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Signal Processing 84 (2004) 133–139

SIGNAL PROCESSING

www.elsevier.com/locate/sigpro

Morphological operators on complex signals

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Received 22 October 2002; received in revised form 23 July 2003

Abstract

This paper presents a novel approach to the generalization of mathematical morphology to complex signals and images. This generalization is strongly dependent upon the issue of multivariate order relationships. Although there is no natural way to order multivariate data such as complex numbers, it is possible to design an order relationship such that it meets criteria that are appropriate to complex signal processing. We first outline the criteria that the order relationship should meet. We then propose a new order relationship that meets these criteria. Based on this order relationship, we construct the basic operators in mathematical morphology: dilations and erosions.

Keywords: Multivalued morphology; Ordering; Complex functions; Transformations

1. Introduction

Mathematical morphology is an image processing methodology which has been very successful in processing binary and gray-tone images. It is also applicable to 1-D signals. However, it lacks a strong conceptual framework in processing complex signals and images. The goal of this paper is to provide such a framework, thus extending mathematical morphology to complex signals.

Morphological operators are based on order relationships. However, there is no natural way of ordering complex numbers, as Barnett [2] observed, among others. Therefore, any order relationship among complex signals will be somewhat arbitrary. Yet, there are some guidelines we can use in order to design an order relationship which would be robust, physically relevant and reasonably implementable.

We first outline the criteria that make an order relationship appropriate to our specific context. We then propose a new order relationship that fulfills these criteria. We define the set complementation, based on this order relationship. As Serra did for gray-tone functions [8], we define the umbra on complex signals. Finally, we present the dilation and erosion operators, which constitute the basis onto which all the morphological operators are built upon.

There have been some extensions of morphology to color, or multi-valued images, Comer and Delp's work [4,5] being typical. They proposed two methods: the first one was to perform standard morphological operations on each of the red (r), green (g) and blue (b) (RGB) components separately. The problem with this approach is that the transformations introduce values that do not exist in the input image. It is the same situation with complex signals. New phases and amplitudes would be created in an

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doi:10.1016/j.sigpro.2003.10.002

DRDC Ottawa SL 2002-117

uncontrolled manner. Moreover, this method does not provide a strong enough formalism to assess and predict the behavior of such transformations. Their second approach partially corrected these problems. They transformed the vector-valued colors to a scalar, usually the Euclidean distance $d = \sqrt{r^2 + g^2 + b^2}$. They then used d to order pixels. They finally defined a maximum $\vee$ and a minimum $\wedge$ operator that used this order relationship. However, using a single scalar to order multivariate data has ambiguity problems. Two pixels with the same amplitude could very well not have the same color. Although Astola et al. [1] mentioned it was of little practical importance for color image processing, it is a different situation in communication signals, where we often strive to transmit and receive constant amplitude signals. Moreover, the relationship is not an ordering because it violates the antisymmetry property: $a \leqslant b$ and $b \leqslant a$ implies $a = b$.

Talbot et al. [9] suggested the use of a lexicographic order relationship for multivalued morphology. They proposed simple and efficient algorithms that are appropriate for both quantized and continuous vectors. Chanussot and Lambert [3] proposed ordering relationships based on space filling curves. Both these approaches are vector-preserving, that is, they do not introduce values that do not exist in the input image. However, they are not directly applicable to complex signals.

Complex signals have their own specificities, that we need to take into account when generalizing morphology. First, samples at constant amplitude are more likely than in multi-spectral images. Second, it is absolutely necessary not to create samples that do not exist in the input signals. Specifically, phase is a fundamental characteristic of these samples and we should definitively not modify it in uncontrolled manner. Using standard morphological operators on the real and imaginary components independently is not appropriate. Third, complex signals usually are devoid of DC component. Usually, it has been carefully removed by filtering, because it is an undesirable characteristic. It is also removed by propagation in the environment. In contrast, DC components are omnipresent in multi-spectral images. Finally, signal power is an important selection criterion and has to be taken into account in the design of an order relationship.

2. Order relationship

2.1. Properties

$X \preccurlyeq Y$ is an order relationship between X and Y if the following properties hold:

Property 1.

$$X \preccurlyeq X \text{ is true.} \tag{1}$$

Property 2.

$$\text{If } X \preccurlyeq Y \text{ and } Y \preccurlyeq X \text{ then } X = Y. \tag{2}$$

Property 3.

$$\text{If } X \preccurlyeq Y \text{ and } Y \preccurlyeq Z \text{ then } X \preccurlyeq Z. \tag{3}$$

Property 4. *Trichotomy law: one and only one of the following relationships holds, for all X and Y in $\mathbb{C}$:*

$$X \prec Y, \quad X = Y, \quad X \succ Y. \tag{4}$$

When X, Y and Z are complex samples, we found that we should add the following properties, in order to improve the applicability of the order relationship.

Property 5. *The order relationship should be maintained under attenuation and gain variations λ.*

$$\text{If } X \preccurlyeq Y \text{ then } \lambda X \preccurlyeq \lambda Y, \lambda \in \mathbb{R}^+. \tag{5}$$

Property 6. *Scaling, translation and rotation invariance.*

The order relationship should only concern point-wise values. It should be invariant to scalings, that is, magnification factors and time scalings, as well as translations and rotations.

Property 7. *Comparing X to Y should yield the same result as comparing Y to X.*

This property eliminates implementation depending on the order into which the operands are applied.

$$X \preccurlyeq Y \text{ implies } Y \succcurlyeq X. \tag{6}$$

Property 8. *Signal statistics independence.*

The order relationship should be independent of the signal statistics. It should be a sample-to-sample, or point-wise relationship, regardless of the global characteristics of the signals under consideration. This property eliminates all the relationships that could rely on global measures such as averages. Not having this property could expose us to inconsistencies when processing signals with a DC bias. For instance, Barnett suggested using a scalar function based on some distance measurement between a sample and the average value of the dataset, which would be inapplicable to our context.

2.2. Order relationship

We propose the following order relationship, where $\Re(X)$ is the real part of X and $\Im(X)$ is the imaginary part and $|X|$ is the modulus of X:

$$X \preccurlyeq Y \text{ If : } \begin{cases} |X| < |Y| \\ \text{or} \\ |X| = |Y| \text{ and } \Re(X) < \Re(Y) \\ \text{or} \\ |X| = |Y| \text{ and } \Re(X) = \Re(Y) \text{ and} \\ \quad \Im(X) \leqslant \Im(Y). \end{cases} \tag{7}$$

We can also define a strict inequality based on this order relationship:

$$X \prec Y \text{ If : } \begin{cases} |X| < |Y| \\ \text{or} \\ |X| = |Y| \text{ and } \Re(X) < \Re(Y) \\ \text{or} \\ |X| = |Y| \text{ and } \Re(X) = \Re(Y) \text{ and} \\ \quad \Im(X) < \Im(Y). \end{cases} \tag{8}$$

Physically, signals which have more power than others should be considered larger. This is intuitively appealing in signal processing. Signals which have the same power level are considered larger than others if their real part is larger, in the usual sense. If signals are identical both for amplitude and for their real part, we then examine their imaginary part. The largest signal is then the one which has a positive imaginary component. This approach is somewhat arbitrary, especially when we put emphasis on the real component in the comparison. However, it is not possible to remove such arbitrariness.

For instance, we could have resolved the amplitude ambiguity by comparing phases instead of components. However, because phases vary between 0 and 2π, it is not possible to decide which phase is the largest without being arbitrary. Phase unrolling techniques could be devised, in order to free ourselves of the $0 - 2\pi$ circularity, but that would violate time translation and time scaling invariance, which would then make the order relationship impractical.

Speaking in terms of phase, for equal signal amplitudes, our order relationship arbitrarily decides that signals whose phases are closer to zero are the largest. Signals that are symmetrically positioned with respect to the real axis are considered larger if their phases are between 0 and π.

2.3. Proofs

Eq. (7) is an order relationship because Properties 1–4 hold.

Property 1. *It is true because the relation reduces to* $|X| = |X|$ *and* $\Re(X) = \Re(X)$ *and* $\Im(X) = \Im(X)$.

Property 2. *It is true because* $|X| < |Y|$ *and* $|Y| < |X|$ *is not possible. Therefore, the order relationship reduces to first comparing the real values of* X *and* Y, *followed by a comparison of the imaginary values, should the real values be equal. Comparing the real values, we find that* $\Re(X) < \Re(Y)$ *and* $\Re(Y) < \Re(X)$ *is not possible. Therefore,* $\Re(X) = \Re(Y)$ *and the order relationship reduces to* $\Im(X) \leqslant \Im(Y)$, *and* $\Im(Y) \leqslant \Im(X)$, *which is the same as* $\Im(X) = \Im(Y)$ *and therefore* $X = Y$.

Property 3. *There are two situations. In one case, there is at least one modulus not equal to the others. In the second case, all the modulus are equal.*

When there is at least one modulus different from the others, for instance, $|X| < |Y|$ *and that* $|Y| \leqslant |Z|$, *it is obvious that* $|X| < |Z|$ *and therefore that* $X \preccurlyeq Z$. *By symmetry, if* $|Y| < |Z|$ *and* $|X| \leqslant |Y|$, *we then have the same final result. The final case is* $|X| < |Y| < |Z|$ *and the proof is* *trivial. Therefore, whenever the modulus are not the same, it is easy to demonstrate that Property 3 holds.*

In the second case all the modulus are equal, that is, $|X| = |Y| = |Z|$. The order relationship then reduces to

$$\Re(x) < \Re(Y) \quad or \quad \begin{pmatrix} \Re(X) = \Re(Y) \\ and \\ \Im(X) \leqslant \Im(Y) \end{pmatrix} \quad and$$

$$\begin{pmatrix} \Re(Y) = \Re(Z) \\ and \\ \Im(Y) \leqslant \Im(Z)) \end{pmatrix}. \tag{9}$$

If $\Re(X) < \Re(Y)$, then $\Re(X) < \Re(Z)$ and therefore $X \preccurlyeq Z$. By symmetry, it is the same situation for $\Re(Y) < \Re(Z)$, with the same result. If $\Re(X) = \Re(Y) = \Re(Z)$, the relationship reduces to $\Im(X) \leqslant \Im(Y) \leqslant \Im(Z)$, therefore $X \preccurlyeq Z$.

Property 4. *First, on the complex plane, the set of points X that are strictly smaller than Y, $\{X : X \prec Y\}$, according to our order relationship, is the following union:*

1. *The interior of the circle with radius $|Y|$ centered on the origin.*
2. *The points of the circle with radius $|Y|$ centered on the origin, starting from Y (Y excluded), passing through point $-|Y|$ and ending at Y^* (Y^* excluded), where Y^* is the complex conjugate of Y.*
3. *Y^*, if $\Im(Y) > 0$.*

Second, only one point for the set of points that fulfills $X = Y$.

Finally, the set of points X that are strictly larger than Y, $\{X : X \succ Y\}$ is the following union:

1. *All the space outside of the circle centered on the origin and with the radius $|Y|$.*
2. *The path on the circle centered on the origin and with the radius $|Y|$, starting from Y (Y excluded), passing through the point $|Y|$, and finishing at Y^*, Y^* excluded.*
3. *Y^*, if $\Im(Y) < 0$.*

Careful observation of this complex plane partition shows us that the three regions corresponding respectively to $X \prec Y, X = Y, X \succ Y$ are mutually exclusive, which means that the trichotomy law holds.

Therefore, our relationship $X \preccurlyeq Y$ is an order relationship. We then verify the other properties of this order relationship as follows:

Property 5. *Substitution of Eq. (5) into (7) yields*

$$\lambda|X| < \lambda|Y| \quad or \quad \begin{pmatrix} \lambda|X| = \lambda|Y| \\ and \\ \lambda\Re(X) < \lambda\Re(Y) \end{pmatrix} \quad or$$

$$\begin{pmatrix} \lambda|X| = \lambda|Y| \\ and \\ \lambda\Re(X) = \lambda\Re(Y) \\ and \\ \lambda\Im(X) \leqslant \lambda\Im(Y) \end{pmatrix}. \tag{10}$$

It is obvious that this equation is true for $\lambda > 0$, if $X \preccurlyeq Y$.

Property 6. *There is no parameter in Eq. (7) that takes into account spatial or temporal scalings. Therefore, the order relationship is invariant to these parameters.*

Property 7. *There is no part of Eq. (7) that depends on the order of the operands.*

Property 8. *The order relationship is exclusively a point-wise comparison. Therefore, no signal statistics is involved.*

It should be noted that this order relationship, in general, does not simplify to the order relationship in $\mathbb{R}$, when $\Im(X) = 0$. It however does when $\Re(X) \in \mathbb{R}^+$ and $\Im(X) = 0$.

3. Complementation

The complementation procedure is a generalization of Serra's complementation on real-valued functions [8]. For signals in general, Serra defined the complementation as a simple negation, $X^c = -X$. In image processing, signals are generally bounded to some maximum value, $M \in \mathbb{R}^+$ because of limited dynamic range of the pixel representation, or because of detector limitations. Then, the definition of the complementation on an image sample $P \in \mathbb{R}^+$, P^c is defined as follows:

$$P^c = M - P. \tag{11}$$

Heijmans [6, Section 10.21] suggested an extension of complementation to color images which negates each component of the image:

$$P^c(r,g,b) = (M-r, M-g, M-b). \tag{12}$$

We generalize the complementation to complex numbers. There are two fundamental properties associated with complementation:

Property 9. Order relationship reversal under complementation,

$$X \preccurlyeq Y \textit{ implies } X^c \succcurlyeq Y^c. \tag{13}$$

Property 10.

$$X^{c^c} = X. \tag{14}$$

A simple negation such as Serra's would not be appropriate in our case, because Property 9 would not hold, according to our order relationship. Eq. (11) and its generalization to color images, 12 are appropriate only if the samples are positive. Moreover, Property 9 does not hold with our order relationship. Another type of generalization is needed.

First, we use the trigonometric representation of a complex sample:

$$X = A\mathrm{e}^{\mathrm{j}\theta}, \tag{15}$$

where $A = |X|$ and θ is the phase.

We define the complement of $X \in \mathbb{C}$, X^c, as

$$X^c = (M - A)\mathrm{e}^{\mathrm{j}(\theta+\pi)}. \tag{16}$$

It should be noted that this definition does not simplify to Serra's, when $\Im(X)=0$. This is to be expected, because our order relationship is based on signal amplitudes, that is, it usually ignores the sign of a sample and uses it merely to resolve ambiguities in case the amplitudes of X and Y are identical.

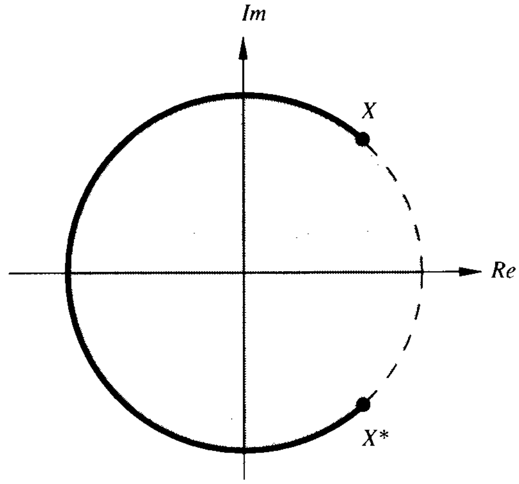

Fig. 1. The umbra of a single complex number, X, on the complex plane.

Property 9 holds because of the following. First, let $X = A\mathrm{e}^{\mathrm{j}\theta}$, and $Y = B\mathrm{e}^{\mathrm{j}\phi}$. If $A < B$, then $M - A > M - B$ and therefore the order relationship is reversed. If $A = B$, then $M - A = M - B$. There are then two cases: $\Re(X) < \Re(Y)$ or $\Re(X) = \Re(Y)$, and $\Im(X) \leqslant \Im(Y)$. When $\Re(X) < \Re(Y)$, $\Re(X^c) = -\Re(X), \Re(Y^c) = -\Re(Y)$ and therefore $\Re(X^c) > \Re(Y^c)$ and the order relationship is reversed. If $\Re(X) = \Re(Y)$, then $\Im(X^c) = -\Im(X), \Im(Y^c) = -\Im(Y)$ and therefore the order relationship is reversed as well.

Property 10 holds because, first $M - (M - A) = A$, and because $\mathrm{e}^{\mathrm{j}(\theta+2\pi)} = \mathrm{e}^{\mathrm{j}\theta}$.

4. Umbra

The umbra U of a real-valued function f is defined as the portion of space that is below the function, union with the function itself. For a single real-valued sample P, it is all the points of the real axis that are smaller than or equal to P. In the image processing practice, the signals are limited between $\pm M$, or between 0 and M, depending on the application.

The umbra of a complex sample X is similarly defined using our order relationship:

$$U = \{Y : Y \preccurlyeq X\}. \tag{17}$$

Fig. 1 illustrates graphically the umbra of a single sample on the complex plane. The umbra is the union of the following regions on the complex plane:

1. The interior of the circle centered on the origin and radius A.
2. The path on the circle centered on the origin and radius A, starting from X, passing by $\theta = \pi$, and ending at X^*, X^* included if $\Im(X) > 0$.

5. Maximum (∨) and minimum (∧)

The max ∨ and the min ∧ operators are defined:

$$X \vee Y = \begin{cases} X & \text{if } Y \preccurlyeq X, \\ Y & \text{otherwise,} \end{cases} \tag{18}$$

$$X \wedge Y = \begin{cases} Y & \text{if } Y \preccurlyeq X, \\ X & \text{otherwise.} \end{cases} \tag{19}$$

These operators are dimensionality preserving [7]:

$$\lambda X \wedge \lambda Y = \lambda(X \wedge Y), \tag{20}$$

$$\lambda X \vee \lambda Y = \lambda(X \vee Y). \tag{21}$$

This property is important because it ensures that the output of these operators have the same physical units as the units of the input signals. This property enables us to predict the properties of the output, when the signals are attenuated or amplified. The ∨ and ∧ operators preserve the dimensionality of the input signal because they merely choose between two samples, based on their order relationship.

6. Dilation and erosion

The dilation of a complex signal F by a flat structuring element B is denoted $\delta_B(F)$ and is defined as the maximum value of the translations of F by the vectors $-b$ of B, $\mathsf{trans}_{-b}(F)$ [8]:

$$\delta_B(F) = \bigvee_{b \in B} \mathsf{trans}_{-b}(F). \tag{22}$$

The erosion a complex signal F by the flat structuring element B uses the minimum instead of the maximum:

$$\varepsilon_B(F) = \bigwedge_{b \in B} \mathsf{trans}_{-b}(F). \tag{23}$$

It should be noted that the dilation and the erosion, using our order relationship, does not create new signal values. As it is the case with the scalar dilation and erosion, this transformation merely chooses among a certain number of samples the one that is the output.

These operators also preserve the dimensionality of the samples, exactly like they do for gray-tone images when we use flat structuring elements. This is because the operators ∨ and ∧ preserve the dimensionality, as mentioned in Eqs. (20) and (21):

$$\lambda\delta_B(F) = \delta_B(\lambda F), \tag{24}$$

$$\lambda\varepsilon_B(F) = \varepsilon_B(\lambda F). \tag{25}$$

They also commute under spatial or time scaling, like their relatives in gray-tone and binary image processing. For signals, the time axis is t. For images, the image plane coordinates are (x, y). In general, let $\mathscr{X}$ be the coordinates, and $\lambda\mathscr{X}$ a uniform scaling λ over these coordinates. $F(\mathscr{X})$ is a function of coordinates $\mathscr{X}$, the dilation $\delta_B(F)(\mathscr{X})$ is also function of $\mathscr{X}$ and so is $B(\mathscr{X})$:

$$\delta_B(F)(\lambda\mathscr{X}) = \delta_{B(\lambda\mathscr{X})}(F(\lambda\mathscr{X})), \tag{26}$$

$$\varepsilon_B(F)(\lambda\mathscr{X}) = \varepsilon_{B(\lambda\mathscr{X})}(F(\lambda\mathscr{X})), \tag{27}$$

where $F(\lambda)$ and $B(\lambda)$ denote spatial or time scaling on the image plane of signal F and structuring element B by the scaling factor λ. This is because such scaling is strictly a space/time operation.

6.1. Non-flat structuring elements

Flat structuring elements translate images and signals only along the image plane or the time axis. Non-flat structuring elements translate both along the image plane, or time axis, and along the value axis. Dilations and erosions of gray-tone image, or real-valued signal, $f(x, y)$ by a real-valued structuring function $g(x, y)$ are then defined:

$$\delta_g(f(x,y)) = \bigvee_{-\infty<j<\infty} \left[\bigvee_{-\infty<i<\infty} f(x-i, y-j) + g(i,j) \right], \tag{28}$$

$$\varepsilon_g(f(x,y)) = \bigwedge_{-\infty<j<\infty} \left[\bigwedge_{-\infty<i<\infty} f(x+i, y+j) - g(i,j) \right]. \tag{29}$$

For complex signals, the definitions still apply. The only differences are that now F and G are complex signals. It should be noted that non-flat structuring elements need to be used with care: one must ensure that the subtraction and the addition are within the $[0 \dots M]$ limits. These signal transformations are usually too sensitive to gain changes, because the effective structuring element shape changes with these changes. Pre-normalization may alleviate the problem, but might be difficult to perform.

7. Conclusion

In this paper, we generalized morphological operators to complex signals. We did so by first exploring the specificities of these signals, leading us to set criteria that make the operators usable in practice. Although there are already multivariate morphological operators, these are mostly applied to color images. Our context is different and this is the first time morphology has been specifically generalized to complex signals.

We also developed a new complementation procedure. This procedure is appropriate for the order relationship we developed. Our approach of the problem also enabled the re-use of all the work already done on gray-tone images. For instance, because we now have an order relationship and a complementation procedure, the theorems in morphological filtering are valid on complex signals.

The order relationship is also usable to generate rank filters, such as the median filter. However, this is beyond the scope of this paper.

Acknowledgements

This work was supported by Defense R&D Canada.

References

[1] J. Astola, P. Haavisto, Y. Neuvo, Vector median filters, Proc. IEEE 78 (4) (April 1990) 678–689.

[2] V. Barnett, The ordering of multivariate data, J. Roy. Statist. Soc. A 139 (pt 3) (April 1976) 318–355.

[3] J. Chanussot, P. Lambert, Total ordering based on space filling curves for multivalued morphology, in: H. Heijmans, J. Roerdink (Eds.), Mathematical Morphology and its Applications to Image and Signal Processing, Kluwer Academic Press, Amsterdam, 1998, pp. 51–58.

[4] M.L. Comer, E.J. Delp, An empirical study of morphological operators in color image enhancement, in: SPIE (Ed.), Proceedings SPIE, Image Processing Algorithms and Techniques III, Vol. 1657, San Jose, CA 1992, pp. 314–325.

[5] M.L. Comer, E.J. Delp, Morphological operations for color image processing, J. Electron. Imaging 8 (3) (July 1999) 279–289.

[6] H. Heijmans, Morphological image operators, Advances in Electronics and Electron Physics, Academic Press, Boston, 1994.

[7] J.-F. Rivest, J. Serra, P. Soille, Dimensionality and image analysis, J. Visual Comm. Image Representation 3 (2) (June 1992) pp. 137–146.

[8] J. Serra, Image Analysis and Mathematical Morphology, Academic Press, London, 1982.

[9] H. Talbot, C. Evans, R. Jones, Complete ordering and multivariate mathematical morphology, in: H. Heijmans, J. Roerdink (Eds.), Mathematical Morphology and its Applications to Image and Signal Processing, Kluwer Academic Press, Amsterdam, 1998, pp. 27–34.

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