0 - \gamma_{n+1}^0)\big[2a_2 + h_1(\gamma_n^0 - \gamma_{n+1}^0)^2\big] \\ &\qquad + h_2\gamma_{n+1}^1\gamma_{n+2}^{-1}\big[2h_1(\gamma_n^0 - \gamma_{n+1}^0) + h_2(\gamma_n^0 - \gamma_{n+2}^0)\big]\big\} \\ &-\phi_1(h_{-1})^2\gamma_{n-1}^1\gamma_n^{-1}\big\{a_3 + h_{-1}(\gamma_n^0 - \gamma_{n-1}^0)\big[2a_2 + h_{-1}(\gamma_n^0 - \gamma_{n-1}^0)^2\big] \\ &\qquad + \phi_2 h_{-2}\gamma_{n-2}^1\gamma_{n-1}^{-1}\big[2h_{-1}(\gamma_n^0 - \gamma_{n-1}^0) + h_{-2}(\gamma_n^0 - \gamma_{n-2}^0)\big]\big\}. \end{aligned} \tag{6–13}$$

We recall the role of ϕ_m as an *inclusion switch*: a term bearing ϕ_m is to be included if and only if $n \geq m$, where n is the rank of the eigenvalue being approximated.

We start by developing convenient expressions for the combinations of quantities γ_j^k that occur above.

To begin with we note that

$$(N+2n)(N+2n+2) = (z-1)(z+1) = z^2 - 1 \stackrel{\text{def}}{\equiv} \omega. \tag{6–14}$$

This quantity ω will soon emerge as the natural variable for the case under study. On the basis of Eq. (6–3) we can express a_0 in terms of it:

$$a_0(N,n) = \frac{4\omega + 3}{4}. \tag{6–15}$$

In particular, for $n = 0$, $n = 1$ and $n = 2$ we have $\omega = N(N+2)$, $\omega = (N+2)(N+4)$ and $\omega = (N+4)(N+6)$, respectively, so that

$$\left.\begin{aligned} a_0(N,0) = \frac{(2N+1)(2N+3)}{4}, \qquad a_0(N,1) = \frac{(2N+5)(2N+7)}{4}, \\ a_0(N,2) = \frac{(2N+9)(2N+11)}{4}, \end{aligned}\right\} \tag{6–16}$$

in agreement with the rightmost part of Eq. (6–3), of course. Likewise for a_1, from Eqs. (6–9) and (6–7) we find that

$$\begin{aligned} a_1(N,n) = \frac{\omega + N^2}{2\omega} &= \frac{(N+n)(N+n+1) + n(n+1)}{(N+2n)(N+2n+2)}, \\ &= \frac{N^2 + (2n+1)N + 2n(n+1)}{(N+2n)(N+2n+2)}, \end{aligned} \tag{6–17}$$

which, according to Eq. (6–7), is valid only if $n > 0$, while

$$a_1(N,0) = \frac{N+1}{N+2}. \tag{6–18}$$

But if $n = 0$,

$$\frac{\omega + N^2}{2\omega} = \frac{N(N+2)+N^2}{2N(N+2)} = \frac{2N(N+1)}{2N(N+2)} = \frac{N+1}{N+2}. \tag{6–19}$$

Thus the general formula turns out to be valid even when $n = 0$—provided it remains computable. It fails to do so only for $N = 0$, which makes ω vanish. Hence we have the following choice: either we use general formula (6–17) in all instances except the special case $N = n = 0$, for which $a_1 = \frac{1}{2}$; or else we use the general formula only if $n > 0$, and special formula (6–18) otherwise.

Now, from Eq. (6–7) we have that, if $n > 0$ and $n + j > 0$,

$$\gamma_n^0 - \gamma_{n+j}^0 = \frac{N^2}{2}\left[\frac{1}{\omega} - \frac{1}{(z-1+2j)(z+1+2j)}\right] = \frac{2N^2 j(z+j)}{\omega(z-1+2j)(z+1+2j)}. \tag{6–20}$$

Thus, for $n \geq 1$,

$$\gamma_n^0 - \gamma_{n+1}^0 = \frac{2N^2}{\omega(z+3)} \tag{6–21}$$

and

$$\gamma_n^0 - \gamma_{n+2}^0 = \frac{4N^2(z+2)}{\omega(z+3)(z+5)}; \tag{6–22}$$

for $n \geq 2$,

$$\gamma_n^0 - \gamma_{n-1}^0 = -\frac{2N^2}{\omega(z-3)}; \tag{6–23}$$

and, for $n \geq 3$,

$$\gamma_n^0 - \gamma_{n-2}^0 = -\frac{4N^2(z-2)}{\omega(z-3)(z-5)}. \tag{6–24}$$

On the other hand if $n = 0$,

$$\gamma_n^0 - \gamma_{n+1}^0 = \gamma_0^0 - \gamma_1^0 = \frac{N+1}{N+2} - \frac{(N+1)(N+2)+2}{(N+2)(N+4)} = \frac{2N}{(N+2)(N+4)} \tag{6–25}$$

and

$$\gamma_n^0 - \gamma_{n+2}^0 = \gamma_0^0 - \gamma_2^0 = \frac{N+1}{N+2} - \frac{(N+2)(N+3)+6}{(N+4)(N+6)} = \frac{4N(N+3)}{(N+2)(N+4)(N+6)}; \tag{6–26}$$

if $n = 1$,

$$\gamma_n^0 - \gamma_{n-1}^0 = \gamma_1^0 - \gamma_0^0 = -\left(\gamma_0^0 - \gamma_1^0\right) = -\frac{2N}{(N+2)(N+4)}; \tag{6–27}$$

and, if $n = 2$,

$$\gamma_n^0 - \gamma_{n-2}^0 = \gamma_2^0 - \gamma_0^0 = -\left(\gamma_0^0 - \gamma_2^0\right) = -\frac{4N(N+3)}{(N+2)(N+4)(N+6)}. \tag{6–28}$$

To summarize,

$$\gamma_n^0 - \gamma_{n+1}^0 = \begin{cases} \dfrac{2N}{(N+2)(N+4)} & \text{if } n = 0, \\ \dfrac{2N^2}{\omega(z+3)} & \text{if } n \geq 1; \end{cases} \tag{6–29a}$$

$$\gamma_n^0 - \gamma_{n+2}^0 = \begin{cases} \dfrac{4N(N+3)}{(N+2)(N+4)(N+6)} & \text{if } n = 0, \\ \dfrac{4N^2(z+2)}{\omega(z+3)(z+5)} & \text{if } n \geq 1; \end{cases} \tag{6–29b}$$

$$\gamma_n^0 - \gamma_{n-1}^0 = \begin{cases} -\dfrac{2N}{(N+2)(N+4)} & \text{if } n = 1, \\ -\dfrac{2N^2}{\omega(z-3)} & \text{if } n \geq 2; \end{cases} \tag{6–29c}$$

$$\gamma_n^0 - \gamma_{n-2}^0 = \begin{cases} -\dfrac{4N(N+3)}{(N+2)(N+4)(N+6)} & \text{if } n = 2, \\ -\dfrac{4N^2(z-2)}{\omega(z-3)(z-5)} & \text{if } n \geq 3. \end{cases} \tag{6–29d}$$

Quantity $\gamma_n^0 - \gamma_{n-1}^0$ does not arise if $n = 0$, nor does quantity $\gamma_n^0 - \gamma_{n-2}^0$ if $n = 0$ or $n = 1$.

From Eqs. (6–6) and (6–8), on the other hand, we find that, if $n + j \geq 0$,

$$\gamma_{n+j}^1 \gamma_{n+j+1}^{-1} = \frac{(n+j+1)^2(N+n+j+1)^2}{(N+2n+2j+1)(N+2n+2j+2)^2(N+2n+2j+3)}. \tag{6–30}$$

But, as we established in the preceding section,

$$(n+j)(N+n+j) = x + j(j-1) + jz, \tag{6–31}$$

where we recognize canonical variable $x \stackrel{\text{def}}{\equiv} n(N+n)$. Thus,

$$\gamma_{n+j}^1 \gamma_{n+j+1}^{-1} = \frac{[x + j(j+1) + (j+1)z]^2}{(z+2j)(z+2j+1)^2(z+2j+2)} \qquad (n + j \geq 0). \tag{6–32}$$

In particular, for $n \geq 0$,

$$\gamma_n^1 \gamma_{n+1}^{-1} = \frac{(n+1)^2(N+n+1)^2}{(N+2n+1)(N+2n+2)^2(N+2n+3)} = \frac{(x+z)^2}{z(z+1)^2(z+2)} \tag{6–33}$$

and

$$\gamma_{n+1}^1 \gamma_{n+2}^{-1} = \frac{(n+2)^2(N+n+2)^2}{(N+2n+3)(N+2n+4)^2(N+2n+5)} = \frac{(x+2+2z)^2}{(z+2)(z+3)^2(z+4)}; \tag{6–34}$$

for $n \geq 1$,

$$\gamma_{n-1}^1 \gamma_n^{-1} = \frac{n^2(N+n)^2}{(N+2n-1)(N+2n)^2(N+2n+1)} = \frac{x^2}{(z-2)(z-1)^2 z}; \tag{6–35}$$

and, for $n \geq 2$,

$$\gamma^1_{n-2}\gamma^{-1}_{n-1} = \frac{(n-1)^2(N+n-1)^2}{(N+2n-3)(N+2n-2)^2(N+2n-1)} = \frac{(x+2-z)^2}{(z-4)(z-3)^2(z-2)}. \quad (6\text{–}36)$$

Quantity $\gamma^1_{n-1}\gamma^{-1}_n$ does not arise if $n = 0$, nor does quantity $\gamma^1_{n-2}\gamma^{-1}_{n-1}$ if $n = 0$ or $n = 1$.

We may now substitute from Eqs. (6–5), (6–29), and (6–33)–(6–36) into expressions (6–10)–(6–13) for coefficients a_2, a_3, a_4 and a_5. We note that due to ϕ_1 and ϕ_2, the case $n = 0$ in every instance, and the case $n = 1$ for a_4 and a_5, would appear to require separate treatment. Actually, that will not be necessary: it will be sufficient to consider the general case, because the numerators of $\gamma^1_{n-1}\gamma^{-1}_n$ and $\gamma^1_{n-2}\gamma^{-1}_{n-1}$ vanish when $n = 0$ and $n = 1$, respectively [see Eqs. (6–35) and (6–36)]. Admittedly, the general expression may include some denominators which, in these special cases, also have the potential to vanish.[6] However, such denominators will then be found to drop out, as argued in Annex A. Therefore, we may proceed from the most general expressions for coefficients $a_2 \ldots a_5$, confident that any special formulæ we require will emerge from the general results we shall obtain. We start with coefficients a_2 and a_3, as they occur in the expressions for a_4 and a_5.

From Eq. (6–10) we obtain

$$a_2 = -\frac{1}{4(z+1)}\frac{(x+z)^2}{z(z+1)^2(z+2)} + \frac{1}{4(z-1)}\frac{x^2}{(z-2)(z-1)^2 z}. \quad (6\text{–}37)$$

Let us introduce the handy notation

$$\omega_j \overset{\text{def}}{\equiv} z^2 - j^2 = z^2 - 1 - j^2 + 1 = \omega - (j^2 - 1). \quad (6\text{–}38)$$

Thus,

$$\omega_1 \equiv \omega, \qquad \omega_2 = \omega - 3, \qquad \omega_3 = \omega - 8, \qquad \omega_4 = \omega - 15, \qquad \omega_5 = \omega - 24, \quad (6\text{–}39)$$

and in terms of such quantities we may rewrite Eq. (6–37) as follows:

$$4\,z\,a_2 = A/B, \quad (6\text{–}40)$$

where

$$B = \omega^3\omega_2 = \omega^3(\omega - 3) \quad (6\text{–}41)$$

and

$$\begin{aligned} A &= x^2(z+1)^3(z+2) - (x+z)^2(z-1)^3(z-2), \\ &= x^2(z^4 + 5z^3 + 9z^2 + 7z + 2) - (x+z)^2(z^4 - 5z^3 + 9z^2 - 7z + 2), \\ &= x^2\big[(z^4 + 9z^2 + 2) + z(5z^2 + 7)\big] - (x+z)^2\big[(z^4 + 9z^2 + 2) - z(5z^2 + 7)\big], \\ &= \big[x^2 + (x+z)^2\big]z(5z^2 + 7) + \big[x^2 - (x+z)^2\big](z^4 + 9z^2 + 2). \end{aligned} \quad (6\text{–}42)$$

Our immediate goal will be to express A as a function of ω. We start with

$$x^2 - (x+z)^2 = -z(z + 2x). \quad (6\text{–}43)$$

[6] i.e., they will do so for certain small values of N.

We recall from Section 5 that

$$z + 2x = 2n(N + n + 1) + N + 1. \tag{6–44}$$

Hence

$$\begin{aligned} \omega - N^2 = z^2 - 1 - N^2 &= (N + 2n + 1)^2 - 1 - N^2, \\ &= 4n(N + n + 1) + 2N \;=\; 2(z + 2x) - 2, \end{aligned} \tag{6–45}$$

and it follows that

$$z + 2x = \tfrac{1}{2}(\omega - N^2 + 2). \tag{6–46}$$

More generally,

$$z + 2x + j = \tfrac{1}{2}\big[\omega - N^2 + 2(j + 1)\big]. \tag{6–47}$$

Substituting from Eq. (6–46) into Eq. (6–43), we obtain

$$x^2 - (x + z)^2 = -\tfrac{1}{2}z(\omega - N^2 + 2). \tag{6–48}$$

Next we turn to

$$x^2 + (x + z)^2 = z^2 + 2x(x + z) = \omega + 1 + 2x(x + z). \tag{6–49}$$

In view of result (6–46) we have

$$\begin{aligned} x(x + z) &= \tfrac{1}{4}(z + 2x - z)(z + 2x + z), \\ &= \tfrac{1}{4}\big[(z + 2x)^2 - z^2\big] \;=\; \tfrac{1}{16}\big[(\omega - N^2 + 2)^2 - 4(\omega + 1)\big], \end{aligned} \tag{6–50}$$

whence

$$x^2 + (x + z)^2 = \tfrac{1}{8}\big[(\omega - N^2 + 2)^2 + 4(\omega + 1)\big]. \tag{6–51}$$

Finally, it can easily be shown that

$$5z^2 + 7 = 5\omega + 12 \tag{6–52}$$

and

$$z^4 + 9z^2 + 2 = \omega^2 + 11\omega + 12. \tag{6–53}$$

It will prove convenient that we establish the following notation:

$$\big[x^2 + (x + z)^2\big]za + \big[x^2 - (x + z)^2\big]b \;=\; \tfrac{1}{8}zR(a, b), \tag{6–54}$$

where, in view of Eqs. (6–48) and (6–51),

$$R(a, b) = \big[(\omega - N^2 + 2)^2 + 4(\omega + 1)\big]a - 4(\omega - N^2 + 2)b. \tag{6–55}$$

Substituting from Eqs. (6–52) and (6–53) into expression (6–42) for A, and using the above notation, we find successively that

$$\begin{aligned} 8A/z &= R(5\omega + 12, \omega^2 + 11\omega + 12), \\ &= \omega^3 - 6N^2\omega^2 + 5N^4\omega + 12N^4, \\ &= \omega(\omega - N^2)(\omega - 5N^2) + 12N^4 \overset{\text{def}}{\equiv} T(N, \omega), \end{aligned} \tag{6–56}$$

yielding

$$a_2(N,n) \;=\; \frac{8A/z}{32B} \;=\; \frac{T(N,\omega)}{32\,\omega^3\omega_2} \tag{6–57}$$

or, explicitly,

$$a_2(N,n) \;=\; \frac{\omega(\omega-N^2)(\omega-5N^2)+12N^4}{32\,\omega^3(\omega-3)}. \tag{6–58}$$

As stated previously, the above formula is valid generally—i.e., not only for $n > 0$. However, it fails to be computable—due to $0/0$ indeterminacy—when $n = 0$ and $N = 0$ or $N = 1$. This is because for $n = 0$

$$B \;=\; (N-1)N^3(N+2)^3(N+3), \tag{6–59}$$

while

$$T(N,\omega) \;=\; T\big(N, N(N+2)\big) \;=\; -8(N-1)N^3(N+1). \tag{6–60}$$

After simplification, though, we find

$$a_2(N,0) \;=\; -\frac{N+1}{4(N+2)^3(N+3)}. \tag{6–61}$$

Hence, as with a_1, we have a choice: either we use general formula (6–58) in all instances except the following two special cases:

$$a_2(0,0) = -\tfrac{1}{96}, \qquad a_2(1,0) = -\tfrac{1}{216}, \tag{6–62}$$

or else we use the general formula only if $n > 0$, and special formula (6–61) otherwise.

Turning to a_3 now, from Eq. (6–11) we obtain

$$a_3 = -\frac{1}{16(z+1)^2}\,\frac{(x+z)^2}{z(z+1)^2(z+2)}\,\frac{2N^2}{\omega(z+3)} + \frac{1}{16(z-1)^2}\,\frac{x^2}{(z-2)(z-1)^2 z}\,\frac{2N^2}{\omega(z-3)}, \tag{6–63}$$

or

$$8\,z\,a_3 = N^2 C/D, \tag{6–64}$$

where

$$D \;=\; \omega^5\omega_2\,\omega_3 \;=\; \omega^5(\omega-3)(\omega-8) \tag{6–65}$$

and

$$\begin{aligned}
C &= x^2(z+1)^4(z+2)(z+3) - (x+z)^2(z-1)^4(z-2)(z-3),\\
&= \qquad x^2(z^6+9z^5+32z^4+58z^3+57z^2+29z+6)\\
&\quad - (x+z)^2(z^6-9z^5+32z^4-58z^3+57z^2-29z+6),\\
&= \qquad x^2\big[(z^6+32z^4+57z^2+6)+z(9z^4+58z^2+29)\big]\\
&\quad - (x+z)^2\big[(z^6+32z^4+57z^2+6)-z(9z^4+58z^2+29)\big],\\
&= \big[x^2+(x+z)^2\big]z(9z^4+58z^2+29) + \big[x^2-(x+z)^2\big](z^6+32z^4+57z^2+6).
\end{aligned} \tag{6–66}$$

It can be shown that

$$9z^4 + 58z^2 + 29 = 9\omega^2 + 76\omega + 96 \tag{6–67}$$

and

$$z^6 + 32z^4 + 57z^2 + 6 = \omega^3 + 35\omega^2 + 124\omega + 96. \tag{6–68}$$

Thus,

$$\begin{aligned} 8C/z &= R(9\omega^2 + 76\omega + 96, \omega^3 + 35\omega^2 + 124\omega + 96), \\ &= 5\omega^4 - 14N^2\omega^3 + 9N^4\omega^2 - 48N^2\omega^2 + 76N^4\omega + 96N^4, \\ &= (\omega^2 - N^2\omega - 7N^2)(5\omega^2 - 9N^2\omega - 13N^2) + 5N^4 \overset{\text{def}}{\equiv} U(N,\omega), \end{aligned} \tag{6–69}$$

yielding

$$a_3(N,n) = \frac{N^2(8C/z)}{64D} = \frac{N^2U(N,\omega)}{64\omega^5\omega_2\omega_3} \tag{6–70}$$

or, explicitly,

$$a_3(N,n) = \frac{N^2\big[(\omega^2 - N^2\omega - 7N^2)(5\omega^2 - 9N^2\omega - 13N^2) + 5N^4\big]}{64\omega^5(\omega-3)(\omega-8)}. \tag{6–71}$$

Computability problems with the above general expression, in the form of 0/0 indeterminacy, arise if $n = 0$ or $n = 1$—specifically, when $(N,n) \in \big[(0,0),(1,0),(2,0),(0,1)\big]$. This is because for $n = 0$

$$D = (N-2)(N-1)N^5(N+2)^5(N+3)(N+4), \tag{6–72}$$

while

$$N^2U(N,\omega) = N^2U\big(N, N(N+2)\big) = -8(N-2)(N-1)N^6(N+1). \tag{6–73}$$

After simplification, though, we find

$$a_3(N,0) = -\frac{N(N+1)}{8(N+2)^5(N+3)(N+4)}. \tag{6–74}$$

Likewise for $n = 1$,

$$D = N(N+1)(N+2)^5(N+4)^5(N+5)(N+6), \tag{6–75}$$

while

$$N^2U(N,\omega) = N^2U\big(N,(N+2)(N+4)\big) = -8N^2(N+1)u_1(N), \tag{6–76}$$

where

$$u_1(N) = 3N^6 + 11N^5 - 122N^4 - 1{,}088N^3 - 3{,}520N^2 - 5{,}120N - 2{,}560. \tag{6–77}$$

Again, though, after simplification we find

$$a_3(N,1) = -\frac{Nu_1(N)}{8(N+2)^5(N+4)^5(N+5)(N+6)}. \tag{6–78}$$

Hence we have the same kind of choice as before: either we use general formula (6–71) in all instances except the following four special cases:

$$a_3(0,0) = a_3(0,1) = 0, \qquad a_3(1,0) = -\tfrac{1}{19{,}440}, \qquad a_3(2,0) = -\tfrac{1}{40{,}960}; \tag{6–79}$$

or else we use the general formula only if $n > 1$, and special formulæ (6–74) and (6–78) otherwise.

Having dealt with a_2 and a_3, we may now calculate a_4 and a_5. From Eq. (6–12), first, we obtain

$$\begin{aligned} a_4 = & -\frac{1}{16(z+1)^2}\frac{(x+z)^2}{z(z+1)^2(z+2)} \times \\ & \quad \left[\frac{A/z}{4\omega^3\omega_2} + \frac{1}{4(z+1)}\frac{4N^4}{\omega^2(z+3)^2} + \frac{1}{8(z+2)}\frac{(x+2+2z)^2}{(z+2)(z+3)^2(z+4)}\right] \\ & -\frac{1}{16(z-1)^2}\frac{x^2}{(z-2)(z-1)^2 z} \times \\ & \quad \left[\frac{A/z}{4\omega^3\omega_2} - \frac{1}{4(z-1)}\frac{4N^4}{\omega^2(z-3)^2} - \frac{1}{8(z-2)}\frac{(x+2-z)^2}{(z-4)(z-3)^2(z-2)}\right]. \end{aligned} \tag{6–80}$$

This can be recast as

$$128\,z\,a_4 = E/F, \tag{6–81}$$

where

$$F = \omega^7\omega_2^3\omega_3^2\omega_4 = \omega^7(\omega-3)^3(\omega-8)^2(\omega-15) \tag{6–82}$$

and

$$E = \omega^3 E_1 + 8N^4\omega_2^2\omega_4 E_2 - 2(A/z)\,\omega_2\omega_3^2\omega_4 E_3, \tag{6–83}$$

with

$$\begin{aligned} E_1 = & \qquad\qquad x^2(x+2-z)^2(z+1)^4(z+2)^3(z+3)^2(z+4) \\ & -(x+z)^2(x+2+2z)^2(z-1)^4(z-2)^3(z-3)^2(z-4), \end{aligned} \tag{6–84}$$

$$E_2 = x^2(z+1)^5(z+2)(z+3)^2 - (x+z)^2(z-1)^5(z-2)(z-3)^2, \tag{6–85}$$

and

$$E_3 = x^2(z+1)^4(z+2) + (x+z)^2(z-1)^4(z-2). \tag{6–86}$$

Let

$$x(x+2-z) \stackrel{\text{def}}{\equiv} \zeta, \qquad (x+z)(x+2+2z) \stackrel{\text{def}}{\equiv} \eta, \tag{6–87}$$

as in Section 5. Proceeding as we did for A and C, we find that

$$E_1 = \zeta^2(p_{1,1} + zp_{1,2}) - \eta^2(p_{1,1} - zp_{1,2}) = (\zeta^2+\eta^2)zp_{1,2} + (\zeta^2-\eta^2)p_{1,1}, \tag{6–88}$$

where

$$\begin{aligned} p_{1,1} &= z^{10} + 175z^8 + 2{,}835z^6 + 8{,}777z^4 + 5{,}204z^2 + 288\,, \\ &= \omega^5 + 180\omega^4 + 3{,}545\omega^3 + 18{,}342\omega^2 + 31{,}968\omega + 17{,}280\,, \end{aligned} \tag{6–89}$$

and

$$\begin{aligned} p_{1,2} &= 20z^8 + 882z^6 + 6{,}072z^4 + 8{,}458z^2 + 1{,}848\,, \\ &= 20\omega^4 + 962\omega^3 + 8{,}838\omega^2 + 23{,}328\omega + 17{,}280\,. \end{aligned} \tag{6–90}$$

Therefore, we seek to express $\zeta^2 + \eta^2$ and $\zeta^2 - \eta^2$ as functions of ω. We tackle the latter quantity first. From Eqs. (5–25), (5–23) and (6–47) we have that

$$\eta - \zeta \;=\; 2z(z + 2x + 1) \;=\; z(\omega - N^2 + 4). \tag{6–91}$$

Likewise, from Eqs. (5–26), (5–24) and (6–47) we find that

$$\begin{aligned} 2(\eta + \zeta) \;&=\; 4z(z + x + 1) + 4x(x + 2), \\ &=\; (z + 2x)(z + 2x + 4) + 3z^2, \\ &=\; (z + 2x + 2)^2 + 3z^2 - 4 \;=\; \tfrac{1}{4}(\omega - N^2 + 6)^2 + 3\omega - 1, \end{aligned} \tag{6–92}$$

whence

$$\eta + \zeta \;=\; \tfrac{1}{8}\big[(\omega - N^2 + 6)^2 + 4(3\omega - 1)\big]. \tag{6–93}$$

Thus

$$\zeta^2 - \eta^2 \;=\; -(\eta - \zeta)(\eta + \zeta) \;=\; -\tfrac{1}{8}z(\omega - N^2 + 4)\big[(\omega - N^2 + 6)^2 + 4(3\omega - 1)\big]. \tag{6–94}$$

Turning to $\zeta^2 + \eta^2$, we have that

$$\zeta^2 + \eta^2 = (\eta - \zeta)^2 + 2\zeta\eta, \tag{6–95}$$

where

$$\zeta\eta \;\equiv\; x(x + z)(x + 2 - z)(x + 2 + 2z). \tag{6–96}$$

But

$$4(x + 2 - z)(x + 2 + 2z) = (z + 2x + 4)^2 - 9z^2, \tag{6–97}$$

which, by way of Eq. (6–47), gives

$$(x + 2 - z)(x + 2 + 2z) = \tfrac{1}{16}\big[(\omega - N^2 + 10)^2 - 36(\omega + 1)\big]. \tag{6–98}$$

Substituting from this and expression (6–50) for $x(x + z)$, we find that

$$\zeta\eta \;=\; \tfrac{1}{256}\big[(\omega - N^2 + 2)^2 - 4(\omega + 1)\big]\big[(\omega - N^2 + 10)^2 - 36(\omega + 1)\big]. \tag{6–99}$$

Finally, using the above result and expression (6–91) for $\eta - \zeta$, we obtain

$$\begin{aligned} \zeta^2 + \eta^2 = {}& (\omega + 1)(\omega - N^2 + 4)^2 \\ & + \tfrac{1}{128}\big[(\omega - N^2 + 2)^2 - 4(\omega + 1)\big]\big[(\omega - N^2 + 10)^2 - 36(\omega + 1)\big], \end{aligned} \tag{6–100}$$

which can be further reduced to

$$\zeta^2 + \eta^2 = \tfrac{1}{128}\big[(\omega - N^2 + 6)^2 + 4(11\omega + 7)\big]^2 - 2(\omega + 1)\big[8(\omega + 1) - N^2\big]. \tag{6–101}$$

For later purposes we summarize our findings as follows:

$$(\zeta^2+\eta^2)za+(\zeta^2-\eta^2)b = \tfrac{1}{128}zS(a,b), \tag{6–102}$$

where

$$S(a,b)=\Big\{\big[(\omega-N^2+6)^2+4(11\omega+7)\big]^2-256(\omega+1)\big[8(\omega+1)-N^2\big]\Big\}a \\ -16(\omega-N^2+4)\big[(\omega-N^2+6)^2+4(3\omega-1)\big]b. \tag{6–103}$$

In terms of the function S introduced above, expression (6–88) for E_1 becomes

$$E_1=\tfrac{1}{128}zS(p_{1,2},p_{1,1}). \tag{6–104}$$

As for E_2, we have that

$$E_2=x^2(p_{2,1}+zp_{2,2})-(x+z)^2(p_{2,1}-zp_{2,2}), \\ =\big[x^2+(x+z)^2\big]zp_{2,2}+\big[x^2-(x+z)^2\big]p_{2,1}=\tfrac{1}{8}zR(p_{2,2},p_{2,1}), \tag{6–105}$$

where

$$p_{2,1}=z^8+71z^6+385z^4+293z^2+18 = \omega^4+75\omega^3+604\omega^2+1{,}280\omega+768, \tag{6–106}$$

and

$$p_{2,2}=13z^6+213z^4+431z^2+111=13\omega^3+252\omega^2+896\omega+768. \tag{6–107}$$

Likewise for E_3,

$$E_3=x^2(zp_{3,1}+p_{3,2})+(x+z)^2(zp_{3,1}-p_{3,2}), \\ =\big[x^2+(x+z)^2\big]zp_{3,1}+\big[x^2-(x+z)^2\big]p_{3,2}=\tfrac{1}{8}zR(p_{3,1},p_{3,2}), \tag{6–108}$$

where

$$p_{3,1}=z^4+14z^2+9=\omega^2+16\omega+24 \tag{6–109}$$

and

$$p_{3,2}=6z^4+16z^2+2=6\omega^2+28\omega+24. \tag{6–110}$$

Substituting from expressions (6–104), (6–105) and (6–108) for E_1, E_2 and E_3, respectively, into expression (6–83) for E, we obtain

$$E=\tfrac{1}{128}z\omega^3S(p_{1,2},p_{1,1})+zN^4\omega_2^2\omega_4R(p_{2,2},p_{2,1})-\tfrac{1}{32}z\omega_2\omega_3^2\omega_4(8A/z)R(p_{3,1},p_{3,2}), \tag{6–111}$$

or, after multiplication by $128/z$ and substitution of expression (6–56) for $8A/z$,

$$128E/z=\omega^3S(p_{1,2},p_{1,1})+128N^4(\omega-3)^2(\omega-15)R(p_{2,2},p_{2,1}) \\ -4(\omega-3)(\omega-8)^2(\omega-15)\times \\ \big[\omega(\omega-N^2)(\omega-5N^2)+12N^4\big]R(p_{3,1},p_{3,2}). \tag{6–112}$$

Upon evaluating the right-hand side we find that

$$128E/z \,=\, 2(\omega - 8)V(N,\omega) \,=\, 2\,\omega_3 V(N,\omega), \tag{6–113}$$

where

$$\begin{aligned} V(N,\omega) = \quad & 5\,\omega^9 - (252N^2+7)\omega^8 + (1{,}638N^4 + 2{,}196N^2 - 264)\omega^7 \\ & - (2{,}860N^4 + 5{,}386N^2 + 1{,}440)N^2\omega^6 \\ & + (1{,}469N^4 - 9{,}516N^2 - 47{,}064)N^4\omega^5 \\ & + (16{,}489N^4 + 161{,}424N^2 + 106{,}560)N^4\omega^4 \\ & - (89{,}088N^2 + 36{,}288)N^6\omega^3 - (287{,}280N^2 + 622{,}080)N^6\omega^2 \\ & + 578{,}880N^8\omega + 1{,}036{,}800N^8. \end{aligned} \tag{6–114}$$

At last, substituting from Eq. (6–113) into Eq. (6–81), we secure the desired result:

$$a_4(N,n) \,=\, \frac{128E/z}{16,384F} \,=\, \frac{V(N,\omega)}{8,192\,\omega^7\omega_2^3\,\omega_3\,\omega_4}, \tag{6–115}$$

or, more explicitly,

$$a_4(N,n) \,=\, \frac{V(N,\omega)}{8,192\,\omega^7(\omega-3)^3(\omega-8)(\omega-15)}. \tag{6–116}$$

Computability problems with the above general expression, in the form of 0/0 indeterminacy, arise if $n = 0$ or $n = 1$—specifically, when $(N,n) \in \big[(0,0),(1,0),(2,0),(3,0),(0,1),(1,1)\big]$. This is because for $n = 0$

$$\omega^7\omega_2^3\,\omega_3\,\omega_4 = (N-3)(N-2)(N-1)^3N^7(N+2)^7(N+3)^3(N+4)(N+5), \tag{6–117}$$

while

$$V(N,\omega) \,=\, V\big(N,N(N+2)\big) \,=\, -128(N-3)(N-2)(N-1)^3N^7(N+1)\,v_0(N), \tag{6–118}$$

where

$$v_0(N) = 5N^4 + 27N^3 + 17N^2 - 71N - 44. \tag{6–119}$$

Simplifying, though, we find

$$a_4(N,0) \,=\, -\frac{(N+1)\,v_0(N)}{64(N+2)^7(N+3)^3(N+4)(N+5)}. \tag{6–120}$$

Likewise for $n = 1$,

$$\omega^7\omega_2^3\,\omega_3\,\omega_4 = (N-1)N(N+1)^3(N+2)^7(N+4)^7(N+5)^3(N+6)(N+7), \tag{6–121}$$

while

$$V(N,\omega) \,=\, V\big(N,(N+2)(N+4)\big) \,=\, -128(N-1)N(N+1)^3v_1(N), \tag{6–122}$$

where

$$
\begin{aligned}
v_1(N) = \quad & 15N^{12} + 154N^{11} - 1{,}760N^{10} - 43{,}490N^9 - 378{,}453N^8 \\
& - 1{,}880{,}434N^7 - 5{,}735{,}992N^6 - 9{,}879{,}968N^5 - 4{,}577{,}152N^4 \\
& + 17{,}115{,}648N^3 + 36{,}296{,}704N^2 + 27{,}467{,}776N + 7{,}176{,}192. \qquad (6\text{–}123)
\end{aligned}
$$

Again, though, simplification yields

$$
a_4(N,1) = -\frac{v_1(N)}{64(N+2)^7(N+4)^7(N+5)^3(N+6)(N+7)}. \qquad (6\text{–}124)
$$

Hence we have the same kind of choice as before: either we use general formula (6–116) in all instances except the following six special cases:

$$
\left.
\begin{aligned}
& a_4(0,0) = \tfrac{11}{1{,}105{,}920}, \quad a_4(1,0) = \tfrac{11}{22{,}394{,}880}, \quad a_4(2,0) = -\tfrac{89}{917{,}504{,}000}, \\
& a_4(3,0) = -\tfrac{103}{1{,}512{,}000{,}000}, \quad a_4(0,1) = -\tfrac{73}{7{,}168{,}000}, \quad a_4(1,1) = -\tfrac{60{,}703}{122{,}472{,}000{,}000};
\end{aligned}
\right\} \qquad (6\text{–}125)
$$

or else we use the general formula only if $n > 1$, and special formulæ (6–120) and (6–124) otherwise.

Finally, from Eq. (6–13) we obtain

$$
\begin{aligned}
a_5 = & -\frac{1}{16(z+1)^2}\frac{(x+z)^2}{z(z+1)^2(z+2)} \times \\
& \left\{ \frac{N^2(C/z)}{8\omega^5\omega_2\omega_3} + \frac{1}{4(z+1)}\frac{2N^2}{\omega(z+3)}\left[\frac{A/z}{2\omega^3\omega_2} + \frac{1}{4(z+1)}\frac{4N^4}{\omega^2(z+3)^2}\right] \right. \\
& \quad + \frac{1}{8(z+2)}\frac{(x+2+2z)^2}{(z+2)(z+3)^2(z+4)} \times \\
& \quad \left. \left[\frac{1}{2(z+1)}\frac{2N^2}{\omega(z+3)} + \frac{1}{8(z+2)}\frac{4N^2(z+2)}{\omega(z+3)(z+5)}\right] \right\} \\
& -\frac{1}{16(z-1)^2}\frac{x^2}{(z-2)(z-1)^2 z} \times \\
& \left\{ \frac{N^2(C/z)}{8\omega^5\omega_2\omega_3} + \frac{1}{4(z-1)}\frac{2N^2}{\omega(z-3)}\left[\frac{A/z}{2\omega^3\omega_2} - \frac{1}{4(z-1)}\frac{4N^4}{\omega^2(z-3)^2}\right] \right. \\
& \quad - \frac{1}{8(z-2)}\frac{(x+2-z)^2}{(z-4)(z-3)^2(z-2)} \times \\
& \quad \left. \left[\frac{1}{2(z-1)}\frac{2N^2}{\omega(z-3)} + \frac{1}{8(z-2)}\frac{4N^2(z-2)}{\omega(z-3)(z-5)}\right] \right\}. \qquad (6\text{–}126)
\end{aligned}
$$

This can be recast as

$$
256\,z\,a_5 = N^2 G/H, \qquad (6\text{–}127)
$$

where

$$
H = \omega^9\omega_2^3\omega_3^3\omega_4\omega_5 = \omega^9(\omega-3)^3(\omega-8)^3(\omega-15)(\omega-24) \qquad (6\text{–}128)
$$

and

$$G = \omega^3 G_1 + 8N^4\omega_2^2\omega_4\omega_5 G_2 - 2\omega_2\omega_3^2\omega_4\omega_5\big[2(A/z)G_3 + (C/z)E_3\big]. \qquad (6\text{–}129)$$

Here

$$\begin{aligned} G_1 = {} & \quad x^2(x+2-z)^2(3z-11)(z+1)^5(z+2)^3(z+3)^3(z+4)(z+5) \\ & - (x+z)^2(x+2+2z)^2(3z+11)(z-1)^5(z-2)^3(z-3)^3(z-4)(z-5), \\ = {} & \zeta^2(q_{1,1}+zq_{1,2}) - \eta^2(q_{1,1}-zq_{1,2}), \\ = {} & (\zeta^2+\eta^2)zq_{1,2} + (\zeta^2-\eta^2)q_{1,1} = \tfrac{1}{128}zS(q_{1,2},q_{1,1}), \end{aligned} \qquad (6\text{–}130)$$

with

$$\begin{aligned} q_{1,1} = {} & 3z^{14} + 815z^{12} + 13{,}042z^{10} - 173{,}898z^8 - 1{,}843{,}121z^6 \\ & - 3{,}333{,}157z^4 - 1{,}251{,}684z^2 - 47{,}520\,, \\ = {} & 3\omega^7 + 836\omega^6 + 17{,}995\omega^5 - 96{,}358\omega^4 - 2{,}391{,}888\omega^3 \\ & - 9{,}763{,}200\omega^2 - 14{,}072{,}832\omega - 6{,}635{,}520\,, \end{aligned} \qquad (6\text{–}131)$$

and

$$\begin{aligned} q_{1,2} = {} & 76z^{12} + 4{,}638z^{10} - 2{,}584z^8 - 751{,}564z^6 \\ & - 2{,}992{,}428z^4 - 2{,}528{,}834z^2 - 364{,}824\,, \\ = {} & 76\omega^6 + 5{,}094\omega^5 + 21{,}746\omega^4 - 714{,}000\omega^3 \\ & - 5{,}215{,}104\omega^2 - 10{,}755{,}072\omega - 6{,}635{,}520\,. \end{aligned} \qquad (6\text{–}132)$$

Also,

$$\begin{aligned} G_2 = {} & x^2(z+1)^6(z+2)(z+3)^3 - (x+z)^2(z-1)^6(z-2)(z-3)^3, \\ = {} & x^2(q_{2,1}+zq_{2,2}) - (x+z)^2(q_{2,1}-zq_{2,2}), \\ = {} & \big[x^2+(x+z)^2\big]zq_{2,2} + \big[x^2-(x+z)^2\big]q_{2,1} = \tfrac{1}{8}zR(q_{2,2},q_{2,1}), \end{aligned} \qquad (6\text{–}133)$$

with

$$\begin{aligned} q_{2,1} = {} & z^{10} + 126z^8 + 1{,}450z^6 + 3{,}172z^4 + 1{,}341z^2 + 54\,, \\ = {} & \omega^5 + 131\omega^4 + 1{,}964\omega^3 + 8{,}288\omega^2 + 12{,}544\omega + 6{,}144\,, \end{aligned} \qquad (6\text{–}134)$$

and

$$\begin{aligned} q_{2,2} = {} & 17z^8 + 536z^6 + 2{,}610z^4 + 2{,}576z^2 + 405\,, \\ = {} & (17z^4 + 94z^2 + 81)(z^4 + 26z^2 + 5), \\ = {} & (17\omega^2 + 128\omega + 192)(\omega^2 + 28\omega + 32), \\ = {} & 17\omega^4 + 604\omega^3 + 4{,}320\omega^2 + 9{,}472\omega + 6{,}144\,. \end{aligned} \qquad (6\text{–}135)$$

And

$$\begin{aligned} G_3 = {} & x^2(z+1)^5(z+2)(z+3) + (x+z)^2(z-1)^5(z-2)(z-3), \\ = {} & x^2(zq_{3,1}+q_{3,2}) + (x+z)^2(zq_{3,1}-q_{3,2}), \\ = {} & \big[x^2+(x+z)^2\big]zq_{3,1} + \big[x^2-(x+z)^2\big]q_{3,2} = \tfrac{1}{8}zR(q_{3,1},q_{3,2}), \end{aligned} \qquad (6\text{–}136)$$

with

$$q_{3,1} = z^6 + 41z^4 + 115z^2 + 35 \,=\, \omega^3 + 44\omega^2 + 200\,\omega + 192\,, \tag{6–137}$$

and

$$\begin{aligned} q_{3,2} &= 10z^6 + 90z^4 + 86z^2 + 6 \,=\, 2(z^2+1)(5z^4 + 40z^2 + 3), \\ &= 10\omega^3 + 120\omega^2 + 296\omega + 192 \,=\, 2(\omega+2)(5\omega^2 + 50\omega + 48). \end{aligned} \tag{6–138}$$

Substituting from expressions (6–130), (6–133), (6–136) and (6–108) for G_1, G_2, G_3 and E_3, respectively, into expression (6–129) for G, we obtain

$$\begin{aligned} G = {} & \tfrac{1}{128} z\omega^3 S(q_{1,2}, q_{1,1}) + zN^4\omega_2^2\omega_4\omega_5 R(q_{2,2}, q_{2,1}) \\ & - \tfrac{1}{32} z\omega_2\omega_3^2\omega_4\omega_5 \big[2(8A/z)R(q_{3,1}, q_{3,2}) + (8C/z)R(p_{3,1}, p_{3,2})\big], \end{aligned} \tag{6–139}$$

or, after multiplication by $128/z$ and substitution of expressions (6–56) and (6–69) for $8A/z$ and $8C/z$,

$$\begin{aligned} 128G/z = {} & \omega^3 S(q_{1,2}, q_{1,1}) + 128N^4(\omega-3)^2(\omega-15)(\omega-24)R(q_{2,2}, q_{2,1}) \\ & - 4(\omega-3)(\omega-8)^2(\omega-15)(\omega-24) \times \\ & \Big\{ 2\big[\omega(\omega - N^2)(\omega - 5N^2) + 12N^4\big] R(q_{3,1}, q_{3,2}) \\ & + \big[(\omega^2 - N^2\omega - 7N^2)(5\omega^2 - 9N^2\omega - 13N^2) + 5N^4\big] R(p_{3,1}, p_{3,2}) \Big\}. \end{aligned} \tag{6–140}$$

Upon evaluating the right-hand side we find that

$$128G/z \,=\, 2(\omega-8)^2 W(N,\omega) \,=\, 2\omega_3^2 W(N,\omega), \tag{6–141}$$

where

$$\begin{aligned} W(N,\omega) = {} & \quad 207\omega^{10} - (2{,}420N^2 + 2{,}837)\omega^9 \\ & + (8{,}242N^4 + 19{,}260N^2 - 1{,}752)\omega^8 \\ & - (10{,}500N^4 - 5{,}474N^2 - 208{,}800)N^2\omega^7 \\ & + (4{,}471N^6 - 110{,}340N^4 - 1{,}225{,}608N^2 - 432{,}000)N^2\omega^6 \\ & + (96{,}187N^4 + 1{,}819{,}440N^2 + 97{,}344)N^4\omega^5 \\ & - (625{,}728N^4 - 5{,}590{,}080N^2 - 5{,}840{,}640)N^4\omega^4 \\ & - (6{,}219{,}792N^2 + 12{,}856{,}320)N^6\omega^3 \\ & - (468{,}288N^2 + 24{,}883{,}200)N^6\omega^2 \\ & + 35{,}417{,}088N^8\omega + 34{,}836{,}480N^8. \end{aligned} \tag{6–142}$$

At last, substituting from Eq. (6–141) into Eq. (6–127), we arrive at our destination:

$$a_5(N,n) \,=\, \frac{N^2(128G/z)}{32{,}768H} \,=\, \frac{N^2 W(N,\omega)}{16{,}384\,\omega^9\omega_2^3\omega_3\omega_4\omega_5}, \tag{6–143}$$

or, more explicitly,

$$a_5(N,n) \,=\, \frac{N^2 W(N,\omega)}{16{,}384\,\omega^9(\omega-3)^3(\omega-8)(\omega-15)(\omega-24)}. \tag{6–144}$$

Computability problems with the above general expression, in the form of 0/0 indeterminacy, arise if $n = 0$ or $n = 1$ or $n = 2$—specifically, when $(N, n) \in \big[(0,0), (1,0), (2,0), (3,0), (4,0), (0,1), (1,1), (2,1), (0,2)\big]$. For $n = 0$ this is because

$$\omega^9 \omega_2^3 \omega_3 \omega_4 \omega_5 = (N-4)(N-3)(N-2)(N-1)^3 N^9 \times \\ (N+2)^9 (N+3)^3 (N+4)(N+5)(N+6), \tag{6–145}$$

while

$$N^2 W(N, \omega) = N^2 W\big(N, N(N+2)\big), \\ = -128(N-4)(N-3)(N-2)(N-1)^3 N^{10} (N+1) w_0(N), \tag{6–146}$$

where

$$w_0(N) = 7N^4 + 31N^3 - 59N^2 - 335N - 214. \tag{6–147}$$

However, simplification yields

$$a_5(N, 0) \;=\; -\frac{N(N+1)w_0(N)}{128(N+2)^9(N+3)^3(N+4)(N+5)(N+6)}. \tag{6–148}$$

For $n = 1$ indeterminacy arises because

$$\omega^9 \omega_2^3 \omega_3 \omega_4 \omega_5 = (N-2)(N-1)N(N+1)^3(N+2)^9 \times \\ (N+4)^9(N+5)^3(N+6)(N+7)(N+8), \tag{6–149}$$

while

$$N^2 W(N, \omega) = N^2 W\big(N, (N+2)(N+4)\big), \\ = -128(N-2)(N-1)N^2(N+1)^3 w_1(N), \tag{6–150}$$

where

$$\begin{aligned} w_1(N) = {} & \;\; 21N^{14} + 98N^{13} - 7{,}392N^{12} - 146{,}518N^{11} - 1{,}373{,}183N^{10} \\ & - 7{,}797{,}082N^9 - 26{,}809{,}232N^8 - 36{,}750{,}720N^7 + 142{,}217{,}728N^6 \\ & + 999{,}052{,}288N^5 + 2{,}963{,}140{,}608N^4 + 5{,}193{,}039{,}872N^3 \\ & + 5{,}452{,}464{,}128N^2 + 3{,}116{,}236{,}800N + 734{,}003{,}200. \end{aligned} \tag{6–151}$$

Again, however, simplification yields a computable formula:

$$a_5(N, 1) \;=\; -\frac{N w_1(N)}{128(N+2)^9(N+4)^9(N+5)^3(N+6)(N+7)(N+8)}. \tag{6–152}$$

Finally, for $n = 2$ we have

$$\omega^9 \omega_2^3 \omega_3 \omega_4 \omega_5 = N(N+1)(N+2)(N+3)^3(N+4)^9 \times \\ (N+6)^9(N+7)^3(N+8)(N+9)(N+10), \tag{6–153}$$

while

$$N^2 W(N, \omega) \;=\; N^2 W\big(N, (N+4)(N+6)\big) \;=\; -128N^2(N+1)(N+2)w_2(N), \tag{6–154}$$

where

$$
\begin{aligned}
w_2(N) = \quad & 35N^{17} + 281N^{16} - 37{,}373N^{15} - 1{,}234{,}019N^{14} - 17{,}867{,}617N^{13} \\
& - 130{,}466{,}577N^{12} - 179{,}075{,}095N^{11} + 6{,}073{,}988{,}475N^{10} \\
& + 67{,}447{,}891{,}250N^{9} + 373{,}810{,}337{,}664N^{8} + 1{,}167{,}061{,}707{,}648N^{7} \\
& + 1{,}266{,}928{,}390{,}656N^{6} - 5{,}678{,}341{,}346{,}304N^{5} \\
& - 30{,}810{,}638{,}020{,}608N^{4} - 72{,}285{,}632{,}299{,}008N^{3} \\
& - 95{,}936{,}369{,}393{,}664N^{2} - 69{,}460{,}103{,}135{,}232N \\
& - 21{,}237{,}655{,}928{,}832 \, .
\end{aligned}
\tag{6–155}
$$

Simplification leads to the following serviceable result:

$$
a_5(N,2) = -\frac{N w_2(N)}{128(N+3)^3(N+4)^9(N+6)^9(N+7)^3(N+8)(N+9)(N+10)} . \tag{6–156}
$$

Hence we have the same kind of choice as for previous coefficients: either we use general formula (6–144) in all instances except the following nine special cases:

$$
\left.
\begin{array}{ll}
\multicolumn{2}{c}{a_5(0,0) = a_5(0,1) = a_5(0,2) = 0,} \\
a_5(1,0) = \frac{19}{564{,}350{,}976}, & a_5(2,0) = \frac{19}{5{,}872{,}025{,}600}, \\
a_5(3,0) = \frac{173}{1{,}134{,}000{,}000{,}000}, & a_5(4,0) = -\frac{71}{884{,}902{,}330{,}368}, \\
a_5(1,1) = -\frac{9{,}530{,}489}{275{,}562{,}000{,}000{,}000}, & a_5(2,1) = -\frac{767{,}729}{226{,}534{,}996{,}574{,}208};
\end{array}
\right\}
\tag{6–157}
$$

or else we use the general formula only if $n > 2$, and special formulæ (6–148), (6–152) and (6–156) otherwise.

Our results have revealed nine instances of n requiring a special formula; they are listed in Table 3 below. These special cases are exactly those which were to be expected, given the caveats on general expressions (6–9)–(6–13).[7] Furthermore, as announced, the special formulæ followed from the general ones. However, to be on the safe side, we made sure they were correct by independent recalculation of each special case, excluding those portions of the general expressions selected by switches ϕ_1 and ϕ_2 and using the particular forms of quantities $\gamma_n^0 - \gamma_{n+j}^0$ specified in Eq. (6–29).

Table 3. Values of n for which the coefficients listed require a special formula.

n	coefficients				
0	a_1	a_2	a_3	a_4	a_5
1			a_3	a_4	a_5
2					a_5

[7] see page 39 and Annex A.

Table 3 shows that special formulæ are not needed for a_1 and a_2 if $n = 1$ or $n = 2$, nor for a_3 and a_4 if $n = 2$. Nevertheless, because such formulæ might prove useful in further studies, we provide them below:

$$a_1(N,1) = \frac{N^2+3N+4}{(N+2)(N+4)}; \tag{6–158}$$

$$a_1(N,2) = \frac{N^2+5N+12}{(N+4)(N+6)}; \tag{6–159}$$

$$a_2(N,1) = -\frac{3N^4+13N^3-4N^2-80N-64}{4(N+2)^3(N+4)^3(N+5)}; \tag{6–160}$$

$$a_2(N,2) = -\frac{5N^5+48N^4+55N^3-684N^2-2{,}160N-1{,}728}{4(N+3)(N+4)^3(N+6)^3(N+7)}; \tag{6–161}$$

$$a_3(N,2) = -\frac{N^2 u_2(N)}{8(N+3)(N+4)^5(N+6)^5(N+7)(N+8)}, \tag{6–162}$$

where

$$u_2(N) = 5N^6+36N^5-617N^4-9{,}552N^3-50{,}976N^2-120{,}960N-103{,}680; \tag{6–163}$$

and

$$a_4(N,2) = -\frac{v_2(N)}{64(N+3)^3(N+4)^7(N+6)^7(N+7)^3(N+8)(N+9)}, \tag{6–164}$$

where

$$\begin{aligned} v_2(N) = {} & 25N^{15}+493N^{14}-7{,}869N^{13}-386{,}147N^{12}-6{,}055{,}321N^{11} \\ & -52{,}291{,}341N^{10}-262{,}244{,}655N^9-599{,}624{,}045N^8 \\ & +1{,}153{,}769{,}100N^7+13{,}583{,}118{,}096N^6+46{,}346{,}392{,}512N^5 \\ & +77{,}749{,}777{,}152N^4+43{,}195{,}991{,}040N^3-59{,}746{,}553{,}856N^2 \\ & -103{,}876{,}411{,}392N-43{,}858{,}132{,}992. \end{aligned} \tag{6–165}$$

These special formulæ take place alongside Eqs. (6–16), (6–18), (6–61), (6–74), (6–78), (6–120), (6–124), (6–148), (6–152) and (6–156), which give $a_0(N,0)$, $a_0(N,1)$, $a_0(N,2)$, $a_1(N,0)$, $a_2(N,0)$, $a_3(N,0)$, $a_3(N,1)$, $a_4(N,0)$, $a_4(N,1)$, $a_5(N,0)$, $a_5(N,1)$ and $a_5(N,2)$, to fill out the 6×3 matrix corresponding to $a_0(N,n) \ldots a_5(N,n)$ and $n = 0$, 1, 2.

The formulæ we obtained may be specialized to any value of N we care; cancellations will guarantee that the resulting expressions are computable for any n. Such expressions are developed in Annex B for $N = 0$, 1 and 2. We duplicate hereunder the formulæ obtained for $N = 0$, as we shall need them further down:

$$a_0(0,n) = \frac{4(2n-1)(2n+3)+15}{4}, \qquad a_2(0,n) = \frac{1}{32(2n-1)(2n+3)}, \tag{6–166}$$

$$a_4(0,n) = \frac{5(2n-1)(2n+3)+48}{8{,}192(2n-3)(2n-1)^3(2n+3)^3(2n+5)}, \tag{6–167}$$

$$a_1(0,n) = \tfrac{1}{2}, \qquad a_3(0,n) = a_5(0,n) = 0. \tag{6–168}$$

Table 4. Combinations of n and N for which the indicated coefficient cannot be computed by means of its general formula.

a_1	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$n = 0$	•				
$n = 1$					
$n = 2$					

a_2	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$n = 0$	•	•			
$n = 1$					
$n = 2$					

a_3	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$n = 0$	•	•	•		
$n = 1$	•				
$n = 2$					

a_4	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$n = 0$	•	•	•	•	
$n = 1$	•	•			
$n = 2$					

a_5	$N = 0$	$N = 1$	$N = 2$	$N = 3$	$N = 4$
$n = 0$	•	•	•	•	•
$n = 1$	•	•	•		
$n = 2$	•				

Table 4 shows for which values of N the special formulæ referenced in Table 3 must be used. Out of the 22 combinations of N and n thus displayed, fully 19 involve $n = 0$ or $N = 0$. Since the corresponding special formulæ are all quite simple, it would make sense in computations to test for these conditions first. Otherwise, except in the three remaining cases of $a_4(1, 1)$, $a_5(1, 1)$ and $a_5(2, 1)$, the general formulæ can be used. We may thus list as follows the set of formulæ best suited to computations according to the foregoing scheme:

If $t \equiv (2n-1)(2n+3)$ and $\omega \equiv (N+2n)(N+2n+2)$, then

$$a_0 = \begin{cases} \dfrac{(2N+1)(2N+3)}{4} & \text{if } n=0, \\ \dfrac{4t+15}{4} & \text{if } N=0, \\ \dfrac{4\omega+3}{4} & \text{otherwise;} \end{cases} \qquad a_1 = \begin{cases} \dfrac{N+1}{N+2} & \text{if } n=0, \\ \dfrac{1}{2} & \text{if } N=0, \\ \dfrac{\omega+N^2}{2\omega} & \text{otherwise;} \end{cases} \tag{6–169}$$

$$a_2 = \begin{cases} -\dfrac{N+1}{4(N+2)^3(N+3)} & \text{if } n=0, \\ \dfrac{1}{32t} & \text{if } N=0, \\ \dfrac{\omega(\omega-N^2)(\omega-5N^2)+12N^4}{32\omega^3(\omega-3)} & \text{otherwise;} \end{cases} \tag{6–170}$$

$$a_3 = \begin{cases} -\dfrac{N(N+1)}{8(N+2)^5(N+3)(N+4)} & \text{if } n=0, \\ 0 & \text{if } N=0, \\ \dfrac{N^2\big\{\big[\omega^2-N^2(\omega+7)\big]\big[5\omega^2-N^2(9\omega+13)\big]+5N^4\big\}}{64\,\omega^5(\omega-3)(\omega-8)} & \text{otherwise;} \end{cases} \tag{6–171}$$

$$a_4 = \begin{cases} -\dfrac{(N+1)(5N^4+27N^3+17N^2-71N-44)}{64(N+2)^7(N+3)^3(N+4)(N+5)} & \text{if } n=0, \\ \dfrac{5t+48}{8{,}192\,t^3(t-12)} & \text{if } N=0, \\ -\dfrac{60{,}703}{122{,}472{,}000{,}000} & \text{if } n=N=1, \\ \dfrac{V(N,\omega)}{8,192\,\omega^7(\omega-3)^3(\omega-8)(\omega-15)} & \text{otherwise;} \end{cases} \tag{6–172}$$

$$a_5 = \begin{cases} -\dfrac{N(N+1)(7N^4+31N^3-59N^2-335N-214)}{128(N+2)^9(N+3)^3(N+4)(N+5)(N+6)} & \text{if } n=0, \\ 0 & \text{if } N=0, \\ -\dfrac{9{,}530{,}489}{275{,}562{,}000{,}000{,}000} & \text{if } n=N=1, \\ -\dfrac{767{,}729}{226{,}534{,}996{,}574{,}208} & \text{if } n=1,\ N=2, \\ \dfrac{N^2\,W(N,\omega)}{16{,}384\,\omega^9(\omega-3)^3(\omega-8)(\omega-15)(\omega-24)} & \text{otherwise.} \end{cases} \tag{6–173}$$

In Eqs. (6–172) and (6–173), V and W are as defined by Eqs. (6–114) and (6–142). Coefficients $a_4(1,1)$, $a_5(1,1)$ and $a_5(2,1)$ were evaluated by means of special formulæ (6–124) and (6–152)—see Eqs. (6–125) and (6–157).

Slepian has provided explicit expressions in terms of N and n for a_0 and a_1 $\big($[1], Eq. (38)$\big)$:

$$\left.\begin{aligned} a_0 &= \left(2n+N+\frac{1}{2}\right)\left(2n+N+\frac{3}{2}\right), \\ a_1 &= \frac{1}{2}\left[1+\frac{N^2}{(2n+N)(2n+N+2)}\right]. \end{aligned}\right\} \tag{6–174}$$

These results agree with our own [see Eqs. (6–2); (6–9) and (6–7)]. Heurtley, on the other hand, provided expressions for the same two coefficients plus a_2 and a_3, with n substituted for our N and m substituted for our $N+2n$ $\big($[2], Eqs. (18) and (20a)–(20d)$\big)$:

$$\left.\begin{aligned} a_0 &= m(m+2)+\frac{3}{4}, \\ a_1 &= \frac{1}{2}\,\frac{m(m+2)+n^2}{m(m+2)}, \\ a_2 &= \frac{1}{64}\left[\frac{(m-n)^2(m+n)^2}{(m-1)m^3(m+1)}-\frac{(m-n+2)^2(m+n+2)^2}{(m+1)(m+2)^2(m+3)}\right], \\ a_3 &= \frac{1}{128}\left[\frac{(m-n)^2(m+n)^2n^2}{(m-1)^2m^5(m+1)(m+2)}-\frac{(m-n+2)^2(m+n+2)^2n^2}{m(m+1)(m+2)^5(m+3)(m+4)}\right]. \end{aligned}\right\} \tag{6–175}$$

Conversion to our notation shows these expressions to be equivalent to our own [see Eqs. (6–15); (6–17); (6–10), (6–5), (6–33) and (6–35); (6–11), (6–5), (6–29), (6–33) and (6–35)]. However, a misprint appears to have slipped into the expression for a_2: $(m+2)^2$ should read $(m+2)^3$. Also, in the expression for a_3, $(m-1)^2$ should be replaced by $(m-2)(m-1)$. Note that neither Slepian's formulæ nor Heurtley's foresee instances where a denominator vanishes, which makes some of them non-computable in certain cases. All of our results regarding a_4 and a_5 seem to be entirely new.

To provide some idea of how, in this case of $\alpha=1$, coefficients $a_j(N,n)$ change with N and n, we list below several concrete instances of Eq. (6–1):

$$\chi_{0,0}(c)=\tfrac{3}{4}+\tfrac{1}{2}c^2-\tfrac{1}{96}c^4+\tfrac{11}{1{,}105{,}920}c^8+O(c^{12}). \tag{6–176}$$

$$\chi_{0,1}(c)=\tfrac{35}{4}+\tfrac{1}{2}c^2+\tfrac{1}{160}c^4-\tfrac{73}{7{,}168{,}000}c^8+O(c^{12}). \tag{6–177}$$

$$\chi_{0,2}(c)=\tfrac{99}{4}+\tfrac{1}{2}c^2+\tfrac{1}{672}c^4+\tfrac{17}{75{,}866{,}112}c^8+O(c^{12}). \tag{6–178}$$

$$\chi_{0,3}(c)=\tfrac{195}{4}+\tfrac{1}{2}c^2+\tfrac{1}{1{,}440}c^4+\tfrac{91}{8{,}211{,}456{,}000}c^8+O(c^{12}). \tag{6–179}$$

$$\chi_{0,4}(c)=\tfrac{323}{4}+\tfrac{1}{2}c^2+\tfrac{1}{2{,}464}c^4+\tfrac{433}{243{,}094{,}691{,}840}c^8+O(c^{12}). \tag{6–180}$$

$$\chi_{0,5}(c)=\tfrac{483}{4}+\tfrac{1}{2}c^2+\tfrac{1}{3{,}744}c^4+\tfrac{211}{459{,}214{,}479{,}360}c^8+O(c^{12}). \tag{6–181}$$

$$\chi_{0,6}(c)=\tfrac{675}{4}+\tfrac{1}{2}c^2+\tfrac{1}{5{,}280}c^4+\tfrac{97}{625{,}591{,}296{,}000}c^8+O(c^{12}). \tag{6–182}$$

$$\chi_{0,7}(c)=\tfrac{899}{4}+\tfrac{1}{2}c^2+\tfrac{1}{7{,}072}c^4+\tfrac{1{,}153}{18{,}480{,}471{,}646{,}208}c^8+O(c^{12}). \tag{6–183}$$

$$\chi_{1,0}(c) = \tfrac{15}{4} + \tfrac{2}{3}c^2 - \tfrac{1}{216}c^4 - \tfrac{1}{19{,}440}c^6 + \tfrac{11}{22{,}394{,}880}c^8 + \tfrac{19}{564{,}350{,}976}c^{10} + O(c^{12}). \quad (6\text{–}184)$$

$$\chi_{1,1}(c) = \tfrac{63}{4} + \tfrac{8}{15}c^2 + \tfrac{11}{6{,}750}c^4 + \tfrac{1{,}033}{21{,}262{,}500}c^6 - \tfrac{60{,}703}{122{,}472{,}000{,}000}c^8 - \tfrac{9{,}530{,}489}{275{,}562{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}185)$$

$$\chi_{1,2}(c) = \tfrac{143}{4} + \tfrac{18}{35}c^2 + \tfrac{279}{343{,}000}c^4 + \tfrac{5{,}953}{2{,}521{,}050{,}000}c^6 + \tfrac{171{,}697}{79{,}060{,}128{,}000{,}000}c^8 + \tfrac{1{,}360{,}341{,}083}{1{,}521{,}907{,}464{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}186)$$

$$\chi_{1,3}(c) = \tfrac{255}{4} + \tfrac{32}{63}c^2 + \tfrac{118}{250{,}047}c^4 + \tfrac{19{,}561}{54{,}584{,}009{,}865}c^6 + \tfrac{41{,}146{,}789}{27{,}730{,}423{,}699{,}735{,}680}c^8 + \tfrac{1{,}848{,}323{,}753}{81{,}760{,}381{,}236{,}300{,}678{,}912}c^{10} + O(c^{12}). \quad (6\text{–}187)$$

$$\chi_{1,4}(c) = \tfrac{399}{4} + \tfrac{50}{99}c^2 + \tfrac{2{,}375}{7{,}762{,}392}c^4 + \tfrac{1{,}214{,}425}{13{,}846{,}415{,}126{,}544}c^6 + \tfrac{8{,}412{,}546{,}925}{17{,}370{,}715{,}475{,}872{,}991{,}232}c^8 + \tfrac{635{,}707{,}009{,}471}{340{,}500{,}764{,}758{,}062{,}374{,}129{,}664}c^{10} + O(c^{12}). \quad (6\text{–}188)$$

$$\chi_{1,5}(c) = \tfrac{575}{4} + \tfrac{72}{143}c^2 + \tfrac{1{,}251}{5{,}848{,}414}c^4 + \tfrac{101{,}641}{3{,}587{,}826{,}536{,}580}c^6 + \tfrac{1{,}661{,}119{,}589}{9{,}391{,}035{,}500{,}355{,}125{,}760}c^8 + \tfrac{2{,}425{,}806{,}503{,}821}{9{,}140{,}974{,}763{,}465{,}869{,}613{,}313{,}024}c^{10} + O(c^{12}). \quad (6\text{–}189)$$

$$\chi_{1,6}(c) = \tfrac{783}{4} + \tfrac{98}{195}c^2 + \tfrac{9{,}359}{59{,}319{,}000}c^4 + \tfrac{9{,}276{,}337}{843{,}596{,}260{,}650{,}000}c^6 + \tfrac{60{,}227{,}359{,}073}{821{,}190{,}343{,}967{,}136{,}000{,}000}c^8 + \tfrac{313{,}279{,}853{,}583{,}781}{5{,}932{,}894{,}937{,}576{,}565{,}816{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}190)$$

$$\chi_{1,7}(c) = \tfrac{1{,}023}{4} + \tfrac{128}{255}c^2 + \tfrac{2{,}008}{16{,}581{,}375}c^4 + \tfrac{1{,}296{,}292}{266{,}316{,}365{,}615{,}625}c^6 + \tfrac{3{,}745{,}170{,}977}{110{,}830{,}218{,}714{,}598{,}500{,}000}c^8 + \tfrac{73{,}618{,}409{,}908{,}187}{5{,}549{,}185{,}928{,}375{,}910{,}946{,}125{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}191)$$

$$\chi_{2,0}(c) = \tfrac{35}{4} + \tfrac{3}{4}c^2 - \tfrac{3}{1{,}280}c^4 - \tfrac{1}{40{,}960}c^6 - \tfrac{89}{917{,}504{,}000}c^8 + \tfrac{19}{5{,}872{,}025{,}600}c^{10} + O(c^{12}). \quad (6\text{–}192)$$

$$\chi_{2,1}(c) = \tfrac{99}{4} + \tfrac{7}{12}c^2 + \tfrac{11}{48{,}384}c^4 + \tfrac{289}{13{,}934{,}592}c^6 + \tfrac{83{,}935}{786{,}579{,}849{,}216}c^8 - \tfrac{767{,}729}{226{,}534{,}996{,}574{,}208}c^{10} + O(c^{12}). \quad (6\text{–}193)$$

$$\chi_{2,2}(c) = \tfrac{195}{4} + \tfrac{13}{24}c^2 + \tfrac{103}{276{,}480}c^4 + \tfrac{881}{318{,}504{,}960}c^6 - \tfrac{3{,}476{,}183}{403{,}609{,}485{,}312{,}000}c^8 + \tfrac{12{,}779{,}873}{92{,}991{,}625{,}415{,}884{,}800}c^{10} + O(c^{12}). \quad (6\text{–}194)$$

$$\chi_{2,3}(c) = \tfrac{323}{4} + \tfrac{21}{40}c^2 + \tfrac{5{,}703}{19{,}712{,}000}c^4 + \tfrac{38{,}177}{63{,}078{,}400{,}000}c^6 - \tfrac{12{,}822{,}865{,}789}{12{,}446{,}448{,}222{,}208{,}000{,}000}c^8 + \tfrac{2{,}690{,}850{,}273{,}443}{199{,}143{,}171{,}555{,}328{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}195)$$

$$\chi_{3,0}(c) = \tfrac{63}{4} + \tfrac{4}{5}c^2 - \tfrac{1}{750}c^4 - \tfrac{1}{87{,}500}c^6 - \tfrac{103}{1{,}512{,}000{,}000}c^8 + \tfrac{173}{1{,}134{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}196)$$

$$\chi_{3,1}(c) = \tfrac{143}{4} + \tfrac{22}{35}c^2 - \tfrac{127}{686{,}000}c^4 + \tfrac{41{,}999}{5{,}042{,}100{,}000}c^6 + \tfrac{15{,}105{,}527}{210{,}827{,}008{,}000{,}000}c^8 - \tfrac{1{,}895{,}547{,}089}{12{,}175{,}259{,}712{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}197)$$

$$\chi_{3,2}(c) = \tfrac{255}{4} + \tfrac{4}{7}c^2 + \tfrac{2}{15{,}435}c^4 + \tfrac{157}{74{,}875{,}185}c^6 - \tfrac{246{,}779}{105{,}663{,}861{,}072{,}000}c^8 - \tfrac{7{,}109}{3{,}146{,}861{,}898{,}835{,}200}c^{10} + O(c^{12}). \quad (6\text{–}198)$$

$$\chi_{3,3}(c) = \tfrac{399}{4} + \tfrac{6}{11}c^2 + \tfrac{31}{191{,}664}c^4 + \tfrac{23{,}249}{37{,}987{,}421{,}472}c^6 - \tfrac{18{,}931{,}105}{21{,}180{,}570{,}615{,}300{,}096}c^8 + \tfrac{183{,}358{,}949}{46{,}131{,}282{,}800{,}123{,}609{,}088}c^{10} + O(c^{12}). \quad (6\text{–}199)$$

$$\chi_{4,4}(c) = \tfrac{675}{4} + \tfrac{23}{42}c^2 + \tfrac{293}{3{,}259{,}872}c^4 + \tfrac{48{,}299}{230{,}016{,}568{,}320}c^6 - \tfrac{295{,}247{,}627}{1{,}669{,}252{,}317{,}888{,}798{,}720}c^8 + \tfrac{255{,}776{,}363}{673{,}042{,}534{,}572{,}763{,}643{,}904}c^{10} + O(c^{12}). \quad (6\text{–}200)$$

$$\chi_{5,5}(c) = \tfrac{1{,}023}{4} + \tfrac{28}{51}c^2 + \tfrac{53}{928{,}557}c^4 + \tfrac{108{,}595}{1{,}193{,}097{,}317{,}958}c^6 - \tfrac{605{,}370{,}083}{12{,}164{,}724{,}806{,}114{,}331{,}360}c^8 + \tfrac{8{,}831{,}182{,}561}{139{,}217{,}976{,}571{,}094{,}853{,}816{,}384}c^{10} + O(c^{12}). \quad (6\text{–}201)$$

$$\chi_{6,6}(c) = \tfrac{1{,}443}{4} + \tfrac{11}{20}c^2 + \tfrac{193}{4{,}896{,}000}c^4 + \tfrac{17{,}737}{387{,}763{,}200{,}000}c^6 - \tfrac{260{,}127{,}179}{14{,}846{,}181{,}064{,}704{,}000{,}000}c^8 + \tfrac{55{,}431{,}528{,}071}{3{,}741{,}237{,}628{,}305{,}408{,}000{,}000{,}000}c^{10} + O(c^{12}). \quad (6\text{–}202)$$

$$\chi_{7,7}(c) = \tfrac{1{,}935}{4} + \tfrac{38}{69}c^2 + \tfrac{379}{13{,}140{,}360}c^4 + \tfrac{60{,}557}{2{,}377{,}327{,}650{,}480}c^6 - \tfrac{3{,}395{,}526{,}343}{470{,}847{,}808{,}867{,}707{,}648{,}000}c^8 + \tfrac{66{,}223{,}121{,}111}{15{,}243{,}603{,}642{,}530{,}261{,}562{,}470{,}400}c^{10} + O(c^{12}). \quad (6\text{–}203)$$

Finally, as we did at the end of the previous section, we consider approximations of $\chi_{N,n}$ of successive orders p, that is,

$$\chi_{N,n}(c) \approx \sum_{j=0}^{p} a_j(N,n)\, c^{2j}, \qquad p = 0, 1, \ldots, 5. \quad (6\text{–}204)$$

Table 5 lists such numbers for $N = n = 0$, $c = \frac{1}{2}$ and $c = 2$. The approximations of orders 3 and 5 are not shown because they are the same as those of orders 2 and 4, due to a_3 and a_5 being zero when $N = 0$. Again, for comparison we provide values exact to fifteen significant digits, as computed using Bouwkamp's scheme. With the smaller value of c the fourth-order approximation is accurate up to the eleventh significant digit, while with the larger value it is accurate only up to the fourth significant digit. As a general rule the accuracy of the approximation is found to rise/drop according as c decreases/increases. Thus, there exists a higher threshold of c above which all accuracy is lost. Conversely, a lower threshold can always be found below which a specified accuracy will obtain. *Both these thresholds increase with N and n*, as Table 6 illustrates for $N = 2$ and $n = 3$: now, with $c = 2$ the fifth-order approximation produces eleven digits of accuracy: with $c = 5$, seven. Further tests show that for $N = 20$ and $n = 50$, the accuracy of the fifth-order approximation at $c = 20$ reads fourteen digits; by $c = 80$ it is still seven digits ($\chi \approx 18{,}004.05$).

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Table 5. Progression of successive approximations of $\chi_{0,0}(c)$ toward exact value — $c = \frac{1}{2}, 2$.

Approx.	$\chi_{0,0}(\frac{1}{2})$	$\chi_{0,0}(2)$
$p = 0$	$7.50000000000000 \times 10^{-1}$	$7.50000000000000 \times 10^{-1}$
$p = 1$	$8.75000000000000 \times 10^{-1}$	$2.75000000000000 \times 10^{0}$
$p = 2$	$8.74348958333333 \times 10^{-1}$	$2.58333333333333 \times 10^{0}$
$p = 4$	$8.74348997186731 \times 10^{-1}$	$2.58587962962963 \times 10^{0}$
exact	$8.74348997181586 \times 10^{-1}$	$2.58579682607078 \times 10^{0}$

Table 6. Progression of successive approximations of $\chi_{2,3}(c)$ toward exact value — $c = 2, 5$.

Approx.	$\chi_{2,3}(2)$	$\chi_{2,3}(5)$
$p = 0$	$8.07500000000000 \times 10^{1}$	$8.07500000000000 \times 10^{1}$
$p = 1$	$8.28500000000000 \times 10^{1}$	$9.38750000000000 \times 10^{1}$
$p = 2$	$8.28546290584416 \times 10^{1}$	$9.40558225953734 \times 10^{1}$
$p = 3$	$8.28546677932224 \times 10^{1}$	$9.40652793289779 \times 10^{1}$
$p = 4$	$8.28546675294802 \times 10^{1}$	$9.40648768903175 \times 10^{1}$
$p = 5$	$8.28546675433166 \times 10^{1}$	$9.40650088448028 \times 10^{1}$
exact	$8.28546675432684 \times 10^{1}$	$9.40650073818606 \times 10^{1}$

7 Algorithm for numerical determination of perturbation coefficients

In this section we turn the recursive formulæ of Slepian's perturbation scheme into an algorithm for the automatic computation of coefficients $a_j(N, n)$. We recall from Sections 3 and 4 that two sets of coefficients may be computed, according as c is considered "small" or "large"—a choice governed by parameter α, with values of 1 and 2 respectively. The first two coefficients have simple expressions: $a_0 = \lambda_n$, $a_1 = \gamma_n^0$, with the definitions of λ_n and γ_j^k given by Eqs. (6–2) and (6–6)–(6–8) if $\alpha = 1$, and by Eqs. (5–2) and (5–7)–(5–9) if $\alpha = 2$. Subsequent coefficients—a_2, a_3, ...—result from formula (4–6), which we duplicate below for the sake of convenience:

$$a_j = A_1^{j-1}\gamma_{n+\alpha}^{-1} + A_{-1}^{j-1}\gamma_{n-\alpha}^{1}, \qquad j = 2, 3, \ldots. \tag{7–1}$$

Each quantity A_q^p must first be tested for special conditions (4–4), which we also repeat:

$$A_q^0 = A_0^q = \delta_{q0}, \tag{7–2a}$$

$$A_q^p = 0 \quad \text{if} \quad |q| > p, \tag{7–2b}$$

$$A_q^p = 0 \quad \text{if} \quad n < -\alpha q. \tag{7–2c}$$

When none of these conditions obtains, A_q^p must be calculated by means of formula (4–7):

$$A_q^p = h_{\alpha q}\left[\sum_{k=1}^{p} a_k\, A_q^{p-k} - \sum_{k=-1}^{1} A_{q-k}^{p-1}\gamma_{n+\alpha(q-k)}^{k}\right]. \tag{7–3}$$

As we remarked earlier, this formula eventually becomes recursive; more precisely, it presupposes the availability of another quantity $A_{q'}^{p'}$ which itself requires application of the formula. As the order j of coefficient a_j increases this "nesting" becomes longer; however, it does end at some point, and it has a predictable structure. We shall infer the latter by exhibiting the recursions for the first several coefficients. Then we shall use this knowledge as the basis for our algorithm.

Eq. (7–1) states that, ultimately, every coefficient a_j depends on A_1^{j-1} and A_{-1}^{j-1}—a fact we may condense in the following notation:

$$a_j\big(A_1^{j-1}, A_{-1}^{j-1}\big). \tag{7–4}$$

Eq. (7–3) makes a similar statement regarding A_q^p. Indeed, let us expand the two summations therein:

$$\sum_{k=-1}^{1} A_{q-k}^{p-1}\gamma_{n+\alpha(q-k)}^{k} = A_{q+1}^{p-1}\gamma_{n+\alpha(q+1)}^{-1} + A_{q-1}^{p-1}\gamma_{n+\alpha(q-1)}^{1} + A_q^{p-1}\gamma_{n+\alpha q}^{0}, \tag{7–5a}$$

$$\sum_{k=1}^{p} a_k\, A_q^{p-k} = a_1 A_q^{p-1} + a_2 A_q^{p-2} + \cdots + a_{p-1}A_q^1 + a_p A_q^0. \tag{7–5b}$$

Altogether, then,

$$A_q^p\big(A_{q+1}^{p-1}, A_{q-1}^{p-1}, A_q^{p-1}, A_q^{p-2}, \ldots, A_q^0\big). \tag{7–6}$$

Any quantity $A_{q'}^{p'}$ on which A_q^p depends will either satisfy one of conditions (7–2), or it will already have been computed, or, failing these two alternatives, it will require a new application of formula (7–3)—yielding another statement like (7–6). Thus, using such statements, one can track the recursions of formula (7–3) for every coefficient a_j. Let us do this for $j = 2, 3, \ldots, 8$:

To compute $a_2\big(A_1^1, A_{-1}^1\big)$ we require computation of

$$A_1^1\big(A_2^0, A_0^0, A_1^0\big) \tag{7–7a}$$

and

$$A_{-1}^1\big(A_0^0, A_{-2}^0, A_{-1}^0\big). \tag{7–7b}$$

To compute $a_3\big(A_1^2, A_{-1}^2\big)$ we require computation of

$$A_1^2\big(A_2^1, A_0^1, A_1^1, A_1^0\big) \tag{7–8a}$$

and

$$A_{-1}^2\big(A_0^1, A_{-2}^1, A_{-1}^1, A_{-1}^0\big). \tag{7–8b}$$

To compute $a_4\big(A_1^3, A_{-1}^3\big)$ we require computation of

$$\left.\begin{array}{l} A_1^3\big(A_2^2, A_0^2, A_1^2, A_1^1, A_1^0\big) \\ A_2^2\big(A_3^1, A_1^1, A_2^1, A_2^0\big) \end{array}\right\} \tag{7–9a}$$

and

$$\left.\begin{array}{l} A_{-1}^3\big(A_0^2, A_{-2}^2, A_{-1}^2, A_{-1}^1, A_{-1}^0\big) \\ A_{-2}^2\big(A_{-1}^1, A_{-3}^1, A_{-2}^1, A_{-2}^0\big). \end{array}\right\} \tag{7–9b}$$

To compute $a_5\big(A_1^4, A_{-1}^4\big)$ we require computation of

$$\left.\begin{array}{l} A_1^4\big(A_2^3, A_0^3, A_1^3, A_1^2, A_1^1, A_1^0\big) \\ A_2^3\big(A_3^2, A_1^2, A_2^2, A_2^1, A_2^0\big) \end{array}\right\} \tag{7–10a}$$

and

$$\left.\begin{array}{l} A_{-1}^4\big(A_0^3, A_{-2}^3, A_{-1}^3, A_{-1}^2, A_{-1}^1, A_{-1}^0\big) \\ A_{-2}^3\big(A_{-1}^2, A_{-3}^2, A_{-2}^2, A_{-2}^1, A_{-2}^0\big). \end{array}\right\} \tag{7–10b}$$

To compute $a_6\big(A_1^5, A_{-1}^5\big)$ we require computation of

$$\left.\begin{array}{l} A_1^5\big(A_2^4, A_0^4, A_1^4, A_1^3, A_1^2, A_1^1, A_1^0\big) \\ A_2^4\big(A_3^3, A_1^3, A_2^3, A_2^2, A_2^1, A_2^0\big) \\ A_3^3\big(A_4^2, A_2^2, A_3^2, A_3^1, A_3^0\big) \end{array}\right\} \tag{7–11a}$$

and

$$\left.\begin{array}{l} A_{-1}^5\big(A_0^4, A_{-2}^4, A_{-1}^4, A_{-1}^3, A_{-1}^2, A_{-1}^1, A_{-1}^0\big) \\ A_{-2}^4\big(A_{-1}^3, A_{-3}^3, A_{-2}^3, A_{-2}^2, A_{-2}^1, A_{-2}^0\big) \\ A_{-3}^3\big(A_{-2}^2, A_{-4}^2, A_{-3}^2, A_{-3}^1, A_{-3}^0\big). \end{array}\right\} \tag{7–11b}$$

To compute $a_7\big(A_1^6, A_{-1}^6\big)$ we require computation of

$$\left.\begin{aligned}&A_1^6\big(A_2^5, A_0^5, A_1^5, A_1^4, A_1^3, A_1^2, A_1^1, A_1^0\big)\\&A_2^5\big(A_3^4, A_1^4, A_2^4, A_2^3, A_2^2, A_2^1, A_2^0\big)\\&A_3^4\big(A_4^3, A_2^3, A_3^3, A_3^2, A_3^1, A_3^0\big)\end{aligned}\right\} \tag{7–12a}$$

and

$$\left.\begin{aligned}&A_{-1}^6\big(A_0^5, A_{-2}^5, A_{-1}^5, A_{-1}^4, A_{-1}^3, A_{-1}^2, A_{-1}^1, A_{-1}^0\big)\\&A_{-2}^5\big(A_{-1}^4, A_{-3}^4, A_{-2}^4, A_{-2}^3, A_{-2}^2, A_{-2}^1, A_{-2}^0\big)\\&A_{-3}^4\big(A_{-2}^3, A_{-4}^3, A_{-3}^3, A_{-3}^2, A_{-3}^1, A_{-3}^0\big).\end{aligned}\right\} \tag{7–12b}$$

To compute $a_8\big(A_1^7, A_{-1}^7\big)$ we require computation of

$$\left.\begin{aligned}&A_1^7\big(A_2^6, A_0^6, A_1^6, A_1^5, A_1^4, A_1^3, A_1^2, A_1^1, A_1^0\big)\\&A_2^6\big(A_3^5, A_1^5, A_2^5, A_2^4, A_2^3, A_2^2, A_2^1, A_2^0\big)\\&A_3^5\big(A_4^4, A_2^4, A_3^4, A_3^3, A_3^2, A_3^1, A_3^0\big)\\&A_4^4\big(A_5^3, A_3^3, A_4^3, A_4^2, A_4^1, A_4^0\big)\end{aligned}\right\} \tag{7–13a}$$

and

$$\left.\begin{aligned}&A_{-1}^7\big(A_0^6, A_{-2}^6, A_{-1}^6, A_{-1}^5, A_{-1}^4, A_{-1}^3, A_{-1}^2, A_{-1}^1, A_{-1}^0\big)\\&A_{-2}^6\big(A_{-1}^5, A_{-3}^5, A_{-2}^5, A_{-2}^4, A_{-2}^3, A_{-2}^2, A_{-2}^1, A_{-2}^0\big)\\&A_{-3}^5\big(A_{-2}^4, A_{-4}^4, A_{-3}^4, A_{-3}^3, A_{-3}^2, A_{-3}^1, A_{-3}^0\big)\\&A_{-4}^4\big(A_{-3}^3, A_{-5}^3, A_{-4}^3, A_{-4}^2, A_{-4}^1, A_{-4}^0\big).\end{aligned}\right\} \tag{7–13b}$$

From the above we infer that *recursion will be avoided* if the required quantities A_q^p are computed in this sequence:

$$\left.\begin{aligned}
&\left.\begin{matrix}A_1^1\\A_{-1}^1\end{matrix}\right\}\to a_2, &&\left.\begin{matrix}A_3^3 \to A_2^4 \to A_1^5\\A_{-3}^3 \to A_{-2}^4 \to A_{-1}^5\end{matrix}\right\}\to a_6,\\
&\left.\begin{matrix}A_1^2\\A_{-1}^2\end{matrix}\right\}\to a_3, &&\left.\begin{matrix}A_3^4 \to A_2^5 \to A_1^6\\A_{-3}^4 \to A_{-2}^5 \to A_{-1}^6\end{matrix}\right\}\to a_7,\\
&\left.\begin{matrix}A_2^2 \to A_1^3\\A_{-2}^2 \to A_{-1}^3\end{matrix}\right\}\to a_4, &&\left.\begin{matrix}A_4^4 \to A_3^5 \to A_2^6 \to A_1^7\\A_{-4}^4 \to A_{-3}^5 \to A_{-2}^6 \to A_{-1}^7\end{matrix}\right\}\to a_8.\\
&\left.\begin{matrix}A_2^3 \to A_1^4\\A_{-2}^3 \to A_{-1}^4\end{matrix}\right\}\to a_5, &&
\end{aligned}\right\} \tag{7–14}$$

More generally, the appropriate sequence may be written as follows:

$$\left.\begin{matrix}A_v^u \to A_{v-1}^{u+1} \to A_{v-2}^{u+2} \to \ldots \to A_1^{j-1}\\A_{-v}^u \to A_{-(v-1)}^{u+1} \to A_{-(v-2)}^{u+2} \to \ldots \to A_{-1}^{j-1}\end{matrix}\right\}\to a_j, \quad j = 2, 3, \ldots, \tag{7–15}$$

with

$$u = \mathrm{INT}\left(\frac{j+1}{2}\right) = \begin{cases} j/2 & \text{if } j \text{ even,} \\ (j+1)/2 & \text{if } j \text{ odd,} \end{cases} \tag{7–16}$$

and

$$v = \mathrm{INT}\left(\frac{j}{2}\right) \qquad = \begin{cases} j/2 & \text{if } j \text{ even,} \\ (j-1)/2 & \text{if } j \text{ odd,} \end{cases} \tag{7–17}$$

where $\mathrm{INT}(x)$ denotes the integer part of x.

We may now proceed to devise an algorithm for computing all coefficients a_j from $j = 0$ up to some j_{end}. We take α, N, n and j_{end} as givens. We start by computing $a_0 = \lambda_n$ and $a_1 = \gamma_n^0$; these two numbers are stored in some 1-d, real array assigned to coefficients a_j. Then, for $j = 2, 3, \ldots, j_{\text{end}}$, we compute by means of formula (7–3), and store, all quantities A_q^p with index pairs (p, q) formed from the following two sequences, in accordance with Eq. (7–15):

$$p = u,\, u+1,\, u+2,\, \ldots,\, j-1; \qquad q = sv,\, s(v-1),\, s(v-2),\, \ldots,\, s\cdot 1. \tag{7–18}$$

Here u and v are as in Eqs. (7–16) and (7–17), and s takes on the values 1 and -1 successively. Note that on account of condition (7–2c), A_q^p is computed only if $n \geq -\alpha q$. Also, quantity A_q^0 in Eq. (7–5b) will always have $q \neq 0$,[8] so that by virtue of condition (7–2a) it will always be zero; therefore, a step can be saved by initializing the first sum in Eq. (7–3) to zero and performing it only if $p > 1$, for k up to $p - 1$:

$$\sum_{k=1}^{p} a_k\, A_q^{p-k} = \begin{cases} 0 & \text{if } p = 1, \\ \displaystyle\sum_{k=1}^{p-1} a_k\, A_q^{p-k} & \text{if } p > 1. \end{cases} \tag{7–19}$$

Once A_1^{j-1} and A_{-1}^{j-1} have become available, a_j is computed as per Eq. (7–1), then stored, and the whole procedure is repeated for the next value of j.

An important feature of our algorithm is that only those quantities A_q^p which meet none of conditions (7–2) are stored. Therefore, all references to A_q^p in formulæ (7–3) and (7–1) must be understood as entailing *first* a test with respect to conditions (7–2); only if none of the latter are satisfied should A_q^p be retrieved from memory. Furthermore, only such a stored, presumably nonzero A_q^p, and $A_0^0 = 1$, need enter the computations.

The set of computed A_q^p can be stored as two successive column vectors according to the following sequence:

This space intentionally left blank.

[8] See the rightmost element in Eqs. (7–7)–(7–13).

$$
\left.\begin{aligned}
&A_1^1, A_1^2, A_1^3, A_1^4, A_1^5, A_1^6, A_1^7, \ldots, A_1^{j_{\text{end}}-1},\\
&A_2^2, A_2^3, A_2^4, A_2^5, A_2^6, \ldots, A_2^{j_{\text{end}}-2},\\
&A_3^3, A_3^4, A_3^5, \ldots, A_3^{j_{\text{end}}-3},\\
&\ldots,\\
&A_{\text{INT}(j_{\text{end}}/2)}^{\text{INT}[(j_{\text{end}}+1)/2]},\\
&A_{-1}^1, A_{-1}^2, A_{-1}^3, A_{-1}^4, A_{-1}^5, A_{-1}^6, A_{-1}^7, \ldots, A_{-1}^{j_{\text{end}}-1},\\
&A_{-2}^2, A_{-2}^3, A_{-2}^4, A_{-2}^5, A_{-2}^6, \ldots, A_{-2}^{j_{\text{end}}-2},\\
&A_{-3}^3, A_{-3}^4, A_{-3}^5, \ldots, A_{-3}^{j_{\text{end}}-3},\\
&\ldots,\\
&A_{-\text{INT}(j_{\text{end}}/2)}^{\text{INT}[(j_{\text{end}}+1)/2]}.
\end{aligned}\right\} \qquad (7\text{–}20)
$$

It can be shown that each vector contains $T = \text{INT}\big(j_{\text{end}}^2/4\big)$ elements. Thus, the minimal storage requirement is a 2-d, real array—say, F—dimensioned as $F(T, 2)$. It can also be shown that indices $i \in (1, 2, \ldots, T)$ and $j \in (1, 2)$ locating A_q^p in such an array, as per (7–20), are

$$
i = (|q| - 1)(j_{\text{end}} - |q|) + p, \qquad j = \begin{cases} 1 & \text{if } q > 0, \\ 2 & \text{if } q < 0. \end{cases} \qquad (7\text{–}21)
$$

The algorithm described above may be expressed in pseudocode as follows:

<u>To compute a_j, $j = 0, 1, \ldots, j_{\text{end}}$ (**input:** α, N, n, j_{end}):</u>

$$
\left.\begin{aligned}
&z \leftarrow N + 2n + 1;\\
&\text{store } a_0 \leftarrow \lambda_n; \quad \text{if } j_{\text{end}} = 0\text{: stop};\\
&\text{store } a_1 \leftarrow \gamma_n^0; \quad \text{if } j_{\text{end}} = 1\text{: stop};\\
&\text{for } j = 2, 3, \ldots, j_{\text{end}}:\\
&\quad \text{for } s = 1, -1:\\
&\quad\quad u \leftarrow \text{INT}[(j+1)/2]; \quad v \leftarrow \text{INT}(j/2);\\
&\quad\quad \text{while } v > 0:\\
&\quad\quad\quad p \leftarrow u; \quad q \leftarrow sv;\\
&\quad\quad\quad \text{if } n \geq -\alpha q:\\
&\quad\quad\quad\quad X \leftarrow 0.0;\\
&\quad\quad\quad\quad \text{if } p > 1\text{: for } k = 1, 2, \ldots, p-1\text{: } X \leftarrow X + A_q^{p-k} a_k;\\
&\quad\quad\quad\quad \text{for } k = -1, 0, 1\text{: } X \leftarrow X - A_{q-k}^{p-1} \gamma_{n+\alpha(q-k)}^k;\\
&\quad\quad\quad\quad \text{store } A_q^p \equiv Y \leftarrow X / h_{\alpha q};\\
&\quad\quad\quad u \leftarrow u + 1; \quad v \leftarrow v - 1;\\
&\quad \text{store } a_j \leftarrow A_1^{j-1} \gamma_{n+\alpha}^{-1} + A_{-1}^{j-1} \gamma_{n-\alpha}^1;\\
&\text{stop};
\end{aligned}\right\} \qquad (7\text{–}22)
$$

Quantity z is required for computing λ_n and h_j, the latter being the reciprocal of h_j as defined by Eq. (4–9) (which avoids an unnecessary multiply):

To compute λ_n:

$$\left.\begin{array}{l} \lambda_n \leftarrow 2z\,; \\ \text{if } \alpha = 1\text{: } \lambda_n \leftarrow (\lambda_n^2 - 1)/4\,; \end{array}\right\} \tag{7–23}$$

To compute h_j:

$$\left.\begin{array}{l} h_j \leftarrow 4j\,; \\ \text{if } \alpha = 1\text{: } h_j \leftarrow h_j(z + j)\,; \end{array}\right\} \tag{7–24}$$

Factor γ_j^k is computed in accordance with the expressions given at the beginning of Sections 5 and 6:

To compute γ_j^k (**input:** α, N, k, j):

$$\left.\begin{array}{lll}
\text{if } \alpha = 1\text{:} & & \\
\quad \text{if } k = 1\text{:} & & \\
\quad\quad \text{if } j \geq 0\text{:} & g \leftarrow -\dfrac{(N+j+1)^2}{(N+2j+1)(N+2j+2)}\,; & \\
\quad\quad \text{else:} & g \leftarrow 0.0\,; & \\
\quad \text{else if } k = 0\text{:} & & \\
\quad\quad \text{if } j > 0\text{:} & g \leftarrow \dfrac{(N+j)(N+j+1) + j(j+1)}{(N+2j)(N+2j+2)}\,; & \\
\quad\quad \text{else if } j = 0\text{:} & g \leftarrow (N+1)/(N+2)\,; & \\
\quad\quad \text{else:} & g \leftarrow 0.0\,; & \\
\quad \text{else:} & & \\
\quad\quad \text{if } j \geq 1\text{:} & g \leftarrow -\dfrac{j^2}{(N+2j)(N+2j+1)}\,; & \\
\quad\quad \text{else:} & g \leftarrow 0.0\,; & \\
\text{else:} & & \\
\quad \text{if } k = 1\text{:} & g \leftarrow (j+1)(j+2)\,; & \\
\quad \text{else if } k = 0\text{:} & g \leftarrow -\dfrac{8j\,(N+j+1) + 4N + 5}{4}\,; & \\
\quad \text{else:} & g \leftarrow (N+j)(N+j-1)\,; & \\
\text{return } g\,; & &
\end{array}\right\} \tag{7–25}$$

Storage, testing and retrieval of A_q^p, in accordance with the guidelines supplied above, proceed as follows:

To store $A_q^p \equiv Y$ into array F (**input:** j_{end}, p, q, Y):

$$\left.\begin{array}{ll} i \leftarrow (|q|-1)(j_{\text{end}} - |q|) + p\,; & \\ \text{if } q > 0\text{:} & F(i,1) \leftarrow Y\,; \\ \text{else:} & F(i,2) \leftarrow Y\,; \\ \text{return}\,; & \end{array}\right\} \tag{7–26}$$

To test A_q^p for special cases;
if appropriate, to retrieve it from array F;
and, to store its value into scalar Y
(**input:** α, n, j_{end}, p, q; **output:** *zero*, Y):

$$\left.\begin{array}{l} \text{if } (p = 0 \text{ and } q \neq 0) \text{ or} \\ \quad (q = 0 \text{ and } p \neq 0) \text{ or} \\ \quad (|q| > p) \quad\quad \text{or} \\ \quad (n < -\alpha q)\text{:} \quad\quad Y \leftarrow 0.0\,; \quad \mathit{zero} \leftarrow \texttt{.T.}\,; \\ \text{else if } (p = 0 \text{ and } q = 0)\text{:} \ Y \leftarrow 1.0\,; \quad \mathit{zero} \leftarrow \texttt{.F.}\,; \\ \text{else:} \\ \quad\quad i \leftarrow (|q|-1)(j_{\text{end}} - |q|) + p\,; \\ \quad\quad \text{if } q > 0\text{:} \ Y \leftarrow F(i,1)\,; \\ \quad\quad \text{else:} \quad\ Y \leftarrow F(i,2)\,; \\ \quad\quad \mathit{zero} \leftarrow \texttt{.F.}\,; \\ \text{return}\,; \end{array}\right\} \tag{7–27}$$

Logical scalar *zero* provides a switch for deciding whether A_q^p is zero (`.T.`) and therefore to be disregarded, or not zero (`.F.`) and therefore to be taken into account by the calling module. In the latter case real scalar Y will contain either $A_0^0 = 1$ or else the previously computed value of A_q^p.

A `FORTRAN` subroutine was written to implement the foregoing algorithm. Spotchecks revealed that it duplicates, essentially, the coefficient values obtained from the analytical formulæ. We computed the two sets of coefficients $a_0, \ldots, a_5$—for the "small-c" ($\alpha = 1$) and the "large-c" ($\alpha = 2$) perturbation series—both ways—analytically and numerically—to a precision of fifteen digits, for all combinations of N and $n \in (0, 1, 2, 3, 7)$. No difference could be found between the coefficients of type $\alpha = 2$. On the other hand small deviations, typically in the last one or two digits, occurred in about one out of three coefficients of type $\alpha = 1$, which became slightly more apparent as $N + n$ increased. We attribute these deviations to the algorithm then having to perform subtractions of small fractions, whereas such operations were removed from the analytical formulæ.[9] As N and n increase the operand fractions become smaller and smaller, so it is not suprising that the observed deviations should display a slight, accompanying rise in amplitude and frequency of occurrence.

[9] Indeed, every formula amounts to an exact ratio of two integers generated through integer calculus.

In any case, these discrepancies are minute; they can be reduced to any desired level by increasing the precision of the computations; and they should not obscure the fact that use of the algorithm has validated the calculations of Sections 5 and 6.

Our subroutine can, of course, compute coefficients of orders beyond 5. It is then seen that successive coefficients of type $\alpha = 1$ decrease monotonically in size by one or more orders of magnitude; moreover, the decrease accelerates if N or n is raised. Conversely, successive coefficients of type $\alpha = 2$ are seen to *increase* monotonically, in the same, accelerating manner. Thus, the ultimate coefficient order attainable with a given computer is determined by the latter's under/overflow condition—and certainly not by storage requirements. We were able to compute coefficients up to order 30 without any problem.

It is known from previous work [7] that both perturbation series start to diverge as soon as the ratio of the current term to the current partial sum stops decreasing. This condition may be expressed symbolically as follows: The current partial sum is, by definition,

$$X_p = \sum_{j=0}^{p} a_j(N,n)\,\epsilon^{\,j-\alpha+1}, \qquad p = 0, 1, 2, \ldots, \tag{7–28}$$

where $\epsilon = c^2$ if $\alpha = 1$ and $\epsilon = 1/c$ if $\alpha = 2$.[10] We track successive ratios

$$R_p = \left| \frac{a_p\,\epsilon^{\,p-\alpha+1}}{X_p} \right|, \qquad p = 0, 1, 2, \ldots. \tag{7–29}$$

Computations show that both series converge as long as $R_p < R_{p-1}$,[11] and start to diverge when the latter condition becomes untrue. We used this criterion to define the ultimate order, k, to which any chosen eigenvalue could be approximated, given coefficients up to order 30. Tables 7 and 8 show the resulting approximations of the same eigenvalues we considered at the end of Section 5 ($\alpha = 2$). Tables 9 and 10 do likewise for the eigenvalues considered at the end of Section 6 ($\alpha = 1$). In all cases except $\chi_{0,0}(10)$ the series turned out to converge all the way up to order 30. Substantial gains in accuracy are seen to have accrued, reaching in one case twelve additional digits.

Table 11 lists, for a range of values of c, the exact eigenvalue $\chi_{0,0}(c)$ rounded at the sixth place, together with k_1 and k_2—the values of k for the small-c and the large-c series, respectively—and the corresponding approximations, denoted $\chi_{0,0}^{(\alpha)}(c)$. Up to $c = 3$ the small-c series is clearly best, but at $c = 4$ the large-c series should obviously take over. Interestingly, $c = 4$ marks *the point of closest approach of the two approximations*. Table 12 lists the same kind of data for $\chi_{2,3}(c)$. The large-c series becomes preferable to the small-c series at $c = 14$—right next to the point of closest approach, now located at $c = 15$. This suggests a mechanism for automatically switching between series when seeking, as c is made to change, the best approximation to refine using Bouwkamp's scheme.

In Annex C we provide a listing of our subroutine, which may be of interest to the reader.

[10] Coefficients a_j also depend on α, of course.

[11] In cases where $R_{p-1} = 0$ because $a_{p-1} = 0$ (which arise for $n = 3, 5, 7, \ldots$ when $\alpha = 1$ and $N = 0$), R_{p-2} is substituted for R_{p-1}.

Table 7. Comparison of approximations of order 5 and ultimate order k with exact value of $\chi_{0,0}(c)$ — $\alpha = 2;\ c = 100, k = 30;\ c = 10, k = 19$.

Approx.	$\chi_{0,0}(100)$	$\chi_{0,0}(10)$
$p = 5$	$1.98744923297344 \times 10^2$	$1.86903796875000 \times 10^1$
$p = k$	$1.98744923295735 \times 10^2$	$1.86901096871968 \times 10^1$
exact	$1.98744923295734 \times 10^2$	$1.86901099396909 \times 10^1$

Table 8. Comparison of approximations of order 5 and ultimate order k with exact value of $\chi_{2,3}(c)$ — $\alpha = 2;\ c = 100, 50;\ k = 30$.

Approx.	$\chi_{2,3}(100)$	$\chi_{2,3}(50)$
$p = 5$	$1.75979299500313 \times 10^3$	$8.58700905050000 \times 10^2$
$p = k$	$1.75979295052608 \times 10^3$	$8.58699269327762 \times 10^2$
exact	$1.75979295052608 \times 10^3$	$8.58699269327762 \times 10^2$

Table 9. Comparison of approximations of order 4 and ultimate order k with exact value of $\chi_{0,0}(c)$ — $\alpha = 1;\ c = \frac{1}{2}, 2;\ k = 30$.

Approx.	$\chi_{0,0}(\frac{1}{2})$	$\chi_{0,0}(2)$
$p = 4$	$8.74348997186731 \times 10^{-1}$	$2.58587962962963 \times 10^0$
$p = k$	$8.74348997181586 \times 10^{-1}$	$2.58579682607078 \times 10^0$
exact	$8.74348997181586 \times 10^{-1}$	$2.58579682607078 \times 10^0$

Table 10. Comparison of approximations of order 5 and ultimate order k with exact value of $\chi_{2,3}(c)$ — $\alpha = 1;\ c = 2, 5;\ k = 30$.

Approx.	$\chi_{2,3}(2)$	$\chi_{2,3}(5)$
$p = 5$	$8.28546675433166 \times 10^1$	$9.40650088448028 \times 10^1$
$p = k$	$8.28546675432684 \times 10^1$	$9.40650073818606 \times 10^1$
exact	$8.28546675432684 \times 10^1$	$9.40650073818606 \times 10^1$

Table 11. Variation of approximations $\chi_{0,0}^{(\alpha)}(c)$ of ultimate order k_α with c.
The true eigenvalue is listed in the second column.

c	$\chi_{0,0}(c)$	k_1	$\chi_{0,0}^{(1)}(c)$	k_2	$\chi_{0,0}^{(2)}(c)$
2	2.58580	30	2.58580	3	2.31250
3	4.46227	30	4.46227	5	4.38137
4	6.52086	29	6.56865	7	6.50453
5	8.58692	1	13.2500	9	8.58417
6	10.6289	1	18.7500	11	10.6284
7	12.6541	1	25.2500	13	12.6540
8	14.6704	1	32.7500	15	14.6704
9	16.6817	1	41.2500	17	16.6817
10	18.6901	1	50.7500	19	18.6901
11	20.6966	1	61.2500	21	20.6966
12	22.7018	1	72.7500	23	22.7018
13	24.7061	1	85.2500	25	24.7061
14	26.7097	1	98.7500	27	26.7097
15	28.7127	1	113.250	29	28.7127
16	30.7153	1	128.750	30	30.7153
17	32.7176	1	145.250	30	32.7176

Table 12. Variation of approximations $\chi_{2,3}^{(\alpha)}(c)$ of ultimate order k_α with c.

c	$\chi_{2,3}(c)$	k_1	$\chi_{2,3}^{(1)}(c)$	k_2	$\chi_{2,3}^{(2)}(c)$
4	89.2265	30	89.2265	0	72.0000
5	94.0650	30	94.0650	2	32.7500
6	100.052	15	100.052	2	53.7500
7	107.239	7	107.239	3	63.1786
8	115.691	7	115.691	3	85.2969
9	125.501	4	125.450	4	100.613
10	136.794	4	136.645	5	118.558
11	149.740	4	149.362	6	138.073
12	164.509	4	163.713	7	158.312
13	181.132	4	179.819	9	177.881
14	199.316	4	197.801	11	198.055
15	218.467	4	217.775	14	217.981
16	237.931	4	239.840	16	237.856
17	257.274	4	264.061	19	257.260
18	276.354	4	290.453	22	276.350
19	295.195	4	318.955	25	295.193
20	313.860	4	349.401	27	313.860
25	405.872	2	521.889	30	405.872
30	496.940	2	787.596	30	496.940

8 Conclusion

This report has taken an in-depth look at Slepian's perturbation scheme for approximating the Sturmian eigenvalues associated with the eigenfunctions of the Fourier transformation over a circle. The eigenfunctions critically depend on an adjustable scaling constant, c; this fact motivated us to begin by reviewing some key properties of the eigenfunctions in the limits $c \to 0$ and $c \to \infty$. Next, the central equations of the perturbation scheme were derived *ab initio*; they were seen to provide a recursive process for calculating the coefficients of the perturbation series. From this process it proved possible to work out explicit expressions of the coefficients up to fifth order. These expressions were then specialized to the two specific cases at hand, where the perturbation parameter (or series variable) is either $1/c$ or c^2 according as, in the Sturm-Liouville equation, c may be considered relatively large or small. In the two resulting fifth-order series the coefficients are given by analytical combinations of the eigenvalue's order N and rank n, which amount, in the end, to exact ratios of integers, so that the coefficients are, in effect, known with infinite accuracy. Previous authors have published similar formulæ up to third order: some of these were shown, here, to be in error. Furthermore, our calculation of the fourth- and fifth- order terms appears to be an entirely novel contribution to the field. We provided several concrete examples of these fifth-order series, as well as illustrations of their employment to approximate various eigenvalues. We confirmed that the accuracy of the series in c^2 drops as c increases, while that of the series in $1/c$ does the same as c *de*creases, as a result of which a transition zone exists where both fifth-order series perform poorly. However, having converted Slepian's recursive process to a *non*-recursive algorithm that allows computation of the perturbation-series coefficients up to any desired order, we supplied evidence suggesting that the additional terms this algorithm provides would permit an eigenvalue to be reasonably approximated even into the transition zone.

The essence of the results presented here is a set of tools adequate for approximating the desired eigenvalues with enough accuracy that they can thereafter be refined by means of Bouwkamp's scheme. This, then, becomes the obvious next step and sets the program of work for Part II: We shall adapt Bouwkamp's technique to refinement of our particular set of eigenvalues (thus filling a gap in the literature), and we shall use the resulting algorithm to compute a table of highly accurate eigenvalues covering a suitably wide range of the parameters that define them (scaling constant c, order N and rank n). Because Bouwkamp's scheme is also the preferred method of computing the coefficients of the orthogonal-polynomial expansion it is based upon, the opportunity should be seized to construct a table of expansion coefficients for the eigenfunctions themselves. Finally, code should be provided enabling the reader to extend such computations for his own purposes—e.g., to some other value of c.

Once eigenfunctions $\psi_{N,n}$ have become available they will find numerous applications, some of which, being of especial interest to the author, are outlined below:

Instrumental optics. In some sense, the eigenfunctions of the Fourier transformation over a circle are the fields that a rotationally-symmetric optical system will naturally seek to propagate. For this reason they can be deemed to provide the most economical and efficient representation of whatever waveform is fed through such a system. It would therefore be most advantageous to take up again, by means of these powerful mathematical tools, the study of the combined influence of aberrations and diffraction. Itoh [8] has already shown that use of the FFSTs allows precise determination of the effect of *any* amount of third- or fifth-order spherical aberration. His results were limited, however, by the paucity of numerical data concerning the self-transforms. More generally, it should be possible to find combinations of filtering[12] and defocus apt not only to compensate fully for aberrations, but even to improve imagery beyond what the supposedly optimal aberration-free pupil can deliver [9]. The completeness and orthogonality of the self-transforms, which are key to these developments, would also make it possible to perform analytically—and with some degree of elegance—many calculations pertaining to transfer functions and currently done by numerical integration. The self-transforms could also find application in calculations relating to the complex degree of coherence of extended objects and aimed at their reconstruction; new sampling theorems formed upon this unique basis might lead to more compact object representations.

Optical data processing. Large areas of this field, such as holography, image restoration and image extrapolation, rely on a theoretical formalism based on *infinite* transformations. This is obviously a crude treatment of phenomena that have finite spatial extent. Here use of the FFSTs would lead to superior theoretical formulations, replacing approximate methods with exact and possibly no less rapid ones.

Antenna design. Recently it was shown, using the zero-order self-transforms—functions $\psi_{0,n}(c,r)$—that the directivity pattern from a disc source optimized in some sense could be duplicated by means of a properly weighted, concentric-ring source with adjustable radii [10]. If the full set of self-transforms—functions $\psi_{N,n}(c,r,\theta)$ with $N \geq 0$—were brought into play, this result could probably be extended to azimuthally-discretized sources, i.e., ring arrays. In effect, one would find how to minimize, over a circular domain, the number of weighted radiating elements required for matching a given directivity pattern already optimized in some respect.

Mitigation of diffraction. Frieden has proposed expansions of point- and ring-like impulses in series of FFSTs [4]. The author applied the former to a theoretical study of superresolution in rotationally-symmetric optics [7, 11, 12]. No one, however, seems to have investigated the possibilities associated with the ring-like impulses. These might allow more resolution gain by relaxing some constraints on the generating aperture's amplitude distribution. In another connection, Durnin [13] has shown that by applying a J_0-like weighting to a circular source, the spreading normally associated with propagation of the beam is entirely prevented over distances several orders of magnitude the radiation wavelength. The weighting is typically achieved by uniformly illuminating a thin-ring aperture. However, later investigators found that varying the illumination has an effect on the beam's axial uniformity. This suggests that Frieden's ring-like impulses could serve as a device for optimizing or controlling some properties of the non-spreading beam. More directly, expansions in terms of functions $\psi_{N,n}$ might facilitate the search for weightings that outperform J_0.

[12] a.k.a. shade screen, coating, weighting function, etc.

The theory of eigenfunctions of the Fourier transformation over a circle has by no means been exhausted. We shall conclude by mentioning a few theoretical issues that have aroused the author's curiosity:

Mathematical properties. Since self-transforms $\phi_{N,n}$ in the limits $c \to 0$ and $c \to \infty$—that is, the Zernike polynomials and the associated Gauss-Laguerre functions—satisfy a three-term recurrence, perhaps this is also true of the self-transforms in the intermediate, general case where $0 < c < \infty$. Thus, a recurrence relation might exist that links three self-transforms of consecutive ranks—$\phi_{N,n-1}$, $\phi_{N,n}$, and $\phi_{N,n+1}$—and which involves the corresponding Sturmian eigenvalues—$\chi_{N,n-1}$, $\chi_{N,n}$, and $\chi_{N,n+1}$. These latter would play a role similar to that of the combinations of N and n appearing in the recurrences for the Zernike polynomials and the AGLFs. The existence of such a recurrence relation would simplify the computation of the self-transforms, and hence confer even greater worth to a large, accurate table of eigenvalues.

Analogs. The eigenfunctions considered here—the $\psi_{N,n}(c, r, \theta)$—have various analogs resulting from simplifications or generalizations. In the Introduction we briefly mentioned the well-known prolate spheroidal wave functions—the $\psi_n(c, x)$—, which are the 1-d analog of the 2-d $\psi_{N,n}$. Rather less understood are the eigenfunctions associated with reciprocal volumes of space of $D > 2$ dimensions. Slepian [1] has expressed the solutions of the relevant integral equation in terms of D-dimensional spherical harmonics combined with *the very eigenfunctions considered here*. A numerical exploration of these higher-dimensional entities remains to be done, and will become feasible once the computations we propose to do in Part II have been carried out. Another analog results from formulating a *discrete* problem of self-transformation between the time and frequency domains. This leads to the definition of certain *discrete prolate spheroidal sequences* (DPSSs) and *discrete prolate spheroidal wave functions* (DPSWFs), which Slepian has investigated thoroughly from a mathematical standpoint [14]. They solve the problem of simultaneous concentration of a discrete, band-limited signal and its finite spectrum, just as the PSWFs do for continuous band-limited signals. There is also a Fourier-Fresnel self-transform problem, which arises from the insertion of a phase factor $\exp[i(y/2)(r')^2]$ into the kernel of integral equation (1–7). A physical realization is found in resonators with plane-parallel mirrors; their modes have, so far, only been simulated through multiple-pass numerical quadrature. Some generalization of the differential operator of Eq. (1–10) must exist that commutes with the properly symmetrized new integral operator: it would open the way to analytical determination of the Fourier-Fresnel self-transforms. Finally, geometrical modifications of the transformation domains lead to further generalizations. For instance, one may formulate a self-transform problem between reciprocal *annular* domains, of possible relevance to telescopic applications. Solutions might be built from the difference between self-transforms over two concentric circles, or they could be expressed in terms of polynomials orthogonal over annular domains [15]. Another such generalization follows from considering focusing configurations with too wide an angular aperture for a scalar treatment to be valid. This leads to the problem of the wide-aperture confocal resonator, featuring vector modes. One approach would use the angular plane-wave spectrum and result in a system of three coupled Fredholm integral equations.

Relevance to field theory. Certain reformulations of the general problem defined by Slepian [1] might shed new light upon the relationship between fields and sources. Both the divergence theorem and the theorem of Stokes link a scalar field in some domain to the field's vector source on the periphery of the domain. On their basis a connection might be found between the D-dimensional eigenfunctions of the finite Fourier transformation and equivalent edge sources. Such a result might have a bearing on the design of innovative radiators, or the understanding of puzzling structures found in nature [16].

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Annex A: An argument about denominators

In the case $\alpha = 1$, the general expressions for calculating the perturbation coefficients—Eqs. (6–9)–(6–13)—include denominators that vanish under certain circumstances. Here we present an argument showing that these denominators must drop out from the formulæ the general expressions lead to, whenever the said circumstances arise.

Let $a(n)$ stand for, say, $a_5(N, n)$ as expressed in Eq. (6–13), but with only as many denominators as required for the purpose of our argument. This simplified version of a_5 states that

$$a(0) = k_0\left(r_0 + \frac{s_0}{d_0}\right) \equiv g, \tag{A–1}$$

$$a(1) = g + k_1\left(r_1 + \frac{s_1}{d_1}\right) = \frac{d_1 g + k_1(d_1 r_1 + s_1)}{d_1}, \tag{A–2}$$

and

$$a(n) = f(n), \qquad n \geq 2, \tag{A–3}$$

where

$$\begin{aligned} f(n) &= g + k_1\left[r_1 + \frac{s_1}{d_1} + k_2\left(r_2 + \frac{s_2}{d_2}\right)\right], \\ &= \frac{d_1 d_2 g + k_1[d_1 d_2 r_1 + d_2 s_1 + k_2(d_1 d_2 r_2 + d_1 s_2)]}{d_1 d_2}. \end{aligned} \tag{A–4}$$

Here the k_j, r_j, s_j and d_j represent polynomials in N and n; $k_1 = 0$ if $n = 0$; and $k_2 = 0$ if $n = 1$. Furthermore, d_1 might be zero if $n = 0$, and likewise, d_2 might be zero if $n \in (0, 1)$.

Assume, first, that neither d_1 nor d_2 vanishes. Given the special properties of k_2 and k_1, we then have

$$f(1) = \frac{d_2[d_1 g + k_1(d_1 r_1 + s_1)]}{d_1 d_2} = a(1) \tag{A–5}$$

and

$$f(0) = \frac{d_1 d_2 g}{d_1 d_2} = a(0). \tag{A–6}$$

Thus $f(n)$, the general expression supposedly applicable only to $n \geq 2$, may also be used for correctly computing the special cases $n = 1$ and $n = 0$—provided no denominator is zero.

But now assume that d_1 and d_2 vanish, as per the conditions stated above. Upon calculating the numerator and denominator of $f(1)$ or $f(0)$ *independently*, by Eqs. (A–5) and (A–6) we are bound to find that d_2—or $d_1 d_2$—factors out of each. After performing the obvious cancellation analytically, we obtain special formulæ that, again, yield $a(1)$ and $a(0)$.

The above argument readily extends to sums of fractions s_j/d_j. However, it ignores one subtlety, related to the presence of $\gamma_n^0 - \gamma_{n-1}^0$ in the expressions for a_3, a_4 and a_5, and that of $\gamma_n^0 - \gamma_{n-2}^0$ in the expression for a_5. Reference to Eqs. (6–29) shows the general forms for these two quantities arising at $n = 2$ and $n = 3$, respectively, rather than $n = 1$ and $n = 2$. It is readily seen that the general forms, when evaluated for $n = 1$ and $n = 2$, correctly yield the special forms for these two values of n—but only after simplification of a factor N^2/N. Thus their denominators have the potential to vanish for $n = 1$ and $n = 2$, respectively, which contradicts our assumptions regarding d_1 and d_2 and would seem to invalidate our argument. However, we may write $s_1 = C_1 \times d_1$ and $s_2 = C_2 \times d_2$; once this is done we find that the correct expressions for $a(1)$ and $a(2)$ still follow, so that the argument remains valid after all.

Annex B: Additional formulæ

In this annex we specialize to $N = 0$, 1 and 2 the general expressions obtained in Section 6 for $a_0(N,n) \ldots a_5(N,n)$.

By definition, $\omega = (N+2n)(N+2n+2)$. Therefore, if $N = 0$,

$$\omega \;=\; 4n(n+1) \;=\; (2n-1)(2n+3)+3 \;=\; (2n-3)(2n+5)+15. \tag{B–1}$$

Since expression (6–15) for $a_0(N,n)$ is universally valid, we then have

$$a_0(0,n) \;=\; \frac{4\omega+3}{4} \;=\; \frac{4(2n-1)(2n+3)+15}{4} \;=\; \frac{(4n+1)(4n+3)}{4}. \tag{B–2}$$

Furthermore, from Eqs. (6–17), (6–71) and (6–144), plus the associated special formulæ, we find that

$$a_1(0,n) \;=\; \tfrac{1}{2}, \qquad a_3(0,n) \;=\; a_5(0,n) \;=\; 0\,. \tag{B–3}$$

Also, $T(0,\omega) = \omega^3$ and $V(0,\omega) = \omega^7(\omega-8)(5\omega+33)$, so that Eqs. (6–57) and (6–116) yield

$$a_2(0,n) \;=\; \frac{1}{32(\omega-3)} \;=\; \frac{1}{32(2n-1)(2n+3)} \tag{B–4}$$

and

$$a_4(0,n) \;=\; \frac{5\omega+33}{8{,}192(\omega-3)^3(\omega-15)} \;=\; \frac{5(2n-1)(2n+3)+48}{8{,}192(2n-3)(2n-1)^3(2n+3)^3(2n+5)}. \tag{B–5}$$

If $N = 1$,

$$\omega \;=\; (2n+1)(2n+3) \;=\; (2n-1)(2n+5)+8 \;=\; (2n-3)(2n+7)+24. \tag{B–6}$$

We then have

$$a_0(1,n) \;=\; \frac{4\omega+3}{4} \;=\; \frac{4(2n+1)(2n+3)+3}{4} \;=\; \frac{(4n+3)(4n+5)}{4}, \tag{B–7}$$

and since $\omega + 1 = 4(n+1)^2$,

$$a_1(1,n) \;=\; \frac{\omega+1}{2\omega} \;=\; \frac{2(n+1)^2}{(2n+1)(2n+3)}. \tag{B–8}$$

It also turns out that

$$T(1,\omega) = (\omega-4)(\omega-3)(\omega+1), \tag{B–9}$$

$$U(1,\omega) = (\omega-3)(\omega+1)f_3(\omega), \tag{B–10}$$

$$V(1,\omega) = (\omega-15)(\omega-3)^3(\omega+1)f_4(\omega), \tag{B–11}$$

and

$$W(1,\omega) = (\omega-15)(\omega-3)^3(\omega+1)f_5(\omega), \tag{B–12}$$

where

$$f_3(y) \;=\; 5y^2 - 4y - 32 \;=\; (y-3)(5y+11) + 1\,, \tag{B–13}$$

$$f_4(y) \;=\; 5y^4 - 144y^3 - 432y^2 + 1{,}600y + 2{,}560\,, \tag{B–14}$$

and

$$f_5(y) \;=\; 207y^5 - 496y^4 - 14{,}224y^3 + 960y^2 + 93{,}184y + 86{,}016\,. \tag{B–15}$$

Hence

$$a_2(1,n) \;=\; \frac{(\omega-4)(\omega+1)}{32\omega^3} \;=\; \frac{(n+1)^2[(2n+1)(2n+3)-4]}{8(2n+1)^3(2n+3)^3}; \tag{B–16}$$

$$a_3(1,n) \;=\; \frac{(\omega+1)f_3(\omega)}{64\omega^5(\omega-8)} \;=\; \frac{(n+1)^2 f_3\big((2n+1)(2n+3)\big)}{16(2n-1)(2n+1)^5(2n+3)^5(2n+5)}; \tag{B–17}$$

$$a_4(1,n) \;=\; \frac{(\omega+1)f_4(\omega)}{8{,}192\omega^7(\omega-8)} \;=\; \frac{(n+1)^2 f_4\big((2n+1)(2n+3)\big)}{2{,}048(2n-1)(2n+1)^7(2n+3)^7(2n+5)}; \tag{B–18}$$

and

$$\begin{aligned} a_5(1,n) \;&=\; \frac{(\omega+1)f_5(\omega)}{16{,}384\omega^9(\omega-8)(\omega-24)}, \\ &=\; \frac{(n+1)^2 f_5\big((2n+1)(2n+3)\big)}{4{,}096(2n-3)(2n-1)(2n+1)^9(2n+3)^9(2n+5)(2n+7)}. \end{aligned} \tag{B–19}$$

If $N=2$,

$$\omega \;=\; 4(n+1)(n+2) \;=\; (2n+1)(2n+5)+3 \;=\; (2n-1)(2n+7)+15. \tag{B–20}$$

We then have

$$a_0(2,n) \;=\; \frac{4\omega+3}{4} \;=\; \frac{16(n+1)(n+2)+3}{4} \;=\; \frac{(4n+5)(4n+7)}{4}, \tag{B–21}$$

while

$$a_1(2,n) \;=\; \frac{\omega+4}{2\omega} \;=\; \frac{(n+1)(n+2)+1}{2(n+1)(n+2)}. \tag{B–22}$$

Also,

$$T(2,\omega) = 64\,g_2(\omega/4), \tag{B–23}$$

$$U(2,\omega) = 64(\omega-8)g_3(\omega/4), \tag{B–24}$$

$$V(2,\omega) = 16{,}384(\omega-8)g_4(\omega/4), \tag{B–25}$$

and

$$W(2,\omega) = 16{,}384(\omega-24)(\omega-8)g_5(\omega/4), \tag{B–26}$$

where

$$g_2(y) \;=\; y^3 - 6y^2 + 5y + 3 \;=\; (y-4)(y-3)(y+1) - 9\,, \tag{B–27}$$

$$g_3(y) \;=\; 5y^3 - 4y^2 - 11y - 3 \;=\; (y-2)(y+1)(5y+1) - 1\,, \tag{B–28}$$

$$g_4(y) = 20y^8 - 975y^7 + 6{,}732y^6 - 3{,}722y^5 - 22{,}850y^4 \\ + 17{,}805y^3 + 11{,}070y^2 - 5{,}535y - 2{,}025\,, \tag{B–29}$$

and

$$g_5(y) = 828y^8 - 5{,}893y^7 - 5{,}290y^6 + 44{,}070y^5 - 9{,}818y^4 \\ - 50{,}253y^3 + 13{,}419y + 2{,}835\,. \tag{B–30}$$

Thus,

$$a_2(2,n) = \frac{2\,g_2(\omega/4)}{\omega^3(\omega-3)} = \frac{g_2\big((n+1)(n+2)\big)}{32(2n+1)(n+1)^3(n+2)^3(2n+5)}; \tag{B–31}$$

$$a_3(2,n) = \frac{4\,g_3(\omega/4)}{\omega^5(\omega-3)} = \frac{g_3\big((n+1)(n+2)\big)}{256(2n+1)(n+1)^5(n+2)^5(2n+5)}; \tag{B–32}$$

$$\begin{aligned} a_4(2,n) &= \frac{2\,g_4(\omega/4)}{\omega^7(\omega-3)^3(\omega-15)}, \\ &= \frac{g_4\big((n+1)(n+2)\big)}{8{,}192(2n-1)(2n+1)^3(n+1)^7(n+2)^7(2n+5)^3(2n+7)}. \end{aligned} \tag{B–33}$$

and

$$\begin{aligned} a_5(2,n) &= \frac{4\,g_5(\omega/4)}{\omega^9(\omega-3)^3(\omega-15)}, \\ &= \frac{g_5\big((n+1)(n+2)\big)}{65{,}536(2n-1)(2n+1)^3(n+1)^9(n+2)^9(2n+5)^3(2n+7)}. \end{aligned} \tag{B–34}$$

For the sake of completeness we provide below explicit expressions for the functions of n that were introduced above. With a view to computations, such expressions are best laid out in terms of the variables $r \equiv n(n+2)$ and $s \equiv n(n+3)$:

$$f_3\big((2n+1)(2n+3)\big) = 8r(10r+13)+1\,, \tag{B–35}$$

$$f_4\big((2n+1)(2n+3)\big) = 16r(2r(8r(5r-21)-729)-1{,}085)-11\,, \tag{B–36}$$

$$f_5\big((2n+1)(2n+3)\big) = 4r(8r(4r(2r(828r+2{,}609) \\ -773)-48{,}975)-254{,}837)+285\,, \tag{B–37}$$

$$g_2\big((n+1)(n+2)\big) = s(s^2-7)-3\,, \tag{B–38}$$

$$g_3\big((n+1)(n+2)\big) = s(s+3)(5s+11)-1\,, \tag{B–39}$$

$$g_4\big((n+1)(n+2)\big) = s(s(s(s(s(5s(4s-131) \\ -4{,}678)+4{,}122)+93{,}250) \\ +253{,}085)+268{,}060)+99{,}669)+89\,, \tag{B–40}$$

$$g_5\big((n+1)(n+2)\big) = s(s(s(s(s(s(828s+7{,}355) \\ +4{,}944)-143{,}478)-609{,}198) \\ -1{,}028{,}701)-757{,}470)-186{,}065)-95\,. \tag{B–41}$$

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Annex C: Subroutine listing

In this annex we list the FORTRAN subroutine implementing the algorithm of Section 7. It complies with the FORTRAN-90 ANSI standard, but should be acceptable to any FORTRAN-77 compiler that supports the DO WHILE and ENDDO extensions.

```
C
C..ANNA.FOR - CHISLP rev 2.0: subroutine ANNA - R.F. Boivin, DRDC Ottawa
C
       SUBROUTINE ANNA  ( ALPHA, CAPN, N, JEND, COEF )
C      ===============================================
C
C..Computes Slepian's perturbation-series coefficients a_j(N,n), given:
C   - the type of perturbation series, alpha (1: small c; 2: large c);
C   - the eigenvalue order, N (0 or 1 or 2 or ...);
C   - the eigenvalue rank,  n (0 or 1 or 2 or ...); and
C   - the order of the last desired coefficient, j_end (j = 0, 1 ... j_end)
C
C..References:
C
C  1. Slepian, D., 1964, "Prolate Spheroidal Wave Functions, Fourier
C     Analysis and Uncertainty--IV: Extensions to Many Dimensions;
C     Generalized Prolate Spheroidal Functions",
C     Bell Syst. Tech. J., vol. 43, pp. 3009-3057.
C
C  2. Boivin, R.F., 2008, "Eigenfunctions of the Fourier Transformation
C     over a Circle--I: Approximation of Sturmian Eigenvalues",
C     DRDC Ottawa Technical Memorandum 2008-342.
C
C..Input:
C
       INTEGER           ALPHA, CAPN, N, JEND
C
C  entity    nature                  contents                    value
C  ------    ------                  --------                    -----
C  ALPHA   intgr scalr  type of perturbation series, alpha       1 or 2
C  CAPN    intgr scalr  eigenvalue order, N                      = or > 0
C  N       intgr scalr  eigenvalue rank, n                       = or > 0
C  JEND    intgr scalr  order of last desired coefficient, j_end  = or > 0
C
C  NOTE: JEND may not exceed JMAX set in subroutine TRESTA.
C
C..Output:
C
       DOUBLE PRECISION   COEF(JEND+1)
```

```
C
C  entity      nature                        contents
C  ------      ------                        --------
C   COEF  real*8 1-d array  computed coefficients a_j(N,n),
C                           stored as COEF(1)      <-- a_0,     ...,
C                                     COEF(JEND+1) <-- a_(j_end)
C..Subprograms:
C
       DOUBLE PRECISION   GAMMA
       EXTERNAL           GAMMA, TRESTA
C
C..Local entities:
C
       LOGICAL            ZERO
       INTEGER            J, K, P, Q, S, U, V
       DOUBLE PRECISION   DPN, H, X, Y, Z
C
C-----------------------------------------------------------------------
C
C..Convert order N to double precision.
C
       DPN = DBLE( CAPN )
C
C..Compute z [required for computation of a_0 and h_(alpha*q)].
C
       Z = DPN + 2.D0*DBLE(N) + 1.D0
C
C..Compute and store a_0 = lambda_n.  If j_end = 0, we are done and return.
C
       X = 2.D0*Z
       IF( ALPHA.EQ.1 ) X = (X*X - 1.D0) / 4.D0
       COEF(1) = X
       IF( JEND.EQ.0 ) RETURN
C
C..Compute and store a_1 = gamma_n^0. If j_end = 1, we are done and return.
C
       COEF(2) = GAMMA( ALPHA, DPN, 0, N )
       IF( JEND.EQ.1 ) RETURN
C
C..Perform algorithm for all a_j beyond j = 1 successively.
C
       DO J = 2, JEND
C
C..Set up loops for computation of quantities A_q^p required by current a_j
C  [see ref. 2, Eqs. (7-15) to (7-18)]. Outer loop, on S, causes inner loop
```

```
C  to be performed twice--once for positive q, once for negative q. Current
C  A_q^p need be computed only if n >= -alpha*q.
C
          DO S = 1, -1, -2
             U = (J+1) / 2
             V = J / 2
             DO WHILE ( V.GT.0 )
                P = U
                Q = S * V
                IF( N.GE.-ALPHA*Q ) THEN
C
C..Initialize accumulator. Then compute first sum entering general formula
C  for A_q^p [skipped if p=1; see ref. 2, Eqs. (7-3) and (7-19)].
C
                   X = 0.D0
                   IF( P.GT.1 ) THEN
                      DO K = 1, P-1
                         CALL TRESTA( ALPHA, N, JEND, 1,
     +                                   P-K, Q, ZERO, Y )
                         IF( .NOT.ZERO ) X = X + Y*COEF(K+1)
                      ENDDO
                   ENDIF
C
C..Now compute second sum entering general formula for A_q^p.
C
                   DO K = -1, 1
                      CALL TRESTA( ALPHA, N, JEND, 1,
     +                             P-1, Q-K, ZERO, Y )
                      IF( .NOT.ZERO )
     +                   X = X - Y*GAMMA( ALPHA, DPN, K, N+ALPHA*(Q-K) )
                   ENDDO
C
C..Finally, divide A_q^p-so-far by h_(alpha*q) to get desired A_q^p, which
C  is assigned to scalar Y for storage by subroutine TRESTA.
C
                   Y = DBLE( ALPHA*Q )
                   H = 4.D0*Y
                   IF( ALPHA.EQ.1 ) H = H * (Z+Y)
                   Y = X / H
                   CALL TRESTA( ALPHA, N, JEND, 2, P, Q, ZERO, Y )
                ENDIF
C
C..Current A_q^p has been dealt with. Increment and decrement counters that
C  will provide next values of p and q.
C
```

```
                U = U + 1
                V = V - 1
C
C..End loop for computation of quantities A_q^p with q of given sign, then
C  loop on sign of q.
C
             ENDDO
          ENDDO
C
C..Compute current coefficient a_j from A_1^(j-1) and A_(-1)^(j-1) [see
C  ref. 2, Eq. (7-1)]. The result is accumulated in scalar X then stored
C  in the proper element of array COEF.
C
          X = 0.D0
          CALL TRESTA( ALPHA, N, JEND, 1, J-1,  1, ZERO, Y )
          IF( .NOT.ZERO ) X = X + Y*GAMMA( ALPHA, DPN, -1, N+ALPHA )
          CALL TRESTA( ALPHA, N, JEND, 1, J-1, -1, ZERO, Y )
          IF( .NOT.ZERO ) X = X + Y*GAMMA( ALPHA, DPN,  1, N-ALPHA )
          COEF(J+1) = X
C
C..End loop on coefficients a_j.
C
       ENDDO
C
C..Done: return control to calling program.
C
       RETURN
C
       END
C
C*********************************************************************
C*********************************************************************
C
       DOUBLE PRECISION FUNCTION GAMMA( ALPHA, DPN, K, J )
C      ===================================================
C
C..Input:
C
       INTEGER                          ALPHA,      K, J
       DOUBLE PRECISION                        DPN
C
C..Computes quantity gamma_j^k as defined in ref. 2
C  [see Eqs. (5-7) to (5-9), (6-6) to (6-8), and pseudocode (7-25)].
C
C..Local entities:
```

```
C
      DOUBLE PRECISION   DPJ, NPJ, NP2J, JP1, NPJP1, NP2JP1, NP2JP2
C
C-----------------------------------------------------------------------
C
      DPJ    = DBLE( J )
      NPJ    = DPN  + DPJ
      NP2J   = NPJ  + DPJ
      JP1    = DPJ  + 1.D0
      NPJP1  = NPJ  + 1.D0
      NP2JP1 = NP2J + 1.D0
      NP2JP2 = NP2J + 2.D0
C
      IF( ALPHA.EQ.1 ) THEN
C
         IF( K.EQ.1 ) THEN
            IF( J.GE.0 ) THEN
               GAMMA = - (NPJP1*NPJP1) / (NP2JP1*NP2JP2)
            ELSE
               GAMMA = 0.D0
            ENDIF
C
         ELSE IF( K.EQ.0 ) THEN
            IF( J.GT.0 ) THEN
               GAMMA = (NPJ*NPJP1 + DPJ*JP1) / (NP2J*NP2JP2)
            ELSE IF( J.EQ.0 ) THEN
               GAMMA = (DPN+1.D0) / (DPN+2.D0)
            ELSE
               GAMMA = 0.D0
            ENDIF
C
         ELSE
            IF( J.GE.1 ) THEN
               GAMMA = - (DPJ*DPJ) / (NP2J*NP2JP1)
            ELSE
               GAMMA = 0.D0
            ENDIF
C
         ENDIF
C
      ELSE
C
         IF( K.EQ.1 ) THEN
            GAMMA = JP1 * (DPJ+2.D0)
         ELSE IF( K.EQ.0 ) THEN
```

```
            GAMMA = - (8.D0*DPJ*NPJP1 + 4.D0*DPN + 5.D0) / 4.D0
         ELSE
            GAMMA = NPJ * (NPJ-1.D0)
         ENDIF
C
      ENDIF
      RETURN
C
      END
C
C*********************************************************************
C*********************************************************************
C
      SUBROUTINE TRESTA( ALPHA, N, JEND, M, P, Q, ZERO, Y )
C     =====================================================
C
C..Input:
C
      INTEGER            ALPHA, N, JEND, M, P, Q
C
C..Output:
C
      LOGICAL                                  ZERO
C
C..Input/output:
C
      DOUBLE PRECISION                               Y
C
C..Manages Testing/REtrieval (mode M=1) and STorage (mode M.NE.1) of quan-
C  tities A_q^p.
C     In mode M=1 the subroutine first tests a quantity A_q^p for special
C  cases where it is zero [see ref. 2, Eqs. (7-2)]. In such a case control
C  returns with switch ZERO set to .TRUE. so that the particular A_q^p can
C  be ignored. Otherwise, either we are dealing with A_0^0 = 1, or else
C  A_q^p was computed before and was thus, perforce, stored in array F. In
C  the former instance the subroutine sets scalar Y to 1.0, in the latter
C  it retrieves A_q^p from array F and stores same into scalar Y; then
C  control returns with switch ZERO set to .FALSE. so that the particular
C  A_q^p can be taken into account via Y.
C     In mode M.NE.1 TRESTA stores into array F the quantity A_q^p computed
C  by the calling routine, ANNA, and stored there in scalar Y.
C     For computation of storage addresses see ref. 2, Eqs. (7-20)-(7-21).
C
C..NOTE: PARAMETER JMAX below sets the largest value of JEND expected in
C        subroutine ANNA.
```

```
C
C..Local entities:
C
      INTEGER            JMAX, TMAX
      PARAMETER        ( JMAX=50, TMAX=(JMAX*JMAX)/4 )
C
      DOUBLE PRECISION   F(TMAX,2)
C
      INTEGER            I, J
C
C-----------------------------------------------------------------------
C
      J = IABS( Q )
C
      IF( M.EQ.1 ) THEN
C
         IF( ( (P.EQ.0) .AND. (Q.NE.0) ) .OR.
     +       ( (Q.EQ.0) .AND. (P.NE.0) ) .OR.
     +       ( J.GT.P )                  .OR.
     +       ( N.LT.-ALPHA*Q )                ) THEN
                                                  Y    = 0.D0
                                                  ZERO = .TRUE.
                                                  RETURN
C
         ELSE IF( (P.EQ.0) .AND. (Q.EQ.0) )     THEN
                                                  Y    = 1.D0
                                                  ZERO = .FALSE.
                                                  RETURN
C
         ELSE
            I = (J-1)*(JEND-J) + P
            IF( Q.GT.0 ) THEN
               Y = F(I,1)
            ELSE
               Y = F(I,2)
            ENDIF
            ZERO = .FALSE.
            RETURN
C
         ENDIF
C
      ELSE
         I = (J-1)*(JEND-J) + P
         IF( Q.GT.0 ) THEN
            F(I,1) = Y
```

```
          ELSE
             F(I,2) = Y
          ENDIF
          RETURN
C
       ENDIF
C
       END
C
C************************************************************************
C************************************************************************
```

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(Security classification of title, body of abstract and indexing annotation must be entered when document is classified)

1. ORIGINATOR (The name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Centre sponsoring a contractor's report, or tasking agency, are entered in section 8.) Defence R&D Canada – Ottawa 3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4	2. SECURITY CLASSIFICATION (Overall security classification of the document including special warning terms if applicable.) UNCLASSIFIED

3. TITLE (The complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title.)

Eigenfunctions of the Fourier transformation over a circle — I: Approximation of Sturmian eigenvalues

4. AUTHORS (Last name, followed by initials – ranks, titles, etc. not to be used.)

Boivin, R.F.

5. DATE OF PUBLICATION (Month and year of publication of document.) March 2009	6a. NO. OF PAGES (Total containing information. Include Annexes, Appendices, etc.) 102	6b. NO. OF REFS (Total cited in document.) 16

7. DESCRIPTIVE NOTES (The category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

Technical Memorandum

8. SPONSORING ACTIVITY (The name of the department project office or laboratory sponsoring the research and development – include address.)

Defence R&D Canada – Ottawa
3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4

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10a. ORIGINATOR'S DOCUMENT NUMBER (The official document number by which the document is identified by the originating activity. This number must be unique to this document.) DRDC Ottawa TM 2008-342	10b. OTHER DOCUMENT NO(s). (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

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We clarify and expand upon Slepian's perturbation scheme for approximating the eigenvalues of the Sturm-Liouville equation characterizing the eigenfunctions of the finite Fourier transformation over a circle. The eigenvalues are approximated as power series in terms of c^2 or $1/c$, where c represents the adjustable scaling constant that controls mapping of the frequency domain of the transformation to the field domain. Analytical expressions of the series coefficients are worked out up to fifth order. An algorithm is also provided for numerical determination of higher-order coefficients. The accuracy of the series is investigated; prospects for extensive computations of the eigenvalue spectrum are discussed; and some applications of the eigenfunctions are outlined.

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Fourier transformation
Hankel transformation
Finite Fourier self-transforms
Finite Hankel self-transforms
Generalized prolate spheroidal functions
Sturm-Liouville equation
Eigenfunctions
Eigenvalues
Perturbation methods