85559

DEPARTMENT OF NATIONAL DEFENCE

CANADA

OPERATIONAL RESEARCH DIVISION

DIRECTORATE OF AIR OPERATIONAL RESEARCH

STAFF NOTE 79/3

HIT PROBABILITIES OF A SALVO OF WEAPONS AGAINST A CONFIGURATION OF TARGETS

By

D.W. MASON

Staff Notes are written as informal records of data, analyses, tentative views, comments, methodology, or briefing material, which for one reason or another do not warrant or require formal publication. The contents are the responsibility of the author, and do not necessarily reflect the opinion of the Directorate.

OTTAWA, ONTARIO

MARCH, 1979.

ABSTRACT

A formula is derived for the expected number of hits by a salvo of weapons against a configuration of targets. The number of weapons in the salvo, the (x, y) coordinates of each target and the vulnerable area of the target in the x-y plane are assumed given. The weapon impact uncertainties include a salvo aim error, which is assumed to obey a bivariate-normal distribution about a nominal aim point on the x-y plane, and a ballistic dispersion factor for each individual weapon which also is described by a bivariate-normal distribution about the actual aim point.

TABLE OF CONTENTS

I. INTRODUCTION AND SUMMARY

This report investigates the hit probabilities associated with the launch of a salvo of weapons against a fixed configuration of targets. The weapon/target combination that spawned this analysis was that of a salvo of air-launched rockets (e.g. CRV-7) against a collection of hardened aircraft shelters. The theory however is presented in general form and is applicable to any combination satisfying the following assumptions:

- the weapon salvo is launched sufficiently quickly that the aim point of each weapon on the "impact" plane[1] is the same. That is, if no ballistic dispersion took place then all weapons would impact at the same point.
- the actual aim point of the salvo is distributed according to a bivariate normal distribution about a nominal aim point on the impact plane.
- each weapon is ballistically dispersed in a bivariate normal fashion about the actual aim point.
- the targets have fixed coordinates on the impact plane. (Note: This need not imply the targets are not moving, as long as the time between the first and last weapon impact allows relatively little target movement.)
- the size of each target is defined via a vulnerable area in the impact plane and the measure of effectiveness used is: the expected number of targets hit with at least one weapon.

[1] The "impact" plane can be defined as the normal plane or the ground plane, or any other plane as long as the target, aim error and ballistic dispersion parameters are defined with respect to that plane.

Figure 1 depicts this set-up in the impact plane. Table I supplies a list of variable definitions for reference.

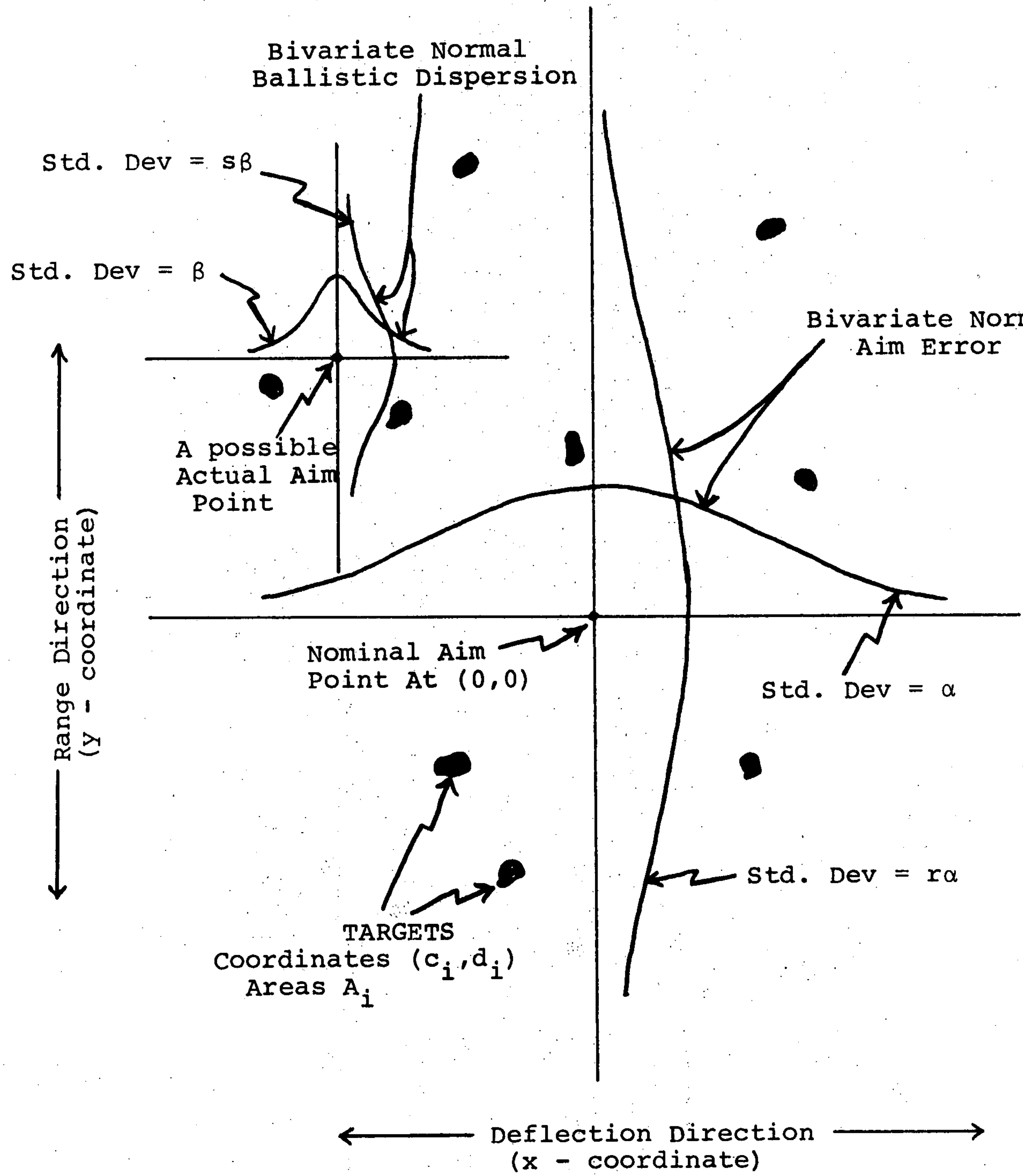
TABLE I

PARAMETER DEFINITIONS

k	-	Number of targets
n	-	Number of weapons in the salvo
A_i	-	Vulnerable area of target i
(c_i, d_i)	-	The deflection direction and range direction coordinates of target i with respect to the nominal aim point.
α	-	The standard deviation of the aim error in the deflection direction.
r	-	The scale factor such that $r\alpha$ is the standard deviation of the aim error in the range direction. (Usually $r>1$.)
β	-	The standard deviation of the ballistic dispersion in the deflection direction
s	-	The scale factor such that $s\beta$ is the standard deviation of the ballistic dispersion in the range direction. (Usually $s>1$ also.)

FIGURE 1

MODEL ASSUMPTIONS



The first result is the expression for the expected number of targets hit (as opposed to the expected number of weapons hitting some target) given that the actual salvo aim point is known:

$$E_A(\text{targets}) = \sum_{i=1}^{k} (1-(1-p_i)^n) \qquad (1)$$

where k and n are as defined in Table I and p_i is the probability that a single weapon will strike target i. The calcuation of p_i involves the area of target i, the coordinates of the target with respect to the actual aim point, and the ballistic dispersion standard deviation in both the range and deflection directions. The term in the summation (1) is precisely the probability that at least one weapon will hit the i^{th} target. Although (1) is a concise and intuitively logical expression for expected targets hit, its derivation is non-trival. The combinatorial approach of Section II generates (1) with the assistance of the Principle of Inclusion and Exclusion.

The final step is to introduce the aim error and compute the average of the expressions (1) over all possible actual aim points. The actual aim point is distributed according to a bivaraite normal distribution about a nominal aim point.

As is shown in Section III, direct integration of (1) over this range gives rise to the following expression for the overall expected number of targets hit:

$$E(\text{targets}) = \sum_{i=1}^{k} \sum_{j=1}^{n} (-1)^{j+1}(A_i/j)(A_i/2\pi s\beta^2)^{j-1}$$

$$\times\ P_N(c_i,u_j)P_N(d_i,v_j) \qquad (2)$$

where $P_N(g,h)$ is the normal density function

$$P_N(g,h) = (1/\sqrt{2\pi}h) \exp(-g^2/2h^2)$$

$$\text{and } u_j = \sqrt{\alpha^2+\beta^2/j}$$

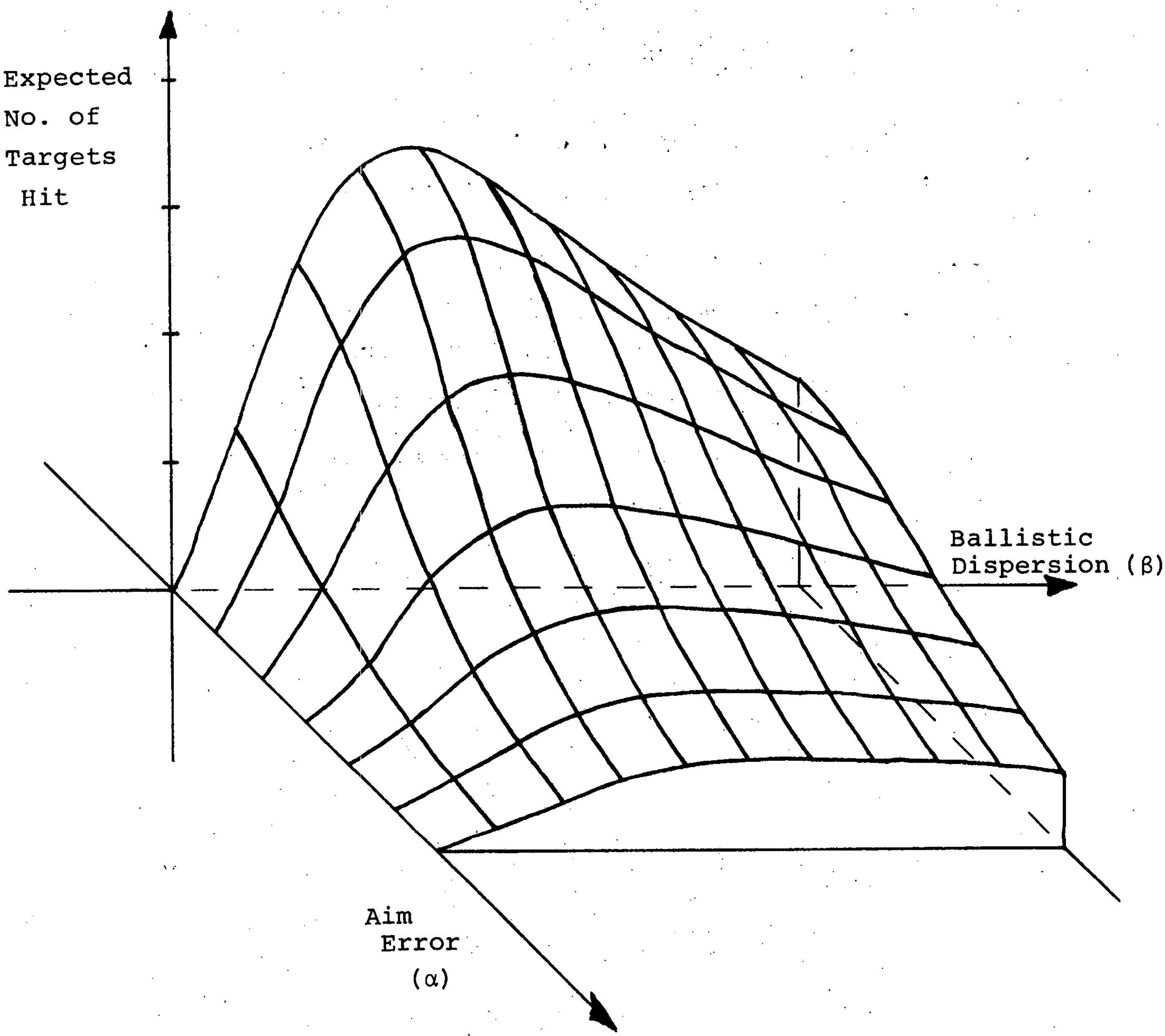
$$v_j = \sqrt{r^2\alpha^2+s^2\beta^2/j}$$

Equation (2) is programmed into a FORTRAN subroutine called AIMBAL. A listing of AIMBAL is given in Annex "D".

The dependence of Equation (2) on the values of the aim error and ballistic dispersion standard deviations is illustrated in Figure 2 for a sample case of a salvo of rockets against an actual configuration of aircraft shelters at an airfield.

FIGURE 2

EXPECTED TARGETS HIT AS A FUNCTION
OF AIM ERROR AND BALLISTIC DISPERSION
STANDARD DEVIATION PARAMETERS



II. DERIVATION OF EXPECTED TARGETS HIT EXPRESSION WITH KNOWN AIM POINT (USING THE PRINCIPLE OF INCLUSION AND EXCLUSION)

The Principle of Inclusion and Exclusion is a useful technique of Combinatorial Mathematics for enumerating objects with certain properties. (see Ref (1), Chapter 4). Application of the technique involves considering a universal set of objects, then subtracting some members having specific properties. However, some members having combinations of these properties may have been excluded more than once, so they are re-included ... etc. The following simple example illustrates the reasoning.

We wish to enumerate all ORAE scientists (a total of, say, N) which do not have brown hair or do not wear glasses. The vertically shaded region of Figure 3 represents this number. The expression

(No. of scientists without brown hair or glasses) =
N - (no.with brown hair) - (no. with glasses) (3)

has excluded some scientists twice (i.e. those with both brown hair and glasses). This number is represented by the horizontally shaded region in Figure 3. The correct expression is thus:

(no.of scientists without brown hair or glasses) =
N - (no. with brown hair) - (no. with glasses)
+ (no. with brown hair and glasses) (4)

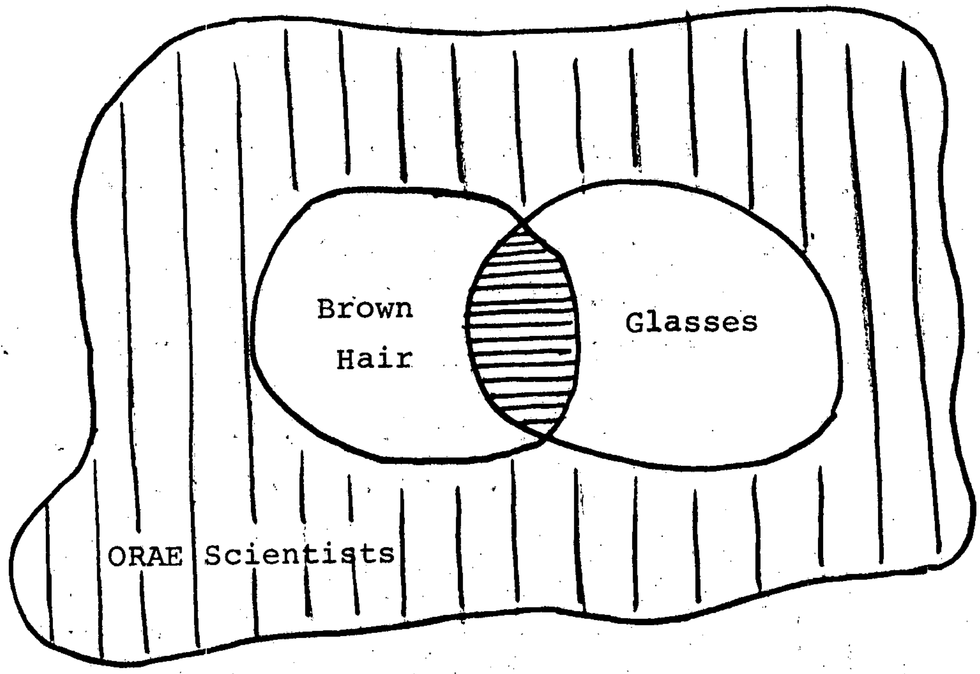


FIGURE 3

Sample Application of Principle of Inclusion and Exclusion

This same principle applies to the probabilities of selecting objects with certain property combinations (just divide the expression above by N, the number of scientists).

The Principle of Inclusion and Exclusion is stated in terms of probabilities and in terms of the problem being considered, as follows.

<u>Principle of Inclusion and Exclusion</u>

Let a_i, $i=1,\ldots,k$ represent the property that target i is hit (after n rounds), and a_i' the property that it is not hit. Consider a subset of m, $0 \leqslant m \leqslant k$, of the targets (re-indexed, if necessary, so that they are the first m in sequence). The Principle states that

$$P(a_1,a_2,\ldots,a_m,X)$$

$$= P(X) - \sum_{i \leqslant m} P(a_i',X) + \sum_{i<j \leqslant m} P(a_i',a_j',X) -$$

$$\ldots + (-1)^m P(a_1',a_2',\ldots,a_m',X) \qquad (5)$$

where X represents the property list $(a_{m+1}',\ldots,a_k')$.

Proof of this statement by induction on m is provided in Annex "A", and is parallel to that in Ref (1), p.98.

The expressions on the right hand side of (5) all can be evaluated easily. For instance

$$P(a_i',a_j',a_k') = (1-p_i-p_j-p_k)^n \qquad (6)$$

Hence in a sample case of k=5 one can write:

$$P(a_1,a_2,a_3',a_4,a_5')$$

$$=(1-p_3-p_5)^n$$

$$-(1-p_1-p_3-p_5)^n-(1-p_2-p_3-p_5)^n-(1-p_4-p_3-p_5)^n$$

$$+(1-p_1-p_2-p_3-p_5)^n+(1-p_1-p_4-p_3-p_5)^n+(1-p_2-p_4-p_3-p_5)^n$$

$$-(1-p_1-p_2-p_3-p_4-p_5)^n$$

$$=(p_0+p_1+p_2+p_4)^n$$

$$-(p_0+p_1+p_2)^n-(p_0+p_2+p_4)^n-(p_0+p_1+p_4)^n$$

$$+(p_0+p_1)^n+(p_0+p_2)^n+(p_0+p_4)^n$$

$$-(p_0)^n \qquad (7)$$

where p_0 represents $1 - \sum p_i$.

If N represents the number of targets hit (after the launching of n weapons) then

$$P(N=i) = \sum_{\substack{\text{all combinations}\\ \text{of } i \text{ targets of } k}} P(\underbrace{a_{j_1}, a_{j_2}, \ldots, a_{j_i}}_{i \text{ hit}}, \underbrace{a'_{m_1}, \ldots, a'_{m_{k-i}}}_{k-i \text{ missed}}) \quad (8)$$

Consider the example (7) above again. In this case (8) will contain $\binom{5}{3} = 10$ terms.

$$\begin{aligned} P(N=3) = {} & P(a_1,a_2,a_3,a'_4,a'_5) + P(a_1, a_2, a'_3,a_4,a'_5) \\ & + P(a_1,a_2,a'_3,a'_4,a_5) + P(a_1,a'_2,a_3,a_4,a'_5) \\ & + P(a_1, a'_2,a_3,a'_4,a_5) + P(a_1,a'_2,a'_3,a_4,a_5) \\ & + P(a'_1,a_2,a_3,a_4,a'_5) + P(a'_1,a_2,a_3,a'_4,a_5) \\ & + P(a'_1,a_2,a'_3,a_4,a_5) + P(a'_1,a'_2,a_3,a_4,a_5) \end{aligned} \quad (9)$$

Each of the ten terms in (9) are an expression in the form of (7) (which itself has 8 terms). However (7) reduces in size considerable when like terms are collected.

For example:

$(p_0+P_1+p_2)^n$ is included in $\binom{k-2}{1} = 3$ of 10 cases (k=5),

$(p_0+p_4)^n$ is included in $\binom{k-1}{1} = 6$ of 10 cases, and

$(p_o)^n$ is included in all $\binom{k}{3} = 10$ cases.

This is given in generalized forms as follows. Let

...../11

$$T_o = (p_0)^n$$

$$T_1 = \sum_i (p_0+p_i)^n$$

$$T_2 = \sum_{i<j} (p_o+p_i+p_j)^n \tag{10}$$

$$\cdots$$

$$T_k = (p_0+p_1+\dots+p_k)^n=1$$

Then

$$P(N = m) = T_m-\binom{k-m+1}{1} +_{m-1} + \binom{k-m+2}{2} T_{m-2} - \cdots$$

$$+(-1)^m\binom{k-m+m}{m}T_0 \tag{11}$$

The expected number of hits is given by

$$E = \sum_{i=1}^{k} (i)P(N=i)$$

$$=(1)\ \left(\binom{k-1}{0}T_1-\binom{k}{1}T_0\right)$$

$$+(2)\ \left(\binom{k-2}{0}T_2-\binom{k-1}{1}T_1+\binom{k}{2}T_0\right)$$

$$+(3)\ \left(\binom{k-3}{0}T_3-\binom{k-2}{1}T_2+\binom{k-1}{2}T_1-\binom{k}{3}T_0\right)$$

$$+\ \cdots$$

$$+(k)\ \left(\binom{k-k}{0}T_k-\binom{1}{1}T_{k-1}+\dots(-1)^k\binom{k}{k}T_0\right) \tag{12}$$

Grouping according to the T_i gives

..../12

$$E = T_0((0)\binom{k}{0}-(1)\binom{k}{1}+(2)\binom{k}{2}-(3)\binom{k}{3}+\ldots+(-1)^k(k)\binom{k}{k})$$

$$+ T_1((1)\binom{k-1}{0}-(2)\binom{k-1}{1}+(3)\binom{k-1}{2}-\ldots+(-1)^{k-1}(k)\binom{k-1}{k-1})$$

$$+T_2((2)\binom{k-2}{0}-(3)\binom{k-2}{1}+(4)\binom{k-2}{2}-\ldots+(-1)^{k-2}(k)\binom{k-2}{k-2})$$

$$+ \ldots$$

$$+T_{k-2}((k-2)\binom{2}{0}-(k-1)\binom{2}{1}+(k)\binom{2}{2})$$

$$+T_{k-1}((k-1)\binom{1}{0}-(k)\binom{1}{1})$$

$$+T_k((k)\binom{0}{0}) \qquad (13)$$

Note that the factor of each T_i (call it C_i) contains each of the $\binom{k-i}{m}$ for m=0,1,...,k-i in an alternating sequence. The enumerating function for the binomial coefficients is (see Ref (1) p. 26).

$$(1 + x)^n = \binom{n}{0}+\binom{n}{1}x +\binom{n}{2}x^2+\ldots+\binom{n}{n}x^n \qquad (14)$$

If we differentiate (14) once with k=n and set x=-1, the RHS of (14) can be recognized as C_0.

$$C_0 = 0\binom{k}{1}+2\binom{k}{2}-3\binom{k}{3}+\ldots+(-1)^k(k)\binom{k}{k}$$

$$= n(1 + x)^{n-1} \text{ evaluated at } -1$$

$$= n(0)^{n-1}$$

$$= 0 \qquad (15)$$

In general C_i is obtained by multiplying (14) by $-x^i$, setting n to k-i, taking the first derivative and evaluating at x=-1.

..../13

$$C_i = \frac{d}{dx}(-x^i)(1+x)^{k-i}\Big|_{x=-1}$$

$$= (-1)(x^{i-1})(1+x)^{k-i-1}[i(1+x)+(k-i)(x)]\Big|_{x=-1}$$

$$= (-1)^{i-1}(0)^{k-i-1}[k-i]$$

$$=0 \qquad \text{if } k-i-1 \geq 1 \quad \text{or} \quad i \leq k-2 \tag{16}$$

C_{k-1} and C_k can be evaluated directly from (13)

$$\left.\begin{aligned} C_{k-1} &= -1 \\ C_k &= k \end{aligned}\right\} \tag{17}$$

Hence the expected number of shelters hit reduces considerably to

$$E = kT_k - T_{k-1}$$

$$\left.\begin{aligned} \text{where} \quad T_k &= (p_0+p_1+\ldots+p_k)^n = 1 \\ \text{and} \quad T_{k-1} &= \sum_{i=1}^{k}(p_0+p_1+\ldots+p_{i-1}+p_{i+1}+\ldots+p_k)^n \\ &= \sum_{i=1}^{k}(1-p_i)^n \end{aligned}\right\} \tag{18}$$

$$\therefore\ E = k - \sum_{i=1}^{k}(1-p_i)^n$$

$$= \sum_{i=1}^{k}(1-(1-p_i)^n) \tag{19}$$

which is the sum of the probabilities of hitting each target with at least one weapon.

..../14

This derivation of (19) has required a sizeable quantity of algebraeic manipulation in relation to the complexity (or lack thereof) of the final answer, which suggest a simple, less "brute-force-like" technique should be able to generate (19). This in fact is true. Given that we now know the form of the answer, through application of this brute-force method, in retrospect we see the suitability of trying an inductive proof on the number of targets. Such a proof is provided in 3 pages (instead of 7) in Annex "B". I thank Mr. E.J. Emond for his contributions on this inductive method.

III. DERIVATION OF EXPECTED TARGETS HIT WITH AIM POINT UNCERTAINTY

Equation (19) of Section II is the expression for expected targets hit given the actual salvo aim point. In this section an uncertainty about the precise location of this point is introduced. The actual aim point on the impact plane, denoted (x, y), is assumed to obey a bivariate normal distribution about the nominal salvo aim point at the origin. The standard deviation in the deflection direction is denoted α, and in the range direction is $r\alpha$. With (c_i, d_i) representing the coordinates of the i^{th} target, let

$$P(c_i|x) = (1/\sqrt{2\pi}\beta) \exp\{-(c_i-x)^2/2\beta^2\} \quad (20)$$

$$P(d_i|y) = (1/\sqrt{2\pi}s\beta) \exp\{-(d_i-y)^2/2s^2\beta^2\} \quad (21)$$

Given the actual aim point (x, y) then, the probability of hitting target i is approximately

$$p_i = (A_i)P(c_i|x)P(d_i|y) \quad (22)$$

This approximation is excellent as long as A_i is only a fraction of $s\beta^2$ - the product of the two ballistic dispersion standard deviations. For rockets against aircraft shelters or smaller targets, for example, this approximation is quite good at typical launch ranges.

As was previously shown the expected number of targets hit (by at least one of n weapons) is (as E is a function of (x,y)

$$E = E(x,y) = \sum_{i=1}^{k} (1-(1-p_i)^n) \quad (23)$$

The total expected number of hits is

$$T = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(x,y)\ p(x)\ p(y)\ dx\ dy \tag{24}$$

where $p(x) = (1/\sqrt{2\pi}\alpha)\ \exp\ \{-x^2/2\alpha^2\}$ (25)

and $p(y) = (1/\sqrt{2\pi}r\alpha)\ \exp\ \{-y^2/2r^2\alpha^2\}$ (26)

Note that

$$1-(1-p)^n$$
$$= 1-1+\binom{n}{1}p-\binom{n}{2}p^2+\ldots+(-1)^{n-1}\binom{n}{n}p^n$$

Hence

$$T = \sum_{i=1}^{k} \sum_{j=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-1)^{j+1}\binom{n}{j}(A_i/2\pi s\beta^2)^j$$

$$\times \exp\ \{-j(c_i-x)^2/2\beta^2-j(d_i-y)^2/2s^2\beta^2\}$$

$$\times(1/2\pi r\alpha^2)\ \exp\ \{-x^2/2\alpha^2-y^2/2r^2\alpha^2\}\ dx\ dy \tag{27}$$

$$= \sum_{i=1}^{k} \sum_{j=1}^{n} (-1)^{j+1}\binom{n}{j}(A_i/2\pi s\beta^2)^j(1/2\pi r\alpha^2)$$

$$\times \int_{-\infty}^{\infty} \exp\ \{-j(c_i-x)^2/2\beta^2-x^2/2\alpha^2\}\ dx$$

$$\times \int_{-\infty}^{\infty} \exp\ \{-j(d_i-y)^2/2s^2\beta^2-y^2/2r^2\alpha^2\}\ dy \tag{28}$$

In Annex "C" it is shown that

$$\int_{-\infty}^{\infty} \exp\{-(a^2y^2+by+c)\}dy$$

$$= (\sqrt{\pi}/a)\exp\{-c+b^2/4a^2\} \tag{29}$$

For the case of the integral over x in (28), call it T_x, we have

$$-a^2 = -j/2\beta^2-1/2\alpha^2, \quad a = \sqrt{(j\alpha^2+\beta^2)/2\alpha^2\beta^2}$$

$$b = -jc_i/\beta^2$$

$$c = jc_i^{\,2}/2\beta^2$$

Hence

$$T_x = \sqrt{2\pi\alpha^2\beta^2/(j\alpha^2+\beta^2)}\ \exp\{-c_i^{\,2}/2(\alpha^2+\beta^2/j)\} \tag{30}$$

In the evaluation of T_y (the integral over y in (28))

$$a = \sqrt{(jr^2\alpha^2+s^2\beta^2)/2(r\alpha s\beta)^2}$$

$$b = -jd_i/s^2\beta^2$$

$$c = jd_i^{\,2}/2s^2\beta^2$$

and

$$T_y = \sqrt{2\pi(r\alpha s\beta)^2/(jr^2\alpha^2+s^2\beta^2)}\ \exp\{-d_i^{\,2}/2(r^2\alpha^2+s^2\beta^2/j)\} \tag{31}$$

Re-writing (28) then gives:

$$T = \sum_{i=1}^{k}\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(A_i/j)(A_i/2\pi s\beta^2)^{j-1}P_N(c_i|u_j)P_N(d_i|v_j) \tag{32}$$

..../18

where $P_N(g,h)$ is the Normal density function.

$$P_N(g,h) = (1/\sqrt{2\pi}h) \exp\{-g^2/2h^2\}$$

and $$u_j = \sqrt{\alpha^2+\beta^2/j}$$

$$v_j = \sqrt{r^2\alpha^2+s^2\beta^2/j}$$

The expression (32) is mechanized in a FORTRAN subroutine called AIMBAL, a listing of which appears as Annex "D".

REFERENCES

1. Introduction to Combinatorial Mathematics, by C.L. Liu, McGraw-Hill, 1968.

PROOF BY INDUCTION OF PRINCIPLE OF INCLUSION AND EXCLUSION

Case m=1: Since a_i either holds or it does not,

$$P(X) = P(a_i, X) + P(a_i', X)$$

Hence $$P(a_i, X) = P(X) - P(a_i', X) \qquad (A1)$$

and this is the m=1 case.

General Case: Assume true for m-1. Therefore

$$P(a_1, a_2, \ldots, a_{m-1}, X) = P(X) - \sum_{i \leq m-1} P(a_i', X)$$
$$+ \sum_{i<j \leq m-1} P(a_i', a_j', X) - \ldots$$
$$+ (-1)^{m-1} P(a_1', a_2', \ldots, a_{m-1}', X) \qquad (A2)$$

Similarly we can state

$$P(a_1, a_2, \ldots, a_{m-1}, a_m', X) = P(a_m', X) - \sum_{i \leq m-1} P(a_i', a_m', X)$$
$$+ \sum_{i<j \leq m-1} P(a_i', a_j', a_m', X) - \ldots$$
$$+ (-1)^{m-1} P(a_1', a_2', \ldots, a_{m-1}', a_m', X) \qquad (A3)$$

Subtracting (A2) - (A3) gives

$$P(a_1, a_2, \ldots, a_{m-1}, X) - P(a_1, a_2, \ldots, a_{m-1}, a_m', X)$$
$$= P(X) - \sum_{i \leq m} P(a_i', X) + \sum_{i<j \leq m} P(a_i', a_j', X) - \ldots$$
$$+ (-1)(-1)^{m-1} P(a_1', a_2', \ldots, a_m', X) \qquad (A4)$$

..../A2

which is precisely the required result if one recognizes that

$$p(a_1, a_2, \ldots, a_{m-1}, a_m, X)$$

$$= P(a_1, a_2, \ldots, a_{m-1}, X) - P(a_1, a_2, \ldots, a_{m-1}, a_m', X)$$

as in (A1).

PROOF OF EXPECTED TARGETS HIT EXPRESSION WITH A KNOWN AIM POINT - BY INDUCTION

Statement: If p_i is the probability of hitting target i (i=1,...,k targets) with a single weapon and n is the number of weapons fired in the salvo, then the expected number of targets hit by at least one weapon is

$$E_A(k,n,\{p_i\}) = \sum_{i=1}^{k} (1-(1-p_i)^n) \qquad (B1)$$

Proof by Induction: Proof is by induction on k - the number of targets. For k=1 the expected number of hits is simply the probability of hitting the target with at least one weapon

$$E_A(1,n,\{p_i\}) = 1-(1-p_1)^n \qquad (B2)$$

This takes care of the initial case.

Now let us assume (B1) is true for k targets and see if this implies its truth for k+1 targets. The n weapons either all miss the $k+1^{st}$ target, or at least one hits it. The expression for expected hits is then

$$
\begin{aligned}
E_A(k+1,n,\{p_i\}) = &\ (\text{Probability that no weapons hit } k+1^{st} \text{ target}) \\
&\times (\text{Expected number of the first } k \text{ targets hit } \underline{\text{given}}\ k+1^{st} \text{ is missed}) \\
&+ (\text{Probability that 1 weapon hits } k+1^{st} \text{ target}) \\
&\times (1 + \text{Expected number of the first } k \text{ targets hit with } n-1 \text{ weapons } \underline{\text{given}} \text{ only 1 hit target } k+1) \\
&+ \ldots \\
&+ (\text{Probability that } n \text{ weapons hit } k+1^{st} \text{ target}) \\
&\times (1+0) \qquad \text{(B3)}
\end{aligned}
$$

This expression becomes

$$
\begin{aligned}
E_A&(k+1,n,\{p_i\}) \\
&= (1-p_{k+1})^n \Big(\sum_{i=1}^{k} (1-(1-p_i/(1-p_{k+1}))^n)\Big) \\
&+ \sum_{j=1}^{n} \Big[1+\sum_{i=1}^{k} (1-(1-p_i/(1-p_{k+1}))^{n-j})\Big] \\
&\times \Big[\binom{n}{j} (p_{k+1})^j (1-p_{k+1})^{n-j}\Big] \qquad \text{(B4)}
\end{aligned}
$$

Note that the p_i values for $i=1,\ldots,k$ are now conditional on target $k+1$ <u>not</u> being hit by the number of weapons $j=0,1,2,\ldots,n$ being considered each time. These values have been replaced by the p_i^* where

$$p_i^* = p_i/(1-p_{k+1}) \qquad \text{(B5)}$$

Equation B4 can be written as

$$E_A(k+1,n,\{p_i\}) = k(1-p_{k+1})^n + (k+1)(1-(1-p_{k+1})^n)$$

$$- \sum_{i=1}^{k} (1-p_{k+1}-p_i)^n$$

$$- \sum_{j=1}^{n} (1-p_{k+1}-p_i)^{n-j} \binom{n}{j} (p_{k+1})^j$$

$$= (k-1) - (1-p_{k+1})^n$$

$$- \sum_{i=1}^{k} \left[(1-p_{k+1}-p_i)^n + \sum_{j=1}^{n} (1-p_{k+1}-p_i)^{n-j} \binom{n}{j} (p_{k+1})^j \right] \tag{B6}$$

The term in the summation above becomes

$$(1-p_{k+1}-p_i)^n \left[1 + \sum_{j=1}^{n} \binom{n}{j} (p_{k+1}/(1-p_{k+1}-p_i))^{\,j} \right]$$

$$= (1-p_{k+1}-p_i)^n \left[1 + (p_{k+1}/(1-p_{k+1}-p_i)) \right]^n$$

$$= (1-p_{k+1}-p_i)^n (1-p_i)^n / (1-p_{k+1}-p_i)^n$$

$$= (1-p_i)^n \tag{B7}$$

Substitution of (B7) into (B6) gives

$$E_A(k+1,n,\{p_i\}) = (k+1) - \sum_{i=1}^{k+1} (1-p_i)^n \tag{B8}$$

completing the proof.

INTEGRAL EVALUATION

$$\int_{-\infty}^{\infty} \exp\{-a^2x^2-bx-c\}\,dx$$

$$= \exp\{-c\} \int_{-\infty}^{\infty} \exp\{-a^2x^2-bx-b^2/4a^2+b^2/4a^2\}dx$$

$$= \exp\{-c+b^2/4a^2\} \int_{-\infty}^{\infty} \exp\{-(ax+b/2a)^2\}dx \qquad \text{(C1)}$$

Substitute $y = ax+b/2a$

Therefore $dy = a\,dx$ and

$$\int_{-\infty}^{\infty} \exp\{-a^2x^2-bx-c\}\,dx$$

$$=(\exp\{-c+b^2/4a^2\}/a) \int_{-\infty}^{\infty} \exp\{-y^2\}\,dy$$

$$= (\exp\{-c+b^2/4a^2\}/a)(\sqrt{2\pi})(1/\sqrt{2})$$

$$\times \int_{-\infty}^{\infty} (\sqrt{2\pi}(1/\sqrt{2}))^{-1} \exp\{-y^2/2(1/\sqrt{2})^2\}\,dy$$

$$= (\sqrt{\pi}/a)\exp\{-c+b^2/4a^2\}. \qquad \text{(C2)}$$

as the integral represents the area under the normal curve.

SUBROUTINE AIMBAL LISTING

```
      SUBROUTINE AIMBAL(ALPHA,R,BETA,S,NWPN,NTGT,X,Y,VULAR,IND,EXHITS)
C
C  THIS ROUTINE CALCULATES EXPECTED NUMBER OF TARGETS HIT BY AT LEAST
C  ONE WEAPON FOR A GIVEN CONFIGURATION OF TARGETS, SALVO SIZE, AIM ERROR
C  AND BALLISTIC DISPERSION.
C
      DIMENSION X(40),Y(40),VULAR(40)
      DOUBLE PRECISION EH,A2,R2,B2,S2,FACTOR,TERM,VARX,VARY,COEF,
     *     PI,PRBGAU,DELEH
      DATA PI/3.14159265359/
C
C  INPUT PARAMETERS
C  ----------------
C  ALPHA  -  THE STD DEV OF THE AIM ERROR IN THE DEFLECTION DIRECTION
C            AT THE TARGET (FEET).
C  R  -  R * ALPHA  IS THE STD DEV (IN FT AT TGT) OF THE AIM ERROR IN
C        THE RANGE DIRECTION.
C  BETA  -  SIMILAR TO ALPHA, BUT REPRESENTS THE BALLISTIC DISPERSION.
C  S  -  SIMILAR TO R, BUT ALSO REPRESENTS BALL. DISP.
C  NWPN  -  NUMBER OF WEAPONS FIRED IN THE SALVO.
C  NTGT  -  NUMBER OF TARGETS.
C  X(I),I=1,NTGT  -  X-COORDINATE OF TARGET  I  (IN FEET) WRT THE NOMINAL AIM
C                    POINT AT (0,0).  THE X-AXIS REPRESENTS THE DEFLECTION
C                    DIRECTION.
C  Y(I),I=1,NTGT  -  THE CORRESPONDING Y-COORDINATE.  THE Y-AXIS REPRESENTS
C                    THE RANGE DIRECTION.
C  VULAR(I),I=1,NTGT  -  THE VULNERABLE AREA OF TARGET  I  (IN SQ FT)
C                        IN THE HORIZONTAL PLANE.
C  IND  -  PRINT INDICATOR.  A VALUE OF ZERO GENERATES NO PRINT-OUT.
C          A VALUE OF 1 DISPLAYS ALL ARGUMENTS BUT THE ARRAYS (X,Y,VULAR).
```

ANNEX "D"

```
C          A VALUE OF 2 DISPLAYS ALL SUBROUTINE ARGUMENT VALUES.
C
C   OUTPUT PARAMETER
C   ----------------
C   EXHITS  -  EXPECTED NUMBER OF THE NTGT TARGETS THAT WOULD BE
C              HIT BY AT LEAST 1 WEAPON.
C
      A2=ALPHA*ALPHA
      R2=R*R
      B2=BETA*BETA
      S2=S*S
      EH=0.0
C
      DO 100   I=1,NTGT
      FACTOR=VULAR(I)/(2.0*PI*S*B2)
      TERM=-1.0/FACTOR
C
      DO 50   J=1,NWPN
      DFJ=DFLOAT(J)
      TERM=TERM*FACTOR*(-1.0)*DFLOAT(NWPN-J+1)/DFJ
      VARX=A2+B2/DFJ
      VARY=R2*A2+S2*B2/DFJ
      COEF=TERM*VULAR(I)/DFJ
      DELEH=COEF*PRBGAU(X(I),VARX)*PRBGAU(Y(I),VARY)
      IF(DABS(DELEH).LT.1.D-20)GO TO 100
C                       (TO AVOID WASTING COMPUTATION TIME AND TO
C                          AVOID POTENTIAL UNDERFLOWS)
      EH=EH+DELEH
   50 CONTINUE
C
```

```
  100 CONTINUE
      EXHITS=EH
      IF(IND.EQ.0)RETURN
C
      WRITE(6,150)ALPHA,R,BETA,S,NWPN,NTGT,EXHITS
  150 FORMAT(/' AIMBAL',9X,'ALPHA',9X,'R',6X,'BETA',9X,'S',6X,'NWPN',
     *     6X,'NTGT',4X,'EXHITS'/' ',6('-'),4X,4F10.3,2I10,F10.4)
      IF(IND.EQ.1)RETURN
      WRITE(6,200)(X(I),I=1,NTGT)
  200 FORMAT(/' ',9X,'X',10F10.2,3(/' ',10X,10F10.2))
      WRITE(6,250)(Y(I),I=1,NTGT)
  250 FORMAT(' ',9X,'Y',10F10.2,3(/' ',10X,10F10.2))
      WRITE(6,300)(VULAR(I),I=1,NTGT)
  300 FORMAT(' ',5X,'VULAR',10F10.1,3(' ',10X,10F10.1))
      RETURN
      END

      DOUBLE PRECISION FUNCTION PRBGAU(X,V)
C
C  EVALUATES THE NORMAL DENSITY FUNCTION AT X WITH MEAN ZERO, VARIANCE V.
C
      DOUBLE PRECISION V,XPNT
      XPNT=(X**2)/(2.0*V)
      PRBGAU=0.0
      IF(XPNT.GT.99.)RETURN
C                 (TO AVOID UNDERFLOW PROBLEMS)
      PRBGAU=DEXP(-XPNT)*.39894228/DSQRT(V)
      RETURN
      END
```