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Simple harmonic oscillator and the Beněs filter

Exact fundamental solution

Bhashyam Balaji

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Defence R&D Canada – Ottawa

Canada

Technical Memorandum
DRDC Ottawa TM 2008-352
March 2009

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Abstract

The solution to the universal continuous-discrete filtering problem requires the solution of the Fokker-Planck-Kolmogorov forward equation (FPKfe) for the state process for an arbitrary initial condition. The fundamental solution for the FPKfe is derived for a state model of arbitrary dimension with Beněs drift and a positive-definite diffusion matrix proportional to the identity matrix, and requires only the computation of elementary transcendental functions and standard linear algebra techniques. In particular, there is no need to solve a partial (or ordinary) differential equation. The measurement process may be a discrete-time nonlinear stochastic process, and the time step size can be arbitrary. These results can also be applied to explicitly solve the continuous-continuous Yau filtering problem with Beněs drift.

Résumé

La solution du problème universel du filtrage continu-discret passe par la résolution de l'équation de Fokker-Planck-Kolmogorov (FPK) pour les processus à états avec une condition initiale arbitraire. Nous calculons la solution fondamentale de l'équation FPK pour un modèle d'états de dimension arbitraire avec un terme de dérive de type Beněs et une matrice de diffusion définie positive proportionnelle à la matrice unité. Cette opération nécessite uniquement le calcul de fonctions transcendantes élémentaires et l'utilisation des méthodes d'algèbre linéaire courantes. En particulier, nul besoin de résoudre une équation aux dérivées partielles (ou équation différentielle ordinaire). Le processus de mesure peut être un processus stochastique non linéaire en temps discret et la grandeur du pas de calcul peut être arbitraire. Les résultats obtenus peuvent également servir à résoudre de façon explicite l'équation du filtrage continu-continu de Yau avec un terme de dérive de type Beněs.

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Executive summary

Simple harmonic oscillator and the Beněs filter

Bhashyam Balaji; DRDC Ottawa TM 2008-352; Defence R&D Canada – Ottawa; March 2009.

Introduction

Many problems in sensor signal processing can be formulated as nonlinear tracking problems. Specifically, the signal of interest is assumed to be evolve according to a continuous-time stochastic process. However, the signal process is not directly measurable. Instead, what is observed is a related discrete-time stochastic process called the measurement process.

Since the problem is nonlinear, the extended Kalman filter is not a reliable solution. A more general approach is called continuous-discrete filtering, which is based on solving a partial differential equation called the Fokker-Planck-Kolmogorov forward equation (FPKfe).

The solution of the universal FPKfe is equivalent to finding the fundamental solution of the FPKfe. That is, from the fundamental solution it is possible to obtain the solution of the FPKfe for an arbitrary initial condition. It is therefore desirable to find ways to obtain exact, closed-form expressions for the fundamental solution of the FPKfe for various state models.

Results

In this report, we demonstrate that the fundamental solution of the FPKfe for a state model of the Beněs type can also be expressed in terms of elementary transcendental functions and that it requires only elementary linear algebra methods. The diffusion matrix is assumed to be a time-independent matrix proportional to the identity matrix. A special case of our results is the affine, linear state model with symmetric drift matrix.

Significance

The proposed solution is considerably simpler than solving the FPKfe, or even ODEs. This is because numerical solutions of differential equations often place restrictions on time step size. In contrast, we have derived closed-form solutions— there is no restriction on the size of the time-steps. Since the differential equation is independent of measurements, the measurement model may be nonlinear and arbitrary.

Future Plans

Further investigations are planned into studying problems that have closed-form solutions, particularly those based on exactly solvable problems in quantum physics. It is also planned

that the exact solutions derived here will be used for benchmarking approximate methods for solving nonlinear filtering problems.

Sommaire

Simple harmonic oscillator and the Beněs filter

Bhashyam Balaji ; DRDC Ottawa TM 2008-352 ; R & D pour la défense Canada – Ottawa ; mars 2009.

Introduction

Beaucoup de problèmes dans le traitement du signal du capteur peut être formulée comme le suivi des problèmes non linéaires. Plus précisément, le signal d'intérêt est supposé être évoluer en fonction d'un temps continu de processus stochastiques. Toutefois, le signal n'est pas directement mesurable. Au lieu de cela, ce qui est observé est lié en temps discret de processus stochastiques appelé le processus de mesure.

Comme le problème est non linaire, le filtre de Kalman étendu n'est pas une solution fiable. Une approche plus générale est appelée discret-continu de filtrage, qui est base sur la résolution d'une équation aux dérivées partielles appelé le Fokker-Planck-Kolmogorov en avant l'équation (FPKfe).

Rsoudre l'équation FPK universelle équivaut à calculer la solution fondamentale de cette équation. Autrement dit, on peut déduire la solution de l'équation FPK de la solution fondamentale étant donné une condition initiale arbitraire. Il est donc souhaitable de trouver des faons de représenter la solution fondamentale de l'équation FPK pour divers modèles d'états au moyen d'une expression exacte en forme analytique fermée.

Résultats

Dans ce rapport, nous faisons la démonstration que la solution fondamentale de l'équation FPK pour un modèle d'états de type Benes peut être calculée à l'aide de fonctions transcendantes élémentaires et des méthodes d'algèbre linéaire courantes. On suppose que la matrice de diffusion est une matrice indépendante du temps proportionnelle à la matrice unité. Le modèle d'états linéaire affine avec matrice de dérive symétrique constitue un cas particulier de nos résultats.

Portée

La solution proposée est beaucoup plus simple que la résolution de l'équation FPK ou même d'équations différentielles ordinaires, parce que la solution numérique des équations différentielles impose souvent des contraintes sur la grandeur du pas de calcul. À l'inverse, nous proposons ici des solutions analytiques, c'est-à-dire que la grandeur du pas de calcul n'est soumise à aucune contrainte. Comme l'équation différentielle est indépendante des observations, le modèle de mesure peut être non linéaire et arbitraire.

Pistes de recherche pour l'avenir

Nous prévoyons mener d'autres études sur les problèmes qui appellent des solutions analytiques, en particulier les problèmes de physique quantique à solution exacte. Nous projetons aussi d'utiliser les solutions exactes obtenues dans cette étude pour étalonner les méthodes d'approximation devant servir à résoudre les problèmes de filtrage non linéaire.

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1 Introduction

Many problems in sensor signal processing can be formulated as nonlinear tracking problems. Specifically, the signal of interest is assumed to be evolve according to a continuous-time stochastic process. However, the signal process is not directly measurable. Instead, what is observed is a related discrete-time stochastic process called the measurement process.

Since the problem is nonlinear, the extended Kalman filter is not a reliable solution. A more general approach is called continuous-discrete filtering, which is based on solving a partial differential equation called the Fokker-Planck-Kolmogorov forward equation (FPKfe). The general framework for tackling tracking problems is called universal nonlinear filtering.

The complete solution of the universal filtering problem is the determination of the evolution of the conditional probability density function for an arbitrary initial condition. An excellent discussion of the subject is in the textbook by Jazwinski [1].

The non-trivial part in the continuous-discrete filtering is the computation of the solution of a partial differential equation known as the Fokker-Planck-Kolmogorov forward equation (FPKfe). The numerical solution of the general FPKfe can be challenging, especially if the dimensionality of the problem is large.

The solution of the universal FPKfe is equivalent to finding the fundamental solution of the FPKfe. That is, from the fundamental solution it is possible to obtain the solution of the FPKfe for an arbitrary initial condition. It is therefore desirable to find ways to obtain exact, closed-form expressions for the fundamental solution of the FPKfe for various state models.

In a previous paper, it is shown that the fundamental solution for the state model with an affine, linear state model drift, and with a positive-definite state-independent diffusion matrix, can be determined using elementary linear algebra methods [2]. There is no need to solve a PDE (e.g., by discretizing the FPKfe differential operator). In this report we demonstrate that the fundamental solution of the FPKfe for a state model of the Beněs type [3] can also be expressed in terms of elementary transcendental functions and requires only elementary linear algebra methods. The diffusion matrix is assumed to be a time-independent matrix proportional to the identity matrix. A special case of our results is the affine, linear state model with symmetric drift matrix. As in [2], there is no restriction on the size of the time-steps, and the measurement model may be nonlinear and arbitrary. Among other uses, the exact solutions derived here can be used for benchmarking approximate methods for solving nonlinear filtering problems.

Note that this is different from the solution obtained by Beněs [3], or the solution obtained by Yau and Lai for the general Yau filtering system [4, 5], which require the solution of a system of ordinary differential equations. In the continuous-discrete case, the measurement model need not be Gaussian, so that the initial condition at some stage need not be Gaussian (e.g., multi-modal). Then the conditional mean is not a meaningful quantity. For instance, such a situation is common in the track-before-detect problem. A universal solution is possible if we can decompose the probability density function at any stage as a sum of Gaussian probability distributon functions. However, the determination of the Gaussian components is not trivial in general, especially for the multidimensional case. In contrast, exact formulas for the transition probability density are derived in this report. From this, the time evolution of the conditional probability density follows in a straightforward manner.

The layout of this report is as follows. In Section 2, we review the fundamental notions of continuous-discrete filtering theory. In Section 3, we derive the Euclidean Schrödinger equation that is equivalent to the FPKfe for the state model with Beněs drift. Then, in Section 4 we discuss the special cases. In Section 5, we show that the FPKfe for the model of interest can be transformed to n uncoupled one-dimensional Euclidean Schrödinger equations for a particle moving in a quadratic potential via elementary linear algebra methods. In Section 6, the fundamental solution is explicitly expressed in terms of elementary transcendental functions. The validity of the formulas derived in this report are illustrated in some examples in Section 7. Extension to the continuous-continuous case is discussed in Section 8. In order to make the report completely self-contained, the appendices include relevant background mathematical material such as useful properties of the Hermite polynomials, and the derivation of the fundamental solution of the one-dimensional Euclidean quantum harmonic oscillator and its variants.

2 Basic Concepts

2.1 The FPKfe

In continuous-discrete filtering theory, the state model is given by the Itô stochastic differential equation of the form[1]

$$d\mathbf{x}(t) = f(\mathbf{x}(t), t)dt + e(\mathbf{x}(t), t)d\mathbf{v}(t). \tag{1}$$

Here $\mathbf{x}(t)$ is a $\mathbb{R}^n-$valued process, $f(\mathbf{x}(t), t) \in \mathbb{R}^n$, $e(\mathbf{x}(t), t) \in \mathbb{R}^{n\times p}$ and $\mathbf{v}$ is a $\mathbb{R}^p-$valued Brownian process with covariance $Q(t)$. We refer to f as the drift and e as the diffusion vielbein, and the quantity eQe^T is referred to as the diffusion matrix. This generates a Markov process that is completely characterized by the initial density function $p(t_0, x)$ and the transition probability density $P(t'', x''|t', x')$. The transition probability density obeys the FPKfe

$$\begin{cases} \dfrac{\partial P}{\partial t}(t, x|t', x') = \mathscr{L}(P(t, x|t', x')), \\ P(t', x|t', x') = \delta^n(x - x'), \end{cases} \tag{2}$$

where $\mathscr{L}$ is the forward diffusion operator of the state process generated by Equation 1:

$$\mathscr{L}(\cdot) = -\sum_{i=1}^{n} \frac{\partial(\cdot f_i)}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2\left[\cdot(eQe^T)_{ij}\right]}{\partial x_i \partial x_j}. \tag{3}$$

From the transition probability density one can compute the probability density $p(t_n, x)$ at time t_n for an arbitrary initial condition $p(t_{n-1}, x)$ at time t_{n-1} as follows:

$$p(t_n, x) = \int P(t_n, x|t_{n-1}, x')p(t_{n-1}, x')\{d^n x'\}. \tag{4}$$

Unless otherwise stated, all integrals are from $-\infty$ to ∞.

2.2 Continuous-Discrete Filtering

The general continuous-discrete filtering problem considers the following signal and measurement processes:

$$\begin{cases} d\mathbf{x}(t) = f(\mathbf{x}(t), t)dt + e(\mathbf{x}(t), t)d\mathbf{v}(t), \\ \mathbf{y}(t_k) = h(\mathbf{x}(t_k), t_k, \mathbf{w}(t_k)). \end{cases} \tag{5}$$

Here $\mathbf{y}$ is a $\mathbb{R}^m-$valued process, $h(\mathbf{x}(t), t, \mathbf{w}(t)) \in \mathbb{R}^m$ and $\mathbf{w}$ is a $\mathbb{R}^q-$valued Brownian process.

The universal continuous-discrete filtering problem is solved as follows. Let the initial distribution be $\sigma_0(x)$ and let the measurements be collected at time instants $t_1, t_2, \ldots, t_k, \ldots$. We use the notation $Y(\tau) = \{y(t_l) : t_0 < t_l \leq \tau\}$. Prior to incorporating the measurements, the state evolves according to the FPKfe; i.e.,

$$\begin{cases} \dfrac{\partial p}{\partial t}(t, x|Y(t_0)) = \mathscr{L}(p(t, x|Y(t_0))), \quad t_0 < t \leq t_1, \\ p(t_0, x|Y(t_0)) = \sigma_0(x). \end{cases}$$

This is the prediction step.

From Bayes' rule (and using $p(t_i, x|Y(t_i)) = p(t_i, x|y(t_i), Y(t_{i-1}))$ at observation t_1, the 'corrected' conditional density at time t_1 is

$$p(t_1, x|Y(t_1)) = \frac{p(y(t_1)|x)p(t_1, x|Y(t_0))}{\int p(y(t_1)|\xi)p(t_1, \xi|Y(t_0))\{d^n\xi\}}. \tag{6}$$

This is then the initial condition of the FPKfe for the next prediction step which results in

$$\begin{cases} \dfrac{\partial p}{\partial t}(t, x|Y(t_1)) = \mathscr{L}(p(t, x|Y(t_1))), \quad t_1 < t \leq t_2, & \text{(Prediction Step)} \\ p(t_2, x|Y(t_2)) = \dfrac{p(y(t_2)|x)p(t_2, x|Y(t_1))}{\int p(y(t_2)|\xi)p(t_2, \xi|Y(t_1))\{d^n\xi\}}, & \text{(Correction Step)}, \end{cases} \tag{7}$$

and so on. Thus, at observation at time t_k, the conditional density is given by

$$p(t_k, x|Y(t_k)) = \frac{p(y(t_k)|x)p(t_k, x|Y(t_{k-1}))}{\int p(y(t_k)|\xi)p(t_k, \xi|Y(t_{k-1}))\{d^n\xi\}}, \tag{8}$$

and $p(t_k, x|Y(t_{k-1}))$ is given by the solution of the PDE

$$\frac{\partial}{\partial t}p(t, x|Y(t_{k-1})) = \mathscr{L}(p(t, x|Y(t_{k-1}))), \quad t_{k-1} \leq t < t_k, \tag{9}$$

with initial condition $p(t_{k-1}, x|Y(t_{k-1}))$.

Often, the signal and measurement model is described by the following system:

$$\begin{cases} d\mathbf{x}(t) = f(\mathbf{x}(t), t)dt + e(\mathbf{x}(t), t)d\mathbf{v}(t), \\ \mathbf{y}(t_k) = h(\mathbf{x}(t_k), t_k)dt + d\mathbf{w}(t_k), \quad k = 1, 2, \ldots, \quad t_{k+1} > t_k \geq 0, \end{cases} \tag{10}$$

where $\mathbf{y}(t) \in \mathbb{R}^{m\times1}$, $h \in \mathbb{R}^{m\times1}$ and the noise process is described by $\mathbf{w}(t) \sim N(0, R(t))$. Then, $p(y(t_k)|x)$ is given by

$$\begin{aligned} p(y(t_k)|x) = {} & \frac{1}{((2\pi)^m \det R(t_k))^{1/2}} \\ & \times \exp\left\{-\frac{1}{2}(y(t_k) - h(x(t_k), t_k))^T(R(t_k))^{-1}(y(t_k) - h(x(t_k), t_k))\right\}. \end{aligned} \tag{11}$$

Thus, we see that for continuous-discrete filtering we need to solve a FPKfe. Note that the FPKfe is independent of measurements, and hence can be computed off-line. Furthermore, observe that the complete information is in the transition probability density, which also satisfies the FPKfe, except with a $\delta-$function initial condition. Thus, for times $t'' > t'$ and $x'', x' \in \mathbb{R}^n$, it is sufficient to solve the FPKfe for $p(t'', x''|t', x')$ and use the above to compute $p(t_k, x|Y(t_{k-1}))$ using

$$p(t_k, x|Y(t_{k-1})) = \int P(t_k, x|t_{k-1}, x')p(t_{k-1}, x'|Y(t_{k-1}))\{d^n x'\}. \tag{12}$$

Alternatively, one may use a convenient set of basis functions[6]. Then, the evolution of each of the basis functions under the FPKfe follows from Equation 4. Since the basis functions are independent of measurements, the computation may also be performed off-line.

Note that the solution is universal, i.e., the initial distribution can be arbitrary. Therefore, the solution of Equation 2 is equivalent to the solution of the universal nonlinear filtering problem.

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3 Equivalent Schrödinger Equation

3.1 Model

The state model we consider in this report is the following:

$$d\mathbf{x}(t) = (\nabla\phi)(\mathbf{x}(t))dt + e(\mathbf{x}(t), t)d\mathbf{v}(t), \tag{13}$$

where the diffusion matrix, eQe^T, is positive-definite and proportional to the identity matrix, i.e., $eQe^T = \hbar_\nu I_n$ with $\hbar_\nu > 0$, and I_n is the $n-$dimensional identiy matrix. Also, ϕ is a C^∞ function on $\mathbb{R}^n$ satisfying the following condition:

$$\begin{aligned}\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}(x) + \sum_{i=1}^{n} f_i^2(x) &= \sum_{i=1}^{n} \left[\frac{\partial^2 \phi}{\partial x_i^2}(x) + \frac{1}{\hbar_\nu} \left(\frac{\partial \phi}{\partial x_i}(x) \right)^2 \right], \\ &= \sum_{i,j=1}^{n} T'_{ij} x_i x_j + \sum_{i=1}^{n} P'_i x_i + r'.\end{aligned} \tag{14}$$

Here T' is a constant nonnegative-definite matrix, i.e., $T' \geq 0$, and P'_i are r' are real constants. Such a model arises in the study of the Beněs filter [3], which is a special case of the general finite-dimensional filter known as the Yau filter [7]. As we show in the next section, this includes the affine, linear model with a symmetric drift matrix. The FPKfe for the state model in Equation 13 is

$$\frac{\partial u}{\partial t}(t,x) = \frac{\hbar_\nu}{2} \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(t,x) - \sum_{i=1}^{n} \left[\frac{\partial \phi}{\partial x_i}(x) \right]^2 \frac{\partial u}{\partial x_i}(t,x) - \sum_{i=1}^{n} \left[\frac{\partial^2 \phi}{\partial x_i^2}(x) \right] u(t,x). \tag{15}$$

Following [8], we now show that the FPKfe for certain state models of this type are related to the Euclidean Schrödinger equation for a particle moving in a quadratic potential.

3.2 Equivalence via Gauge Transformation

Define $\nu(t,x)$ to be related to $u(t,x)$ via a "gauge transformation" as follows (e.g., [8]):

$$u(t,x) = e^{\kappa\phi(x)}\nu(t,x), \qquad \kappa \equiv \frac{1}{\hbar_\nu}. \tag{16}$$

Then

$$\frac{\partial u}{\partial t}(t,x) = e^{\kappa\phi(x)} \frac{\partial \nu}{\partial t}(t,x), \tag{17}$$

and

$$\frac{\partial u}{\partial x_i}(t,x) = e^{\kappa\phi(x)} \left\{ \kappa \left[\frac{\partial \phi}{\partial x_i}(x) \right] \nu(t,x) + \frac{\partial \nu}{\partial x_i}(t,x) \right\}, \tag{18}$$

and

$$\frac{\partial^2 u}{\partial x_i^2}(t,x) = e^{\kappa\varphi(x)}\Big\{\kappa\left[\frac{\partial^2\phi}{\partial x_i^2}(x)\right]\nu(t,x) + \kappa^2\left[\frac{\partial\phi}{\partial x_i}(x)\right]^2\nu(t,x) + 2\kappa\left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial\nu}{\partial x_i}(t,x) + \frac{\partial^2\nu}{\partial x_i^2}(t,x)\Big\}. \tag{19}$$

This implies that

$$\frac{\hbar_\nu}{2}\sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}(t,x) = e^{\kappa\phi(x)}\frac{1}{2}\sum_{i=1}^n\left\{\left[\frac{\partial^2\phi}{\partial x_i^2}(x)\right]\nu(t,x) + \frac{1}{\hbar_\nu}\left[\frac{\partial\phi}{\partial x_i}(x)\right]^2\nu(t,x) + 2\left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial\nu}{\partial x_i}(t,x) + \hbar_\nu\frac{\partial^2\nu}{\partial x_i^2}(t,x)\right\}, \tag{20}$$

and

$$-\sum_{i=1}^n\left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial u}{\partial x_i}(t,x) = -e^{\kappa\phi(x)}\sum_{i=1}^n\left\{\frac{1}{\hbar_\nu}\left[\frac{\partial\phi}{\partial x_i}(x)\right]^2\nu(t,x) + \left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial\nu}{\partial x_i}(t,x)\right\}, \tag{21}$$

and

$$-\sum_{i=1}^n\left[\frac{\partial^2\phi}{\partial x_i^2}(x)\right]u(t,x) = -e^{\kappa\phi(x)}\sum_{i=1}^n\left[\frac{\partial^2\phi}{\partial x_i^2}(x)\right]\nu(t,x). \tag{22}$$

Adding Equation 20, 21, and 22 we obtain

$$\frac{\hbar_\nu}{2}\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}(t,x) - \sum_{i=1}^n\left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial u}{\partial x_i}(t,x) - \sum_{i=1}^n\frac{\partial^2\phi}{\partial x_i^2}(x)u(t,x) = e^{\kappa\phi(x)}\frac{1}{2}\sum_{i=1}^n\left\{\hbar_\nu\frac{\partial^2\nu}{\partial x_i^2}(t,x) - \left[\frac{1}{\hbar_\nu}\left[\frac{\partial\phi}{\partial x_i}(x)\right]^2 + \frac{\partial^2\phi}{\partial x_i^2}(x)\right]\nu(t,x)\right\}. \tag{23}$$

When combined with Equation 17, and incorporating the initial condition, we get the following theorem:

Theorem 1 *The FPKfe*

$$\begin{cases}\dfrac{\partial u}{\partial t}(t,x) = \dfrac{\hbar_\nu}{2}\displaystyle\sum_{i=1}^n\frac{\partial^2 u}{\partial x_i^2}(t,x) - \sum_{i=1}^n\left[\frac{\partial\phi}{\partial x_i}(x)\right]\frac{\partial u}{\partial x_i}(t,x) - \sum_{i=1}^n\left[\frac{\partial^2\phi}{\partial x_i^2}(x)\right]u(t,x),\\ u(t,x) = \sigma_i(x),\end{cases} \tag{24}$$

is equivalent, via the gauge transformation in Equation 16, to the following time-independent Euclidean Schrödinger equation:

$$\begin{cases} \dfrac{\partial \nu}{\partial t}(t,x) = \dfrac{\hbar_\nu}{2}\displaystyle\sum_{i=1}^{n}\dfrac{\partial^2 \nu}{\partial x_i^2}(t,x) - \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\left[\dfrac{1}{\hbar_\nu}\left[\dfrac{\partial \phi}{\partial x_i}(x)\right]^2 + \dfrac{\partial^2 \phi}{\partial x_i^2}(x)\right]\nu(t,x), \\ \nu(t,x) = \sigma_0(x)e^{-\kappa\phi(x)}. \end{cases} \tag{25}$$

Furthermore, when the following Beněs-type condition is satisfied ($T \geq 0$, P_i and r real)

$$\sum_{i=1}^{n}\left[\frac{\partial^2 \phi}{\partial x_i^2}(x) + \frac{1}{\hbar_\nu}\left(\frac{\partial \phi}{\partial x_i}(x)\right)^2\right] = \frac{1}{\hbar_\nu}\left[\sum_{i,j=1}^{n} T_{ij}x_i x_j + \sum_{n=1}^{n} P_i x_i + r\right], \tag{26}$$

the equivalent Euclidean Schrödinger equation is

$$\begin{cases} -\hbar_\nu\dfrac{\partial \nu}{\partial t}(t,x) = -\dfrac{\hbar_\nu^2}{2}\displaystyle\sum_{i=1}^{n}\dfrac{\partial^2 \nu}{\partial x_i^2}(t,x) + \dfrac{1}{2}\left[\displaystyle\sum_{i,j=1}^{n} T_{ij}x_i x_j + \displaystyle\sum_{i=1}^{n} P_i x_i + r\right]\nu(t,x), \\ \nu(t,x) = \sigma_0(x)e^{-\kappa\phi(x)}. \end{cases} \tag{27}$$

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4 Special Cases

4.1 Linear Symmetric Drift Matrix Case

Suppose that

$$\phi(x) = \frac{1}{2}\sqrt{\hbar_\nu}\sum_{k,l=1}^{n} x_k S_{kl} x_l. \tag{28}$$

Observe that the constant matrix S can be taken to be a symmetric matrix (i.e., $S^T = S$). For, if A is an antisymmetric matrix, i.e., $A^T = -A$, then

$$\begin{aligned} x^T A x &= \left(x^T A x\right)^T, \\ &= -x^T A x, \\ &= 0. \end{aligned} \tag{29}$$

Then

$$\begin{aligned} \frac{\partial \phi}{\partial x_i}(x) &= \frac{1}{2}\sqrt{\hbar_\nu}\sum_{k,l=1}^{n}\left[\delta_{ik} S_{kl} x_l + x_k S_{kl} \delta_{il}\right], \\ &= \frac{1}{2}\sqrt{\hbar_\nu}\sum_{l=1}^{n} S_{il} x_l + \frac{1}{2}\sqrt{\hbar_\nu}\sum_{k=1}^{n} x_k S_{ki}, \\ &= \sqrt{\hbar_\nu}\sum_{l=1}^{n} S_{il} x_l. \end{aligned} \tag{30}$$

Consequently,

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}(x) &= \sqrt{\hbar_\nu}\sum_{l=1}^{n} S_{il}\delta_{il}, \\ &= \sqrt{\hbar_\nu}\sum_{l=1}^{n} S_{ll}, \\ &= \sqrt{\hbar_\nu}\,\mathrm{tr} S, \end{aligned} \tag{31}$$

and

$$\begin{aligned} \frac{1}{\hbar_\nu}\sum_{i=1}^{n}\left(\frac{\partial \phi}{\partial x_i}(x)\right)^2 &= \sum_{i=1}^{n} (S_{ij} x_j)^2, \\ &= x^T S^2 x. \end{aligned} \tag{32}$$

Thus, the FPKfe reduces to the following Euclidean Schrödinger equation:

$$\begin{cases} -\hbar_\nu \dfrac{\partial \nu}{\partial t}(t,x) = -\dfrac{\hbar_\nu^2}{2}\displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \nu}{\partial x_i^2}(t,x) + \dfrac{1}{2}\left(\sqrt{\hbar_\nu}\text{tr } S + x^T S^2 x\right)\nu(t,x), \\ \nu(t,x) = \sigma_i(x) e^{-\kappa\phi(x)}. \end{cases} \tag{33}$$

Note that if

$$\tilde{\nu}(t,x) \equiv \exp\left(\sqrt{\hbar_\nu} t \frac{1}{2}\text{tr}S\right)\nu(t,x), \tag{34}$$

then

$$\begin{cases} -\hbar_\nu \dfrac{\partial \tilde{\nu}}{\partial t}(t,x) = -\dfrac{\hbar_\nu^2}{2}\displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \tilde{\nu}}{\partial x_i^2}(t,x) + \dfrac{1}{2}[x^T S^2 x]\tilde{\nu}(t,x), \\ \tilde{\nu}(t,x) = \sigma_i(x) e^{-\kappa\phi(x) + \kappa\sqrt{\hbar_\nu} t \frac{1}{2}\text{tr } S}. \end{cases} \tag{35}$$

4.2 Affine Linear Symmetric Drift Matrix

Suppose

$$\phi(x) = \sqrt{\hbar_\nu}\left(\frac{1}{2}\sum_{i,j=1}^{n} x_i S_{ij} x_j + \sum_{i=1}^{n} l_i x_i\right). \tag{36}$$

Then,

$$\frac{\partial \phi}{\partial x_i}(x) = \sqrt{\hbar_\nu}\left(\sum_{l=1}^{n} S_{il} x_l + l_i\right), \tag{37}$$

and

$$\sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}(x) = \sqrt{\hbar_\nu}\text{tr } S, \tag{38}$$

and

$$\begin{aligned} \frac{1}{\hbar_\nu}\sum_{i=1}^{n}\left(\frac{\partial \phi}{\partial x_i}(x)\right)^2 &= \sum_{i=1}^{n}(S_{ij}x_j + l_i)^2, \\ &= \left(x^T S + l^T\right)(Sx + l) \end{aligned} \tag{39}$$

Therefore, Equation 25 reduces to

$$\begin{cases} -\hbar_\nu \dfrac{\partial \nu}{\partial t}(t,x) = -\dfrac{\hbar_\nu^2}{2}\displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \nu}{\partial x_i^2}(t,x) + \dfrac{1}{2}\left[x^T T x + P^T x + r\right]\nu(t,x), \\ \nu(t,x) = \sigma_i(x) e^{-\kappa\phi(x)}, \end{cases} \tag{40}$$

where T is a $n \times n$ symmetric matrix, P is a $n-$dimensional vector and r a scalar defined as follows:

$$\begin{aligned} T &= S^2, \\ P &= 2l^T S, \\ r &= \sqrt{\hbar_\nu} \text{tr } S + l^T l. \end{aligned} \tag{41}$$

4.3 General Beněs Case

In the general Beněs case the equivalent Euclidean Schrödinger equation is again given by Equation 40, except that T, P and r are not necessarily given by Equation 41. Thus, it includes the affine, linear symmetric drift matrix case as a special case. In addition, there are nonlinear state models that satisfy the Beněs cndition. For example, the one-dimensional problem with drift given by

$$f(x) = \frac{Ae^x - Be^{-x}}{Ae^x + Be^{-x}}, \tag{42}$$

satisfies the Beněs condition. Such a filter is termed the Beněs filter, a special case of the more general Yau filter which is the most general finite dimensional filter of maximal rank. Defining $\tilde{\nu}(t, x) = e^{\frac{1}{2}\kappa r t} \nu(t, x)$ leads to the following PDE:

$$\begin{cases} -\hbar_\nu \dfrac{\partial \tilde{\nu}}{\partial t}(t, x) = -\dfrac{\hbar_\nu^2}{2} \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \tilde{\nu}}{\partial x_i^2}(t, x) + \dfrac{1}{2} \left[x^T T x + P^T x \right] \tilde{\nu}(t, x), \\ \qquad \tilde{\nu}(t, x) = \sigma_i(x) e^{-\kappa \phi(x) + \frac{1}{2}\kappa r t}, \end{cases} \tag{43}$$

In subsequent sections, the fundamental solution of this equation is derived utilizing only linear algebra techniques and the computation of elementary transcendental functions.

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5 Diagonalization via Affine Orthogonal Transformation

In this section, it is shown that the Euclidean Schrödinger equation is related to the $n-$dimensional quantum simple harmonic oscillator.

5.1 Linear Symmetric Drift Matrix

As discussed in Section 4.1, the FPKfe for this case is equivalent to Equation 35. Now S is a real, symmetric matrix and so S and S^2 are diagonalizable, i.e., there exists an orthogonal matrix O such that $O^T SO$ is diagonal:

$$O^T SO = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix}, \qquad O^T S^2 O = \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^2 \end{bmatrix}. \tag{44}$$

Let

$$\bar{x}_i = \sum_{j=1}^{n} O_{ij} x_j. \tag{45}$$

As proved in Appendix A, the Laplacian is invariant under an orthogonal transformation. Therefore, in this case, Equation 35 is equivalent to the following PDE:

$$\begin{cases} -\hbar_\nu \dfrac{\partial \tilde{\nu}}{\partial t}(t, \bar{x}) = \displaystyle\sum_{i=1}^{n} \left[-\frac{\hbar_\nu^2}{2} \frac{\partial^2}{\partial \bar{x}_i^2} + \frac{1}{2} \lambda_i^2 \bar{x}_i^2 \right] \tilde{\nu}(t, \bar{x}), \\ \tilde{\nu}(t, \bar{x}) = \sigma_i(\bar{x}) e^{-\kappa \phi(\bar{x}) + \kappa t \frac{1}{2} \mathrm{tr}\ S}. \end{cases} \tag{46}$$

5.2 General Case: Nonnegative-Definite T

For the general Beněs case, let $x' = Ox$, where O diagonalizes T. Then,

$$x^T T x + P^T x = \bar{x}^T \Lambda \bar{x} + P^T O^T x, \tag{47}$$

$$= \sum_{i=1}^{n} (\lambda_i^2 x_i^2 + \alpha_i x_i). \tag{48}$$

Here Λ is the diagonal matrix of eigenvalues, $\lambda_i^2, i = 1, 2, \ldots, n$, of T and α_i are elements of the vector $P^T O^T$. Then

$$\begin{cases} -\hbar_\nu \dfrac{\partial \tilde{\nu}}{\partial t}(t, \bar{x}) = \displaystyle\sum_{i=1}^{n} \left[-\frac{\hbar_\nu^2}{2} \frac{\partial^2}{\partial \bar{x}_i^2} + \frac{1}{2} [\lambda_i^2 \bar{x}_i^2 + \alpha_i \bar{x}_i] \right] \tilde{\nu}(t, x), \\ \tilde{\nu}(t, \bar{x}) = \sigma_i(\bar{x}) e^{-\kappa \phi(\bar{x}) + \frac{1}{2} \kappa r t}. \end{cases} \tag{49}$$

If T is not nonnegative-definite some of its eigenvalues are 0.

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6 Fundamental Solution

So far it has been demonstrated that the FPKfe for the state model with Beněs drift and diffusion matrix $\hbar_\nu I_n$ is equivalent to Equation 49. Thus, it is sufficient to obtain the fundamental solution to Equation 49. In order to do that, we use the method of separation of variables. Let

$$\tilde{\nu}(t, \bar{x}) = T(t) \prod_{i=1}^{n} \psi^{(i)}(\bar{x}_i). \tag{50}$$

Substituting this into Equation 49 and dividing by $\tilde{\nu}(t, \bar{x})$ leads to

$$-\hbar_\nu \frac{dT}{dt}(t) = \sum_{i=1}^{n} \frac{1}{\psi^{(i)}(\bar{x}_i)} \hat{H}_i \psi^{(i)}(\bar{x}_i), \tag{51}$$

where

$$\hat{H}_i = -\frac{\hbar_\nu^2}{2} \frac{\partial^2}{\partial \bar{x}_i^2} + \frac{1}{2} \left[\lambda_i^2 \bar{x}_i^2 + \alpha_i \bar{x}_i \right]. \tag{52}$$

From the discussion in Appendices D and E, the fundamental solution can be written in terms of the eigenvalues and eigenfunctions of $\hat{H}_i$. Specifically,

$$\bar{P}(t, \bar{x}|t'|\bar{x'}) = \theta(t - t') \sum_{r_0=1}^{\infty} \sum_{r_2=0}^{\infty} \cdots \sum_{r_n=1}^{\infty} \left[\prod_{i=1}^{n} \psi_{r_i}(\bar{x}_i) \psi_{r_i}(\bar{x}'_i) \right] \exp\left(-(t - t')\kappa \sum_{i=1}^{n} E_{r_i} \right). \tag{53}$$

This can be expressed as a product of fundamental solutions of one-dimensional Euclidean Schrödinger equations with a quadratic potential:

$$\bar{P}(t, \bar{x}|t'|\bar{x'}) = \theta(t - t') \prod_{i=1}^{n} \left[\sum_{r=0}^{\infty} \psi_r^{(i)}(\bar{x}_i) \psi_r^{(i)}(\bar{x}'_i) e^{-\kappa E_{r_i}(t-t')} \right]. \tag{54}$$

Therefore,

$$\bar{P}(t, \bar{x}|t', \bar{x}') = \prod_{i=1}^{n} p_i(t, \bar{x}_i|t', \bar{x}_i), \tag{55}$$

where $p_i(t, \bar{x}_i|t', \bar{x}_i)$ is the fundamental solution of the one-dimensional Euclidean Scrödinger equation

$$\begin{cases} -\hbar_\nu \dfrac{\partial p_i}{\partial t}(t, \bar{x}_i|t', \bar{x}'_i) = \left[-\dfrac{\hbar_\nu^2}{2} \dfrac{d^2}{d\bar{x}_i^2} + \dfrac{1}{2} \left[\lambda_i^2 \bar{x}_i^2 + \alpha_i \bar{x}_i \right] \right] p_i(t, \bar{x}_i|t', \bar{x}'_i), \\ p_i(t', \bar{x}_i|t', \bar{x}'_i) = \delta(\bar{x}_i - \bar{x}'_i). \end{cases} \tag{56}$$

Exact, closed-form expressions of the $p_i(t, \bar{x}_i|t', \bar{x}'_i)$ for arbitrary $\bar{x}_i, \bar{x}'_i$ and any $t > t'$ are derived in Appendix E.

Finally, combining the above with Equation 49, the fundamental solution of the original FPKfe is:

$$P(t, x|t', x') = \exp\left(-\kappa\left[\phi(\bar{x}') - \phi(\bar{x})\right] + \frac{1}{2}\kappa r(t' - t)\right)\bar{P}(t, \bar{x}|t', \bar{x}'), \tag{57}$$
$$= \exp\left(-\kappa\left[\phi(\bar{x}') - \phi(\bar{x})\right] + \frac{1}{2}\kappa r(t' - t)\right)\prod_{i=1}^{n} p_i(t, \bar{x}_i|t', \bar{x}_i).$$

Using the results derived in Appendix E, the following detailed expression follows. Let there be n_1 pairs $(\lambda_{i_1} \neq 0, \alpha_{i_1})$, n_2 pairs $(\lambda_{i_2} = 0, \alpha_{i_2} = 0)$, n_3 pairs $(\lambda_{i_3} \neq 0, \alpha_{i_3} \neq 0)$, and n_4 pairs $(\lambda_{i_4} = 0, \alpha_i \neq 0)$, with $n = n_1 + n_2 + n_3 + n_4$. Then

$$P(t, x|t', x') = \exp\left(-\kappa\left[\phi(\bar{x}') - \phi(\bar{x})\right] + \frac{1}{2}\kappa r(t' - t)\right)P_A P_B P_C P_D, \tag{58}$$

where

$$P_A = \prod_{i_1=1}^{n_1}\sqrt{\frac{\lambda_{i_1}}{2\pi\hbar_\nu \sinh(\lambda_{i_1}(t - t'))}}\exp\left(-\frac{\lambda_{i_1}\left[(\bar{x}_{i_1}^2 + \bar{x}'^2_{i_1})\cosh(\lambda_{i_1}(t - t')) - 2\bar{x}_{i_1}\bar{x}'_{i_1}\right]}{2\hbar_\nu \sinh(\lambda_{i_1}(t - t'))}\right), \tag{59}$$

$$P_B = \prod_{i_2=n_1+1}^{n_1+n_2}\frac{1}{\sqrt{2\pi\hbar_\nu(t - t')}}\exp\left(-\frac{(\bar{x}_{i_2} - \bar{x}'_{i_2})^2}{2\hbar_\nu(t - t')}\right),$$

$$P_C = \prod_{i_3=n_1+n_2+1}^{n_1+n_2+n_3}\sqrt{\frac{\lambda_{i_3}}{2\pi\hbar_\nu \sinh(\lambda_{i_3}(t - t'))}}\exp\left(\frac{\alpha_{i_3}^2}{8\lambda_{i_3}^2}(t - t')\right)$$
$$\times \exp\left(-\frac{\lambda_{i_3}}{2\hbar_\nu \sinh(\lambda_{i_3}(t - t'))}\left[(\bar{\xi}_{i_3}^2 + \bar{\xi}'^2_{i_3})\cosh(\lambda_{i_3}(t - t')) - 2\bar{\xi}_{i_3}\bar{\xi}'_{i_3}\right]\right),$$

$$P_D = \prod_{i_4=n_1+n_2+n_3+1}^{n}\sqrt{\frac{1}{2\pi\hbar_\nu(t - t')}}$$
$$\exp\left[-\frac{1}{2\hbar_\nu(t - t')}\left((\bar{x}_{i_4} - \bar{x}'_{i_4})^2 + \frac{\alpha_{i_4}}{2}(\bar{x}_{i_4} + \bar{x}'_{i_4})(t - t')^2 - \frac{\alpha_{i_4}^2}{24}(t - t')^4\right)\right].$$

and where in P_C

$$\bar{\xi}_{i_3} = \bar{x}_{i_3} + \frac{\alpha_{i_3}}{2\lambda_{i_3}^2}. \tag{60}$$

Observe that it is sufficient to calculate the unnormalized transition probability density by ignoring factors independent of the state(s) and then normalizing it.

7 Examples

In this section, we verify the validity of the formulas derived for the fundamental solution. Namely, we compute the fundamental solution using the formulas derived here and compare it with results obtained using simulations. For simplicity, all examples are one-dimensional.

Example 1 (Linear Model) *Consider the model*

$$d\mathbf{x}(t) = -0.5\mathbf{x}dt + \sqrt{10}d\mathbf{v}(t), \qquad x_0 = 45. \tag{61}$$

The fundamental solution is given by Case A in Appendix E. From the transition probability density in Figure 1, it is evident that the agreement with simulations is excellent even for large time steps. The simulated values are calculated from 10^5 *samples.*

Note that one-step explicit finite difference methods cannot be used here since the time steps are so large (i.e., large Courant number). The implicit schemes are unconditionally stable, but they are not accurate (e.g., for instance, [9]). Furthermore, the simplicity of using these fomulas, as opposed to discretizing the original PDE, is self evident.

Example 2 (Affine Linear Model) *In this example, we consider the following affine, linear state model:*

$$d\mathbf{x}(t) = (-0.2\mathbf{x} + 1.2)dt + \sqrt{20}d\mathbf{v}(t), \qquad x_0 = 35. \tag{62}$$

The simulation parameters are as in the previous example. In this case, the relevant formula is given in the discussion of Case C in Appendix E. Figure 2 again shows excellent agreement between computed results for the fundamental solution and simulations. Once again the step sizes are too large for utilization of a one-step explicit finite difference scheme.

Example 3 (A Beněs Model) *We now consider the following model that is a Beněs type model:*

$$d\mathbf{x}(t) = tanh\mathbf{x}(t)dt + \hbar_\nu d\mathbf{v}(t), \qquad x_0 = 10. \tag{63}$$

The state model drift is nonlinear. Since

$$\frac{df}{dx}(x) = sech^2 x, \tag{64}$$

it follows that

$$\frac{df}{dx}(x) + f^2(x) = sech^2 x + tanh^2 x = 1. \tag{65}$$

This instance is covered by Case B in Appendix E. In Figure 3, we plot our results and compare it with simulation results. Note that this nonlinear state model has been solved exactly for any time step.

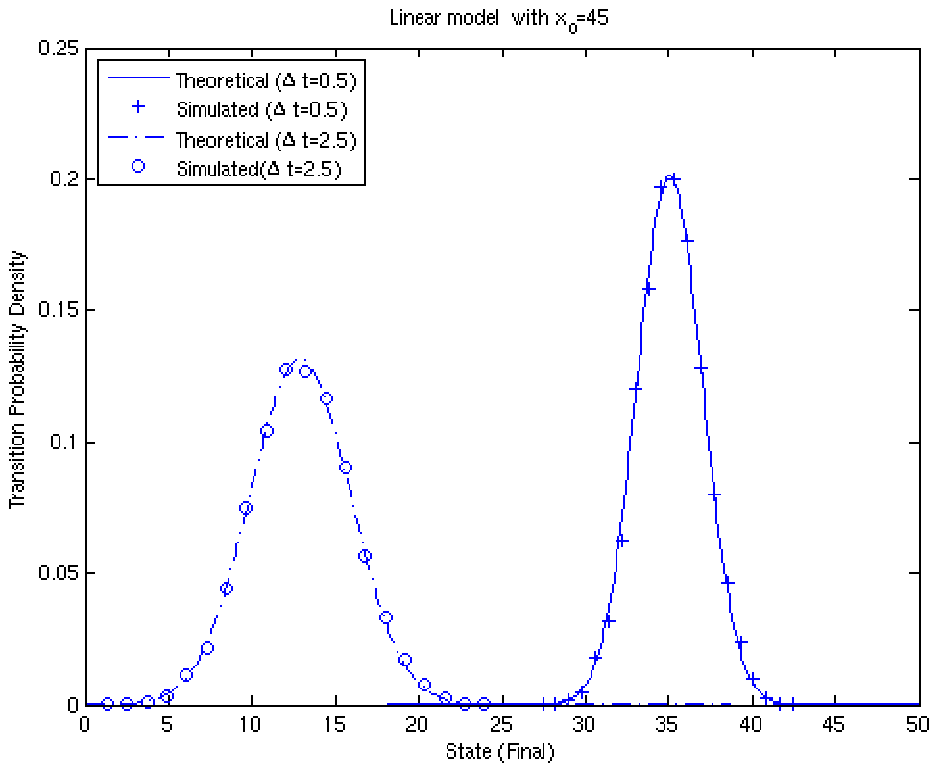

Figure 1: *Linear state model.*

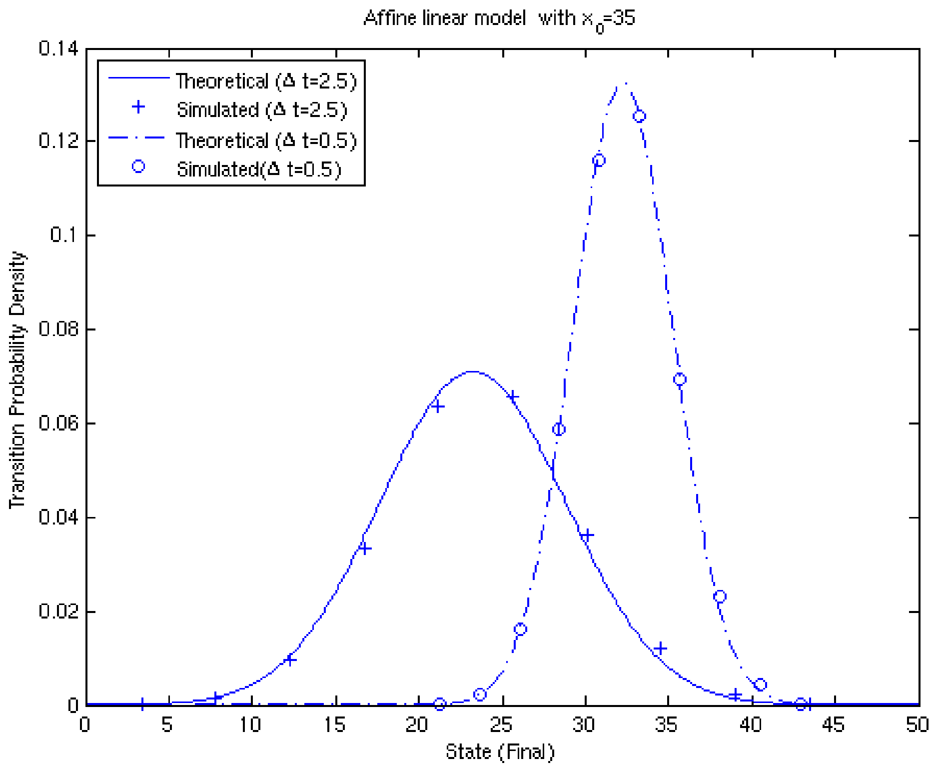

Figure 2: *Affine, linear state model.*

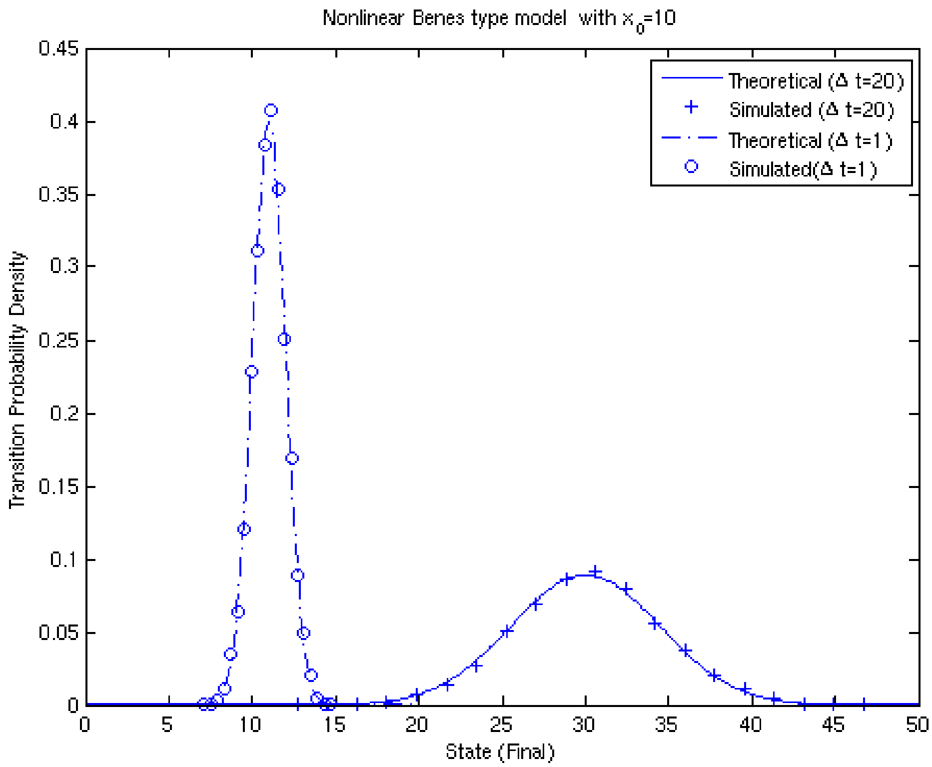

Figure 3: *A state model with Beněs type of drift.*

8 Comment on the Continuous-Continuous Nonlinear Yau Filtering System with Beněs Drift

The continuous-continuous filtering problem considered here is based on the following signal and observation model (see [8]):

$$\begin{cases} d\mathbf{x}(t) & = f(\mathbf{x}(t))dt + e(\mathbf{x}(t))d\mathbf{v}(t), \quad x(t_0) = x_0, \\ d\mathbf{y}(t) & = h(\mathbf{x}(t))dt + d\mathbf{w}(t), \quad y(t_0) = 0, \end{cases} \tag{66}$$

where $\mathbf{x}, \mathbf{v}, \mathbf{y}$, and $\mathbf{w}$ are $\mathbb{R}^n, \mathbb{R}^p, \mathbb{R}^m$, and $\mathbb{R}^m$ valued processes, and $\mathbf{v}$ and $\mathbf{w}$ have components that are independent, standard Brownian processes. It is also assumed that $n = p$, f and h are sufficiently smooth and the diffusion matrix for e is $\hbar_\nu I_n$. The conditional density $\rho(t, x)$ of the state given the observation $\{y(s) : t_0 \leq s \leq t\}$, is given by normalizing $\sigma(t, x)$ that satisfies the DMZ equation

$$\begin{cases} d\sigma(t,x) & = L_0\sigma(t,x)dt + \sum_{i=1}^m h_i(x)\sigma(t,x)dy_i(t) \\ \sigma(t_0,x) & = \sigma_0, \end{cases}$$

where σ_0 is the probability density at t_0, and

$$L_0 = \frac{\hbar_\nu}{2}\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} - \sum_{i=1}^n f_i(x)\frac{\partial}{\partial x_i} - \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(x) - \frac{1}{2}\sum_{i=1}^m h_i^2(x). \tag{67}$$

The function $u(t, x)$ defined as

$$u(t,x) \equiv \exp\left(-\sum_{i=1}^m h_i(x)y_i(t)\right)\sigma(t,x), \tag{68}$$

satisfies the time-varying partial differential equation called the robust DMZ equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = & \dfrac{\hbar_\nu}{2}\displaystyle\sum_{i=1}^n \dfrac{\partial^2 u}{\partial x_i^2}(t,x) + \sum_{i=1}^n\left(-f_i(x) + \sum_{j=1}^m y_j(t)\dfrac{\partial h_j}{\partial x_i}(x)\right)\dfrac{\partial u}{\partial x_i}(t,x) \\ & - \left(\displaystyle\sum_{i=1}^n \dfrac{\partial f_i}{\partial x_i}(x) + \dfrac{1}{2}\sum_{i=1}^m h_i^2(x) - \dfrac{1}{2}\sum_{i=1}^m y_i(t)\Delta h_i(x) + \sum_{i=1}^m\sum_{j=1}^n y_i(t)f_j(x)\dfrac{\partial h_i}{\partial x_j}(x)\right. \\ & \left. - \dfrac{1}{2}\displaystyle\sum_{i,j=1}^m\sum_{k=1}^n y_i(t)y_j(t)\dfrac{\partial h_i}{\partial x_k}(x)\dfrac{\partial h_j}{\partial x_k}(x)\right)u(t,x), \\ u(0,x) = & \sigma_0(x). \end{cases}$$

Let $\mathscr{P} = \{\tau_0 < \tau_1 < \tau_2 < \ldots < \tau_k = \tau\}$ be a partition of $[\tau_0, \tau]$. It was proved in [10] that if $u_{k'}(t, x)$ solves the robust DMZ equation for $\tau_{k'-1} \leq t \leq \tau_{k'}$ with $y(t) = y(\tau_{k'-1})$ (or $y(\tau'_k)$) with initial condition

$$u_{k'}(\tau_{k'-1}, x) = u_{k'-1}(\tau_{k'-1}, x), \tag{69}$$

then

$$u(\tau, x) = \lim_{|\mathscr{P}_k| \to 0} u_k(\tau_k, x), \tag{70}$$

in both the pointwise and L^2 senses. Thus, we just need an algorithm to compute $u_{k'}(\tau_{k'}, x)$, which can be done by computing $\tilde{u}_{k'}(\tau_{k'}, x)$, where, for $\tau_{k'-1} \leq t \leq \tau_{k'}$, $\tilde{u}_{k'}(\tau_{k'}, x)$ satisfies the following Yau Equation

$$\begin{cases} \dfrac{\partial \tilde{u}_{k'}}{\partial t}(t, x) = \dfrac{\hbar_\nu}{2} \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \tilde{u}_{k'}}{\partial x_i^2}(t, x) - \sum_{i=1}^{n} f_i(x) \dfrac{\partial \tilde{u}_k}{\partial x_i}(t, x) - \left(\sum_{i=1}^{n} \dfrac{\partial f_i}{\partial x_i}(x) + \dfrac{1}{2} \sum_{i=1}^{m} h_j^2(x) \right) \tilde{u}_{k'}(t, x), \\ \tilde{u}_{k'}(\tau_{k'-1}, x) = \exp\left[\displaystyle\sum_{j=1}^{m} (y_j(\tau_{k-1}) - y_j(\tau_{k-2})) h_j(x) \right] \tilde{u}_{k'-1}(\tau_{k'-1}, x). \end{cases} \tag{71}$$

Note that

$$u_{k'}(\tau_{k'}, x) = \exp\left[-\sum_{j=1}^{m} y_j(\tau_{k-1}) h_j(x) \right] \tilde{u}_{k'}(\tau_{k'}, x). \tag{72}$$

Finally, the desired quantity, the unnormalized conditional probability density $\sigma(t, x)$ follows from the solution $\tilde{u}_i$ of the Yau Equation as follows[11]:

$$\sigma(\tau, x) = \lim_{|\mathscr{P}_k| \to 0} \tilde{u}_k(\tau_k, x). \tag{73}$$

For the Yau filtering system with Beněs drift, it therefore follows that it is sufficient to solve the following Euclidean Schrödinger equation:

$$\begin{cases} \dfrac{\partial \nu}{\partial t}(t, x) = \dfrac{\hbar_\nu}{2} \displaystyle\sum_{i=1}^{n} \dfrac{\partial^2 \nu}{\partial x_i^2}(t, x) - \dfrac{1}{2} q(x) \nu(t, x) \\ \nu(\tau_{i-1}, x) = \sigma_i(x) e^{-\phi(x)}. \end{cases} \tag{74}$$

where

$$\begin{aligned} q(x) &\equiv \sum_{i=1}^{n} \left[\frac{1}{\hbar_\nu} \left[\frac{\partial \phi}{\partial x_i}(x) \right]^2 + \frac{\partial^2 \phi}{\partial x_i^2}(x) \right] + \sum_{i=1}^{m} h_i^2(x), \\ &= x^T Q x + P^T x + r, \end{aligned} \tag{75}$$

where $Q = Q^T = (q_{ij}), 1 \leq i, j \leq n$, $P = (p_1, \dots, p_n)$ and r is a scalar. In [8] it was shown that when $q(x)$ is a quadratic polynomial the fundamental solution for the Yau filtering system can be found by solving a system of nonlinear ODEs. They proceeded to show that this system of nonlinear ODEs is solved explicitly by the power series method. Observe that $h_i(x)$ need not be a linear function.

However, the case of quadratic $q(x)$ is precisely of the form studied in the previous sections. Therefore, the continuous filtering problem can be solved using the results of the previous sections in the Yau algorithm for this special case.

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9 Conclusion

When the problem is nonlinear, the extended Kalman filter is not a reliable solution. A more general approach is called continuous-discrete filtering, which is based on solving a partial differential equation called the Fokker-Planck-Kolmogorov forward equation (FPKfe). The general framework for tackling tracking problems is universal nonlinear filtering.

In this report, it has been shown that the fundamental solution of the FPKfe that arises in the continous-discrete filtering problem with Beněs drift (and with diffusion matrix proportional to the identity matrix) can be obtained using elementary linear algebra techniques. This method can be summarized as follows:

1. Determine the matrix T, vector P and scalar r defined in Equation 26.
2. Determine the matrix O that diagonalizes T, and hence determine $\bar{x}$, λ_i and α_i, where $i = 1, 2, \ldots, n$.
3. The fundamental solution $P(t, x|t', x')$ follows from Equation 57, where the $p_i(t, \bar{x}_i|t', \bar{x}'_i)$ are given by formulas in Appendix E depending on λ_i and α_i (or from Equations 58 and 59.
4. The continuous-discrete filtering problem can be solved using the prediction and correction steps (see Section 2).

Therefore, the conditional probability density function in the above-mentioned continuous-discrete filtering problem can be determined without discretization of the FPKfe operator. From the conditional probability density function, various estimates are optimal under different criteria (e.g., the conditional mean is the minimum variance estimate). Note that there are no restrictions on the size of the time step or the initial distribution. Furthermore, the measurement process can be non-linear; there is no need to linearize it, as in extended Kalman filtering. Also, there is no need to solve ordinary, or partial, differential equations. Finally, such an exact solution of a nonlinear filtering problem is useful for testing the accuracy of approximate nonlinear filtering algorithms.

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Annex A: Invariance of Laplacian under Affine Orthogonal Transformation

Let O be a real orthogonal matrix, i.e., $O^T O = O O^T = I$ or in components $\sum_{j=1}^{n} O_{ij}(O^T)_{jk} = \delta_{ik}$. Since $O^T = O^{-1}$, the orthogonality property implies that $(O^{-1})_{jk} = O_{kj}$ or $\sum_{j=1}^{n} O_{ij} O_{kj} = \delta_{ik}$.

Let x'_i be related to x_i via an affine orthogonal transformation as follows:

$$x'_i = \sum_{k=1}^{n} O_{ik} x_k + l_i, \tag{A.1}$$

where l_i is a constant. Then, using the chain rule of partial derivatives

$$\begin{aligned} \frac{\partial}{\partial x_j} &= \sum_{i=1}^{n} \frac{\partial x'_i}{\partial x_j} \frac{\partial}{\partial x'_i}, \\ &= \sum_{i,k=1}^{n} O_{ik} \delta_{kj} \frac{\partial}{\partial x'_i}, \\ &= \sum_{i=1}^{n} O_{ij} \frac{\partial}{\partial x'_i}. \end{aligned} \tag{A.2}$$

Likewise,

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} &= \sum_{i,j,k=1}^{n} O_{ij} O_{kj} \frac{\partial^2}{\partial x'_i \partial x'_k}, \\ &= \sum_{i,k=1}^{n} \delta_{ik} \frac{\partial^2}{\partial x'_i \partial x'_k}, \\ &= \sum_{i=1}^{n} \frac{\partial^2}{\partial x'^2_i}. \end{aligned} \tag{A.3}$$

Therefore, the Laplacian is invariant under an affine orthogonal transformation. Note that the affine orthogonal transformation can be a function of time.

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Annex B: One-Dimensional Euclidean Harmonic Oscillator

B.1 Eigenvalues and Eigenfunctions

We are interested in solving the following PDE:

$$-\hbar_\nu \frac{\partial \Phi}{\partial t}(t,\xi) = \left(-\frac{\hbar_\nu^2}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2}\lambda^2 \xi^2 \right) \Phi(t,\xi). \tag{B.1}$$

Following the method of the separation of variables, let

$$\Phi(t,\xi) = T(t)\psi(\xi). \tag{B.2}$$

Then, substituting Equation B.2 into Equation B.1 and dividing by $T(t)\psi(\xi)$ leads to

$$-\frac{\hbar_\nu}{T(t)} \frac{dT}{dt}(t) = \frac{1}{\psi(\xi)} \left(-\frac{\hbar_\nu^2}{2} \frac{d^2}{d\xi^2} + \frac{1}{2}\lambda^2 \xi^2 \right) \psi(\xi). \tag{B.3}$$

Therefore

$$\begin{aligned} E &= -\frac{\hbar_\nu}{T(t)} \frac{dT}{dt}(t), \\ &= \frac{1}{\psi(\xi)} \left(-\frac{\hbar_\nu}{2} \frac{d^2}{d\xi^2} + \frac{1}{2}\lambda^2 \xi^2 \right) \psi(\xi). \end{aligned} \tag{B.4}$$

Solving for $T(t)$ yields

$$\frac{dT}{dt}(t) = -\kappa E T(t), \qquad \kappa \equiv \frac{1}{\hbar_\nu}, \tag{B.5}$$

or

$$T(t) = T(0) e^{-\kappa E t}. \tag{B.6}$$

It remains to solve

$$\left(-\frac{\hbar_\nu^2}{2} \frac{d^2}{d\xi^2} + \frac{1}{2}\lambda^2 \xi^2 \right) \psi(\xi) = E\psi(\xi), \tag{B.7}$$

or

$$\hat{H}\psi(\xi) = E\psi(\xi), \qquad \hat{H} \equiv -\frac{\hbar_\nu^2}{2} \frac{d^2}{d\xi^2} + \frac{1}{2}\lambda^2 \xi^2. \tag{B.8}$$

This is most simply solved using operator methods as follows (see, for instance, [12]). Define the operator $\hat{a}$ by

$$\hat{a} = \frac{1}{\sqrt{2\lambda\hbar_\nu}}\left(\lambda\xi + \hbar_\nu \frac{d}{d\xi}\right), \quad (B.9)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{\xi}{\xi_0} + \xi_0 \frac{d}{d\xi}\right),$$

where

$$\xi_0 \equiv \sqrt{\frac{\hbar_\nu}{\lambda}}. \quad (B.10)$$

Note that $\hat{a}$ is not a Hermitian operator [1]:

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\xi}{\xi_0} - \xi_0 \frac{d}{d\xi}\right). \quad (B.12)$$

The commutator of two operators $\hat{A}$ and $\hat{B}$ is defined by

$$[\hat{A}, \hat{B}]f \equiv \hat{A}(\hat{B}f) - \hat{B}(\hat{A}f), \quad (B.13)$$

where f is any C^∞ real function. Thus

$$\left[\xi, \frac{d}{d\xi}\right] f = \xi\frac{df}{d\xi} - \frac{d}{d\xi}(\xi f), \quad (B.14)$$
$$= \xi\frac{df}{d\xi} - f - \xi\frac{df}{d\xi},$$
$$= -f,$$

or

$$\left[\xi, \frac{d}{d\xi}\right] = -1, \quad (B.15)$$

and

$$\left[\frac{d}{d\xi}, \xi\right] = 1. \quad (B.16)$$

1. The Hermitian conjugate of $\hat{a}$ is denoted by $\hat{a}^\dagger$ and is defined by

$$\int dx(\hat{a}^\dagger\varphi)^*\psi = \int dx\varphi^*\hat{a}\psi, \quad (B.11)$$

for arbitrary φ and ψ.

Clearly

$$[\xi, \xi] = \left[\frac{d}{d\xi}, \frac{d}{d\xi}\right] = 0. \tag{B.17}$$

Then

$$\begin{aligned}
[\hat{a}, \hat{a}^\dagger] &= \frac{1}{2}\left[\frac{\xi}{\xi_0} + \xi_0\frac{d}{d\xi}, \frac{\xi}{\xi_0} - \xi_0\frac{d}{d\xi}\right], \\
&= \frac{1}{2}\left[\frac{d}{d\xi}, \xi\right] - \frac{1}{2}\left[\xi, \frac{d}{d\xi}\right], \\
&= 1.
\end{aligned} \tag{B.18}$$

The operator $\hat{H}$ can be written as

$$\begin{aligned}
\hat{H} &= \frac{1}{2}\hbar_\nu\lambda(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger), \\
&= \hbar_\nu\lambda\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right),
\end{aligned} \tag{B.19}$$

where we used the commutator equation B.18. Therefore, it is sufficient to find the eigenvalues of

$$\hat{N} \equiv \hat{a}^\dagger\hat{a}. \tag{B.20}$$

Let $\psi_\nu(\xi)$ be a normalized eigenfunction of $\hat{N}$ with eigenvalue ν:

$$\hat{N}\psi_\nu(\xi) = \nu\psi_\nu(\xi). \tag{B.21}$$

Using the commutator identity

$$[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}, \tag{B.22}$$

we see that

$$\begin{aligned}
[\hat{N}, \hat{a}^\dagger] &= [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger], \\
&= \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a}, \\
&= \hat{a}^\dagger,
\end{aligned} \tag{B.23}$$

and

$$\begin{aligned}
[\hat{N}, \hat{a}] &= [\hat{a}^\dagger\hat{a}, \hat{a}], \\
&= \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a}, \\
&= -[\hat{a}, \hat{a}^\dagger]\hat{a}, \\
&= -\hat{a}.
\end{aligned} \tag{B.24, B.25}$$

Now

$$\int \psi_\nu^\dagger(\xi)\hat{a}^\dagger\hat{a}\psi_\nu(\xi)d\xi = \int [\hat{a}\psi_\nu(\xi)]^\dagger[\hat{a}\psi(\xi)]d\xi, \tag{B.26}$$

$$= \nu \int \psi_\nu^\dagger(\xi)\psi_\nu(\xi)d\xi,$$

$$\geq 0. \tag{B.27}$$

Therefore $\nu \geq 0$. So the lowest possible eigenvalue is $\nu = 0$. It is also useful to note that

$$\hat{N}\hat{a}^\dagger\psi_\nu(\xi) = (\hat{a}^\dagger\hat{N} + \hat{a}^\dagger)\psi_\nu(\xi), \tag{B.28}$$

$$= (\nu + 1)\hat{a}^\dagger\psi_\nu(\xi),$$

so that $\hat{a}^\dagger\psi_\nu$ is an eigenfunction of $\hat{N}$ with eigenvalue $\nu + 1$, and

$$\hat{N}\hat{a}\psi_\nu(\xi) = (\hat{a}\hat{N} - \hat{a})\psi_\nu(\xi), \tag{B.29}$$

$$= (\nu - 1)\hat{a}\psi_\nu(\xi).$$

so that $\hat{a}\psi_\nu$ is an eigenfunction of $\hat{N}$ with eigenvalue $\nu - 1$. Define $\psi_{\nu+1}$ to be the normalized (up to a phase) eigenfunction of $\hat{N}$ with eigenvalue $\nu + 1$. Since

$$\int (\hat{a}^\dagger\psi_\nu(\xi))^\dagger(\hat{a}^\dagger\psi_\nu(\xi))d\xi = \int \psi_\nu^\dagger(\xi)(\hat{N} + 1)\psi_\nu(\xi)d\xi, \tag{B.30}$$

$$= (\nu + 1)\int \psi_\nu^\dagger(\xi)\psi_\nu(\xi)d\xi,$$

it follows that

$$\hat{a}^\dagger\psi_\nu(\xi) = \sqrt{\nu + 1}\psi_{\nu+1}(\xi). \tag{B.31}$$

Likewise, since

$$\int \left(\hat{a}\psi_\nu(\xi)\right)^\dagger \left(\hat{a}\psi_\nu(\xi)\right) d\xi = \int \psi_\nu^\dagger(\xi)\hat{N}\psi_\nu(\xi)d\xi, \tag{B.32}$$

$$= \nu \int \psi_\nu^\dagger(\xi)\psi_\nu(\xi)d\xi,$$

it follows that

$$\hat{a}\psi_\nu(\xi) = \sqrt{\nu}\psi_{\nu-1}(\xi). \tag{B.33}$$

All the eigenfunctions are thus given by $\psi_n, n = 0, 1, 2, \ldots$ where

$$\psi_n(\xi) = \frac{1}{\sqrt{n}}\hat{a}^\dagger\psi_{n-1}(\xi), \tag{B.34}$$

$$= \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n\psi_0(\xi).$$

From Equation B.34, it follows that we just need to determine ψ_0 to determine all the eigenfunctions. This is easily done by noting that

$$\hat{a}\psi_0(\xi) = 0, \tag{B.35}$$

or

$$\left(\frac{d}{d\xi} + \frac{\xi}{\xi_0^2}\right)\psi_0(\xi) = 0, \tag{B.36}$$

so that

$$\psi_0(\xi) = \frac{1}{\sqrt{\sqrt{\pi}\xi_0}} \exp\left[-\frac{1}{2}\left(\frac{\xi}{\xi_0}\right)^2\right]. \tag{B.37}$$

Hence

$$\begin{aligned}\psi_n(\xi) &= \frac{1}{\sqrt{n!\sqrt{\pi}\xi_0}} \left(\hat{a}^\dagger\right)^n \exp\left[-\frac{1}{2}\left(\frac{\xi}{\xi_0}\right)^2\right], \\ &= \frac{1}{\sqrt{2^n n!\sqrt{\pi}\xi_0}} \exp\left[-\frac{1}{2}\left(\frac{\xi}{\xi_0}\right)^2\right] H_n\left(\frac{\xi}{\xi_0}\right),\end{aligned} \tag{B.38}$$

with eigenvalue $E_n = \hbar_\nu\lambda\left(n + \frac{1}{2}\right)$ and $H_n(\xi)$ are Hermite polynomials defined by

$$\begin{aligned}H_n(\xi) &= e^{\xi^2/2}\left(\sqrt{2}\hat{a}^\dagger\right)^n\Big|_{\xi_0=1} e^{-\xi^2/2}, \\ &= e^{\xi^2} e^{-\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{\xi^2/2} e^{-\xi^2}.\end{aligned} \tag{B.39}$$

Since

$$\begin{aligned}e^{-\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)e^{\xi^2/2} &= e^{-\xi^2/2}\xi e^{\xi^2/2} - e^{-\xi^2/2}\frac{de^{\xi^2/2}}{d\xi} - e^{-\xi^2/2}e^{\xi^2/2}\frac{d}{d\xi}, \\ &= -\frac{d}{d\xi},\end{aligned} \tag{B.40}$$

it follows that

$$e^{-\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)^n e^{\xi^2/2} = \underbrace{e^{-\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)e^{\xi^2/2}\cdots e^{-\xi^2/2}\left(\xi - \frac{d}{d\xi}\right)e^{\xi^2/2}}_{n \text{ times}} \tag{B.41}$$

$$= (-1)^n \frac{d^n}{d\xi^n}. \tag{B.42}$$

Therefore, we may write

$$H_n(\xi) = (-1)^n e^{\xi^2}\frac{d^n}{d\xi^n}e^{-\xi^2}. \tag{B.43}$$

Some relevant properties of Hermite polynomials are summarized in Appendix C

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Annex C: Hermite Polynomials

In this section, we summarize some useful properties of Hermite polynomials. For a good discussion, see, for instance, [13]. Consider the identity

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2+2i\xi t}dt = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-(t-i\xi)^2-\xi^2}dt, \qquad (C.1)$$
$$= e^{-\xi^2},$$

since

$$\int_{-\infty}^{\infty} e^{-a\xi^2-2b\xi}d\xi = \sqrt{\frac{\pi}{a}}e^{b^2/a}. \qquad (C.2)$$

Then,

$$\frac{d^n}{d\xi^n}e^{-\xi^2} = \frac{(2i)^n}{\sqrt{\pi}}\int_{-\infty}^{\infty} t^n e^{-t^2+2i\xi t}dt, \qquad (C.3)$$

and

$$H_n(\xi) = (-1)^n\frac{(2i)^n}{\sqrt{\pi}}e^{\xi^2}\int_{-\infty}^{\infty} t^n e^{-t^2+2i\xi t}dt, \qquad (C.4)$$

and so we get the generating function of the Hermite polynomials:

$$\sum_{n=0}^{\infty}\frac{1}{n!}H_n(\xi)r^n = \frac{e^{\xi^2}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2+2it(\xi-r)}dt, \qquad (C.5)$$
$$= e^{2\xi r-r^2}.$$

Since

$$e^{2\xi r-r^2} = \sum_{p=0}^{\infty}\frac{(2\xi)^p}{p!}r^p\sum_{q=0}^{\infty}(-1)^q\frac{r^{2q}}{q!}, \qquad (C.6)$$

we get

$$H_n(\xi) = \sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{n!}{k!(n-2k)!}(2\xi)^{n-2k}. \qquad (C.7)$$

Differentiating n times, we get

$$\frac{d^n H_n}{d\xi^n}(\xi) = 2^n n!. \qquad (C.8)$$

If $n > m$

$$\int_{-\infty}^{\infty} e^{-\xi^2} H_n(\xi) H_m(\xi) d\xi = (-1)^n \int_{-\infty}^{\infty} \left(\frac{d^n e^{-\xi^2}}{d\xi^n} \right) H_m(\xi) d\xi, \tag{C.9}$$

and integrating by parts n times shows that the integral vanishes. If $n = m$, then

$$\begin{aligned} (-1)^n \int_{-\infty}^{\infty} \frac{d^n e^{-\xi^2}}{d\xi^n} H_n(\xi) d\xi &= \int_{-\infty}^{\infty} e^{-\xi^2} \frac{d^n}{d\xi^n} H_n(\xi) d\xi, \\ &= 2^n n! \sqrt{\pi}. \end{aligned} \tag{C.10}$$

Therefore

$$\int_{-\infty}^{\infty} e^{-\xi^2} H_n(\xi) H_m(\xi) d\xi = 2^n n! \sqrt{\pi} \delta_{mn}. \tag{C.11}$$

Finally, note that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{H_n(\xi) H_n(\xi')}{2^n n!} r^n &= \frac{e^{\xi^2 + \xi'^2}}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-s^2 - t^2 + 2is\xi' + 2it\xi - 2srt} ds dt, \\ &= \frac{e^{\xi^2 + \xi'^2}}{\pi} \int_{-\infty}^{\infty} e^{-t^2 + 2it\xi} dt \int_{-\infty}^{\infty} e^{-s^2 + 2s(i\xi' - rt)} ds, \\ &= \frac{e^{\xi^2 + \xi'^2}}{\pi} \int_{-\infty}^{\infty} e^{-t^2 + 2it\xi} \sqrt{\pi} e^{(i\xi' - rt)^2} dt, \\ &= \frac{e^{\xi^2 + \xi'^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(1-r^2)t^2 + 2it(\xi - r\xi') - \xi'^2} dt, \\ &= \frac{e^{\xi^2}}{\sqrt{\pi}} \sqrt{\frac{\pi}{1 - r^2}} e^{-(\xi - r\xi')^2/(1-r^2)}, \\ &= \frac{1}{\sqrt{1 - r^2}} e^{\left[2\xi\xi' r - (\xi^2 + \xi'^2) r^2\right]/(1-r^2)}. \end{aligned} \tag{C.12}$$

Annex D: Fundamental Solution in terms of the Eigenfunctions

Consider the following PDE:

$$-\hbar_\nu \frac{\partial \Phi}{\partial t}(t,\xi) = \hat{H}\Phi(t,\xi), \tag{D.1}$$

where $\hat{H}$ is a Hermitian operator. An important property of a self-adjoint operator is completeness. Let $\psi_n(x), n = 0, 1, 2, \ldots$ be a complete set of eigenfunctions of the Hermitian operator $\hat{H}$ with eigenvalues $E_n, n = 0, 1, \ldots$. Then, the completeness property can be stated as follows:

$$\sum_{n=0}^{\infty} \psi_n(\xi)\psi_n(\xi') = \delta(\xi - \xi'). \tag{D.2}$$

This is also referred to as the resolution of the identity. Although we assumed the spectrum to be discrete (as in our application), the completeness property is also valid for Hermitian operators with a continuous spectrum with the summation replaced by integration in Equation D.2.

We now show that the fundamental solution of Equation D.1 is given by

$$P(t,\xi|t',\xi') = \theta(t-t')\sum_{r=0}^{\infty} \psi_r(\xi)\psi_r(\xi')e^{-\kappa E_n(t-t')}. \tag{D.3}$$

Observe that since

$$\frac{\partial}{\partial t}\theta(t-t') = \delta(t-t'), \tag{D.4}$$

where $\theta(t-t')$ is the Heaviside step function, and

$$\hat{H}\sum_{r=0}^{\infty} \psi_r(\xi)\psi_r(\xi') = \sum_{r=0}^{\infty} E_r\psi_r(\xi)\psi_r(\xi'), \tag{D.5}$$

we get

$$\left(-\hbar_\nu \frac{\partial}{\partial t} - \hat{H}\right) P(t,\xi|t',\xi) = \delta(t-t')\delta(\xi-\xi'), \tag{D.6}$$

or

$$\left(-\hbar_\nu \frac{\partial}{\partial t} - \hat{H}\right) P(t,\xi|t',\xi') = 0, \qquad \text{when } t \neq t'. \tag{D.7}$$

From Equation D.2, it follows that

$$P(t', \xi | t', \xi') = \delta(\xi - \xi'). \tag{D.8}$$

Thus, we have demonstrated that $P(t, \xi | t', \xi')$ is the fundamental solution of Equation D.1, i.e., it is the solution of the following set of equations:

$$\begin{cases} -\hbar_\nu \dfrac{\partial P}{\partial t}(t, \xi | t', \xi') = \hat{H} P(t, \xi | t', \xi'), \\ P(t', \xi | t', \xi') = \delta(\xi - \xi'). \end{cases} \tag{D.9}$$

Annex E: Fundamental Solution of One-dimensional Schrödinger Equation with Quadratic Potential

Here the expressions for the fundamental solution of the one-dimensional Euclidean Schrödinger equation for a particle moving under a potential that is up to quadratic in ξ, i.e.,

$$-\hbar_\nu \frac{\partial \Phi}{\partial t}(t, \xi) = \left[-\frac{\hbar_\nu^2}{2} \frac{\partial^2}{\partial \xi^2} + \frac{1}{2}(\lambda^2 \xi^2 + \alpha \xi) \right] \Phi(t, \xi), \tag{E.1}$$

is derived. For an alternative derivation using the path integral method, see [14].

E.1 Case A: $\lambda \neq 0$, $\alpha = 0$

From the discussion in Appendix D, the fundamental solution of Equation B.1 is given by $(t > t')$

$$P(t, \xi | t', \xi') = \sum_{n=0}^{\infty} \frac{1}{2^n n! \sqrt{\pi} \xi_0} \exp\left[-\frac{(\xi^2 + \xi'^2)}{2\xi_0^2} \right] H_n\left(\frac{\xi}{\xi_0}\right) H_n\left(\frac{\xi'}{\xi_0}\right) e^{-\kappa E_n (t - t')}. \tag{E.2}$$

Since $E_n = \hbar_\nu \lambda \left(n + \frac{1}{2}\right)$,

$$P(t, \xi | t', \xi') = \sum_{n=0}^{\infty} \frac{1}{2^n n! \sqrt{\pi} \xi_0} \exp\left[-\frac{(\xi^2 + \xi'^2)}{2\xi_0^2} \right] H_n\left(\frac{\xi}{\xi_0}\right) H_n\left(\frac{\xi'}{\xi_0}\right) e^{-\lambda (t - t')} r^n, \tag{E.3}$$

where $r = e^{-\lambda(t-t')}$. The identity in Equation C.12 implies that

$$\begin{aligned} P(t, \xi | t', \xi') = {} & \frac{1}{\sqrt{\pi}\xi_0} \frac{e^{-\frac{1}{2}\lambda(t-t')}}{\sqrt{1 - e^{-2\lambda(t-t')}}} \exp\left\{ -\frac{(\xi^2 + \xi'^2) e^{-2\lambda(t-t')} - 2\xi\xi' e^{-\lambda(t-t')}}{\left(1 - e^{-2\lambda(t-t')}\right)\xi_0^2} \right\} \\ & \times \exp\left[-\frac{(\xi^2 + \xi'^2)}{2\xi_0^2} \right]. \end{aligned} \tag{E.4}$$

Now $\xi_0^2 = \hbar_\nu / \lambda$, and

$$\begin{aligned} \frac{1}{\sqrt{\pi}\xi_0} \frac{e^{-\frac{1}{2}\lambda(t-t')}}{\sqrt{1 - e^{-2\lambda(t-t')}}} &= \sqrt{\frac{\lambda}{\pi \hbar_\nu}} \frac{1}{\sqrt{e^{\lambda(t-t')} - e^{-\lambda(t-t')}}}, \\ &= \sqrt{\frac{\lambda}{2\pi\hbar_\nu \sinh[\lambda(t - t')]}}. \end{aligned} \tag{E.5}$$

Also

$$-\frac{2\lambda\xi\xi' e^{-\lambda(t-t')}}{[1-e^{-2\lambda(t-t')}]\,\xi_0^2} = -\frac{2\xi\xi'}{\hbar_\nu\,[e^{\lambda(t-t')}-e^{-\lambda(t-t')}]}, \qquad \text{(E.6)}$$
$$= -\frac{2\lambda\xi\xi'}{2\hbar_\nu \sinh(\lambda(t-t'))},$$

and

$$\frac{(\xi^2+\xi'^2)}{\xi_0^2}\left[\frac{e^{-2\lambda(t-t')}}{1-e^{-2\lambda(t-t')}}+\frac{1}{2}\right] = \frac{\lambda\,(\xi^2+\xi'^2)}{2\hbar_\nu}\left[\frac{e^{-2\lambda(t-t')}+1}{1-e^{-2\lambda(t-t')}}\right], \qquad \text{(E.7)}$$
$$= \frac{\lambda\,(\xi^2+\xi'^2)}{2\hbar_\nu \sinh[\lambda(t-t')]}\cosh[\lambda(t-t')].$$

Thus, the fundamental solution is

$$P(t,\xi|t',\xi') = \sqrt{\frac{\lambda}{2\pi\hbar_\nu \sinh[\lambda(t-t')]}}\exp\left(-\frac{\lambda\left\{(\xi^2+\xi'^2)\cosh[\lambda(t-t')]-2\xi\xi'\right\}}{2\hbar_\nu \sinh[\lambda(t-t')]}\right). \qquad \text{(E.8)}$$

E.2 Case B: $\lambda = 0$, $\alpha = 0$

The fundamental solution for the case when both $\lambda = 0$ and $\alpha = 0$ is easily obtained by taking the limit $\lambda \to 0$ of the Equation E.8. Specifically, since

$$\lim_{\lambda\to 0}\frac{\sinh[\lambda(t-t')]}{\lambda} = (t-t'), \qquad \lim_{\lambda\to 0}\cosh(\lambda(t-t')) = 1, \qquad \text{(E.9)}$$

we get

$$P(t,\xi|t',\xi') = \frac{1}{\sqrt{2\pi\hbar_\nu(t-t')}}\exp\left[-\frac{(\xi-\xi')^2}{2\hbar_\nu(t-t')}\right]. \qquad \text{(E.10)}$$

E.3 Case C: $\lambda \neq 0$, $\alpha \neq 0$

The fundamental solution for this case follows from the result derived in Section E.1 by noting that with the substitution

$$\bar{\xi} = \xi + \frac{\alpha}{2\lambda^2}, \qquad \text{(E.11)}$$

Equation E.1 becomes

$$\frac{\partial\Phi}{\partial t}(t,\bar{\xi}) = \left[\frac{\hbar_\nu}{2}\frac{\partial^2}{\partial\bar{\xi}^2} - \frac{1}{2}\left(\lambda^2\bar{\xi}^2 - \frac{\alpha^2}{4\lambda^2}\right)\right]\Phi(t,\bar{\xi}). \qquad \text{(E.12)}$$

This is the same as Case A with E shifted by $\frac{\alpha^2}{8\lambda^2}$. Thus, we can deduce from Equation E.8 that the fundamental solution in this case is

$$P(t,\xi|t',\xi') = \sqrt{\frac{\lambda}{2\pi\hbar_\nu \sinh[\lambda(t-t')]}} \exp\left[\frac{\alpha^2}{8\hbar_\nu\lambda^2}(t-t')\right] \times \exp\left(-\frac{\lambda}{2\hbar_\nu \sinh(\lambda(t-t'))}\left[(\bar{\xi}^2+\bar{\xi}'^2)\cosh(\lambda(t-t')) - 2\bar{\xi}\bar{\xi}'\right]\right). \tag{E.13}$$

E.4 Case D: $\lambda = 0$, $\alpha \neq 0$

The fundamental solution in the case where the quadratic term vanishes but the linear term is non-zero follows from taking the limit $\lambda \to 0$ of Equation E.13 (e.g., see [15]). First note that

$$\lim_{\lambda\to 0}\sqrt{\frac{\lambda}{2\pi\hbar_\nu \sinh(\lambda(t-t'))}} = \sqrt{\frac{1}{2\pi\hbar_\nu(t-t')}}. \tag{E.14}$$

In order to compute the $\lambda \to 0$ limit in the remainder of Equation E.13, use the following Taylor series expansions

$$\begin{aligned} \xi\coth\xi &= 1 + \frac{1}{3}\xi^2 - \frac{1}{45}\xi^4 + \cdots, \\ \xi\text{cosech}\xi &= 1 - \frac{1}{6}\xi^2 + \frac{7}{360}\xi^4. \end{aligned} \tag{E.15}$$

As

$$\begin{aligned} \bar{\xi}^2 + \bar{\xi}'^2 &= \xi^2 + \xi'^2 + \frac{\alpha}{\lambda^2}(\xi+\xi') + \frac{\alpha^2}{2\lambda^4}, \\ \bar{\xi}\bar{\xi}' &= \xi\xi' + \frac{\alpha}{2\lambda^2}(\xi+\xi') + \frac{\alpha^2}{4\lambda^4}. \end{aligned} \tag{E.16}$$

we obtain the following:

$$\begin{aligned} &\frac{1}{2\hbar_\nu(t-t')}\left(\bar{\xi}^2+\bar{\xi}'^2\right)\lambda(t-t')\coth(\lambda(t-t')) - 2\bar{x}\bar{\xi}'\lambda\text{cosech}(\lambda(t-t')) \\ &= \frac{1}{2\hbar_\nu(t-t')}\Bigg[\left(\xi^2+\xi'^2+\frac{\alpha}{\lambda^2}(\xi+\xi')+\frac{\alpha^2}{2\lambda^4}\right)\left(1+\frac{1}{3}\lambda^2(t-t')^2 - \frac{1}{45}\lambda^4(t-t')^4+\cdots\right) \\ &\quad - 2\left(\xi\xi'+\frac{\alpha}{2\lambda^2}(\xi+\xi')+\frac{\alpha^2}{4\lambda^4}\right)\left(1-\frac{1}{6}\lambda^2(t-t')^2+\frac{7}{360}\lambda^4(t-t')^4+\cdots\right)\Bigg]. \end{aligned} \tag{E.17}$$

Expanding this as a Laurent series in λ, we get

$$\frac{1}{2\hbar_\nu(t-t')}\left[\left(\frac{\alpha^2}{2\lambda^4}-2\frac{\alpha^2}{4\lambda^4}\right)\right] \tag{E.18}$$
$$+\frac{1}{2\hbar_\nu(t-t')}\left[\left(\frac{\alpha^2}{2\lambda^2}\left(\frac{1}{3}+\frac{1}{6}\right)(t-t')^2+\left(\frac{\alpha^2}{\lambda^2}-2\frac{\alpha^2}{2\lambda^2}\right)(\xi+\xi')\right)\right]$$
$$+\frac{1}{2\hbar_\nu(t-t')}\left[(\xi-\xi')^2+\alpha(\xi+\xi')(t-t')^2\left(\frac{1}{3}+\frac{1}{6}\right)-\frac{\alpha^2}{2}\left(\frac{1}{45}+\frac{7}{360}\right)(t-t')^4\right]+O(\lambda^2).$$

This simplifies to

$$-\frac{\alpha^2}{8\hbar_\nu\lambda^2}(t-t')-\frac{1}{2\hbar_\nu(t-t')}\left[(\xi-\xi')^2+\frac{\alpha}{2}(\xi+\xi')(t-t')^2-\frac{\alpha^2}{24}(t-t')^4\right]+O(\lambda^2). \tag{E.19}$$

The first term cancels the exponential term with an essential singularity in λ. Therefore, the fundamental solution is

$$P(t,\xi|t',\xi')=\sqrt{\frac{1}{2\pi\hbar_\nu(t-t')}} \tag{E.20}$$
$$\times\exp\left[-\frac{1}{2\hbar_\nu(t-t')}\left((\xi-\xi')^2+\frac{\alpha}{2}(\xi+\xi')(t-t')^2-\frac{\alpha^2}{24}(t-t')^4\right)\right].$$

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(Security classification of title, body of abstract and indexing annotation must be entered when document is classified)

1. ORIGINATOR (The name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Centre sponsoring a contractor's report, or tasking agency, are entered in section 8.) Defence R&D Canada – Ottawa 3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4	2. SECURITY CLASSIFICATION (Overall security classification of the document including special warning terms if applicable.) UNCLASSIFIED

3. TITLE (The complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title.)

Simple harmonic oscillator and the Beněs filter

4. AUTHORS (Last name, followed by initials – ranks, titles, etc. not to be used.)

Balaji, B.

5. DATE OF PUBLICATION (Month and year of publication of document.) March 2009	6a. NO. OF PAGES (Total containing information. Include Annexes, Appendices, etc.) 58	6b. NO. OF REFS (Total cited in document.) 15

7. DESCRIPTIVE NOTES (The category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

Technical Memorandum

8. SPONSORING ACTIVITY (The name of the department project office or laboratory sponsoring the research and development – include address.)

Defence R&D Canada – Ottawa
3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4

9a. PROJECT NO. (The applicable research and development project number under which the document was written. Please specify whether project or grant.) 13dz03	9b. GRANT OR CONTRACT NO. (If appropriate, the applicable number under which the document was written.)
10a. ORIGINATOR'S DOCUMENT NUMBER (The official document number by which the document is identified by the originating activity. This number must be unique to this document.) DRDC Ottawa TM 2008-352	10b. OTHER DOCUMENT NO(s). (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

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The solution to the universal continuous-discrete filtering problem requires the solution of the Fokker-Planck-Kolmogorov forward equation (FPKfe) for the state process for an arbitrary initial condition. The fundamental solution for the FPKfe is derived for a state model of arbitrary dimension with Beněs drift and a positive-definite diffusion matrix proportional to the identity matrix, and requires only the computation of elementary transcendental functions and standard linear algebra techniques. In particular, there is no need to solve a partial (or ordinary) differential equation. The measurement process may be a discrete-time nonlinear stochastic process, and the time step size can be arbitrary. These results can also be applied to explicitly solve the continuous-continuous Yau filtering problem with Beněs drift.

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