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DETERMINATION OF THE EARTH'S GRAVITY POTENTIAL ON THE GEOID

by

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TRANSLATOR'S NOTE

The study of the Earth's figure (the geoid) on the basis of the Stokes theorem has been beset with difficulties due to the fact that to use this theorem it is necessary to postulate some hypothesis on the structure (density stratification) of the Earth's interior. In 1945 M.S. Molodenski (ref. 1 of the present paper) put forward a new principle for study of the geoid, requiring no hypothesis on the Earth's internal structure. The "geodetic gravimetry" approach originated in the research of N.K. Migal (ref. 2), and in particular of I.A. Kazanski (1932), which in turn was based on the work of Vening Meinesz. The present paper is a continuation of previous work by Monin (ref. 5) which is available in English translation (as is Molodenski's fundamental paper).

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DETERMINATION OF THE EARTH'S GRAVITY POTENTIAL ON THE GEOID

I.F. Monin

(Presented by S.I. Subbotin, Member of the Academy of Sciences of the Ukr. SSR)

1. In the derivation of the exterior gravitational field and the figure of the earth, the quantity W_0, the earth's gravity potential on the geoid, appears as a parameter in almost all the formulae characterizing field elements. It is determined by means of geodetic, astronomical and gravimetric observations, from the so-called degree measurements [1-3]. It has been considered impossible to derive the potential W_0 from gravity measurements alone. This question has been most fully studied in ref. [1].

We shall show that the terrestrial gravity field on the geoid can, in fact, be determined by means of gravity measurements alone. We shall first consider the spherical problem. It is well known [4] that the relationship between the spherical functions of zero order in the analysis of gravity anomalies Δg and their vertical gradients $d\Delta g/dz$ may be written thus:

$$a^2\left(\frac{\partial \Delta g}{\partial z}\right)_0 = -2a\Delta g_0 + 2(W_0 - U_0), \quad (1)$$

where a = radius of the terrestrial globe, U_0 = normal values of the gravity potential on the terrestrial ellipsoid. Applying to relationship (1) the well-known sequential theorem of spherical functions

$$R_n = \frac{2n+1}{4\pi}\int RP_n(\cos\psi)\, d\sigma; \qquad R = \sum_0^{\infty} R_n,$$

where R = any spherical coordinate function, $P_n(\cos\psi)$ = Legendre polynomial, $d\sigma$ = surface element of sphere, we get

$$W_0 - U_0 = \frac{a}{8\pi}\int\left(2\Delta g + a\frac{\partial \Delta g}{\partial z}\right) d\sigma. \quad (2)$$

Formula (2) enables us to compute the gravity potential W_0 from gravity anomalies and their vertical gradients, when given on the geoid.

Now let us look at the more general problem. For the anomalic gravity potential T of the Earth we have, in the limit,

$$2T + \varrho\frac{\partial T}{\partial \varrho} = -\varrho\Delta g + 2(W_0 - U_0),$$

where ρ is the radius vector to any point of the topographic surface of the Earth (coordinate origin taken at the centre of the general terrestrial ellipsoid); functions T and $\rho\,\partial T/\partial\rho$ are concordant and regular to infinity. So too, consequently, is the function $\rho\Delta g + 2(W_0 - U_0)$. To it, and to the Earth's surface S, let us apply Green's formula

$$\Phi = \frac{1}{2\pi}\int\left(\Phi_S\frac{\partial\frac{1}{r}}{\partial n} - \frac{1}{r}\cdot\frac{\partial\Phi_S}{\partial n}\right) dS, \quad (3)$$

where r = distance of any given point from a fixed point of the surface S and

$$r^2 = \varrho^2 + \varrho_0^2 - 2\varrho\varrho_0 \cos\psi;$$

ρ_0 = radius vector to the fixed point of surface S and ψ = angle between these points. Since

$$\Phi = -\varrho_0 \Delta g + 2(W_0 - U_0),\ \Phi_S = -\varrho\Delta g + 2(W_0 - U_0),\ \frac{\partial \Phi_S}{\partial n} = -\varrho \frac{\partial \Delta g}{\partial n} - \Delta g \cos\alpha,$$

where n = exterior normal to surface S, α = angle between ρ and n, we obtain from formula (3) the result

$$W_0 - U_0 = \frac{1}{4}\varrho_0 \Delta g + \frac{1}{8\pi}\int \frac{\varrho}{r}\frac{\partial \Delta g}{\partial n} dS + \frac{1}{8\pi}\int \Delta g \left\{\left(\frac{3}{2r} + \frac{\varrho^2 - \varrho_0^2}{2r^3}\right)\cos\alpha + \frac{\varrho\varrho_0}{r^3}\sin\psi \sin\alpha \cos\theta\right\} dS. \tag{4}$$

Here we have taken into account the formula adduced in ref. [5]:

$$-\frac{\partial \frac{1}{r}}{\partial n} = \frac{\varrho - \varrho_0 \cos\psi}{r^3}\cos\alpha + \frac{\varrho_0 \sin\psi}{r^3}\sin\alpha\cos\theta,$$

where θ = angle between planes ρr and ρn.

The quantity $W_0 - U_0$ in formula (4) is determined for a relative flattening of approximately 1/300. Consequently, without transgressing the accepted limits of precision, we may in formula (4) put

$$\varrho \approx a + H,\ \varrho_0 \approx a + H_0,$$
$$r^2 \approx (H - H_0)^2 + r_0^2,$$
$$\cos\alpha dS \approx d\sum = a^2 d\sigma,$$
$$r_0 = 2a\sin\frac{\psi}{2}.$$

Then from (4) we obtain a more convenient formula for computation:

$$W_0 - U_0 = \frac{1}{4}a\Delta g + \frac{1}{16\pi}\int \frac{\partial \Delta g}{\partial n}\cdot\frac{\sec\alpha d\Sigma}{\sin\frac{\psi}{2}\cdot\sqrt{1 + \nu_1^2}} + \frac{1}{32\pi a}\int \Delta g\left\{3 + \frac{\nu_1 + \cos\frac{\psi}{2}\operatorname{tg}\alpha\cos\theta}{(1 + \nu_1^2)\sin\frac{\psi}{2}}\right\}\frac{d\Sigma}{\sqrt{1 + \nu_1^2}\sin\frac{\psi}{2}}, \tag{5}$$

where H, H_0 = normal heights of the variable point and fixed point on surface S; $\nu_1 = (H - H_0)/r_0$. Formula (5) is quite general: it describes the gravity potential W_0 of the real Earth. If we put $H = H_0 = O$, $\alpha = O$, then after some simple mathematical conversions, we obtain formula (2). We have, then,

$$W_0 - U_0 = \frac{1}{4} a\Delta g + \frac{3}{16\pi} \int \Delta g \frac{d\Sigma}{r_0} + \frac{1}{8\pi} \int a \frac{\partial \Delta g}{\partial z} \frac{d\Sigma}{r_0} =$$

$$= \frac{a}{2} \sum_0^\infty \frac{n+2}{2n+1} \Delta g_n + \frac{1}{8\pi} \int a \frac{\partial \Delta g}{\partial z} \cdot \frac{d\Sigma}{r_0} =$$

$$= \frac{1}{8\pi a} \int \Delta g \sum_0^\infty (n+2) P_n (\cos \psi)\, d\Sigma + \frac{1}{8\pi a} \int a \frac{\partial \Delta g}{\partial z} \sum_0^\infty P_n (\cos \psi)\, d\Sigma =$$

$$= \frac{1}{8\pi a} \int \left(2\Delta g + a \frac{\partial \Delta g}{\partial z} \right) d\Sigma + \frac{a}{2} \sum_1^\infty \frac{1}{2n+1} \left\{ (n+2)\, \Delta g_n + \right.$$

$$\left. + a \left(\frac{\partial \Delta g}{\partial z} \right)_n \right\} = \frac{1}{8\pi a} \int \left(2\Delta g + a \frac{\partial \Delta g}{\partial z} \right) d\Sigma.$$

Accordingly, gravity anomalies Δg and their derivatives $\partial \Delta g/\partial n$ enable us to compute the potential W_0; that is, to determine by gravimetry the figure of the Earth.

2. As is shown in ref. [5], the anomalic gravity potential T of the Earth for given Δg and $\partial \Delta g/\partial n$ is defined as:

$$T = \frac{f\Delta M}{z} + \frac{1}{4\pi} \int \Delta g \left\{ \left(-\frac{1}{z} + F_1 \right) \cos \alpha + F_3 \sin \alpha \cos \theta \right\} dS +$$

$$+ \frac{1}{4\pi} \int \frac{\partial \Delta g}{\partial n} \varrho \left(-\frac{1}{z} + F_2 \right) dS, \tag{6}$$

where

$$F_1 = \frac{1}{r_1} - \frac{\varrho}{z^2} \cos \psi,$$

$$F_2 = \frac{1}{z^2} \left(\varrho \cos \psi + r_1 + \varrho \cos \psi \ln \frac{z + r_1 - \varrho \cos \psi}{2z} \right),$$

$$F_3 = \frac{\varrho}{z^2} \left(-\frac{\cos 2\psi}{\sin \psi} - \frac{\varrho \cos \psi - z \cos 2\psi}{r_1 \sin \psi} + \sin \psi \ln \frac{z + r_1 - \varrho \cos \psi}{2z} \right),$$

$$r_1^2 = z^2 + \varrho^2 - 2z\varrho \cos \psi.$$

$$f\Delta M = \frac{1}{4\pi} \int \left(\Delta g \cos \alpha + \varrho \frac{\partial \Delta g}{\partial n} \right) dS, \quad \frac{\partial \Delta g}{\partial n} = \frac{\partial \Delta g}{\partial z} \cos \alpha + \frac{\partial \Delta g}{\partial \tau} \sin \alpha$$

This formula is obtained from Green's formula for potentials by applying thereto the function $\partial(z^2 T)/z\, \partial z$, concordant in the exterior space around the Earth. In this we used Dini's [?] method and spherical functions.

We remark that all the integrals in ref. [5] can be computed even without spherical functions.

It should be pointed out that the anomalic potential T is defined by formula (6) not only in exterior space but also at the Earth's surface. In fact all the integrals in (6) converge absolutely for $z = \rho_0$. In the case of the integrals for functions F_1 and F_2 the convergence is obvious. Let us consider the convergence of the integral of F_3. Let $\psi = \Delta$, where $\Delta \psi$ is a very small quantity, but not equal to zero. Then

$$r_1 \to z - \varrho.$$

$$F_3 \to \frac{\varrho}{z^2} \left(-\frac{1}{\Delta\psi} - \frac{\varrho - z}{z - \varrho} \cdot \frac{1}{\Delta\psi} + \Delta\psi \ln \frac{z - \varrho}{z} \right).$$

Thus when $z = \rho_0$ and $\psi = O$, we have $\rho = \rho_0$ and and $F_3 = O$. It is evident that this integral too converges absolutely.

Finally from (6) we shall obtain a generalized Stokes formula. For this, we take $\alpha = 0$, $\rho = a$. Using the results from ref. [5] we write

$$-\frac{1}{z} + F_1 = \sum_2^\infty \frac{a^n}{z^{n+1}} P_n(\cos\psi), \quad -\frac{1}{z} + F_2 = -\sum_2^\infty \frac{a^n}{(n-1)\, z^{n+1}} P_n(\cos\psi).$$

whence

$$T = \frac{f\Delta M}{z} + \frac{1}{4\pi}\int \Delta g \sum_2^\infty \frac{a^n}{z^{n+1}} P_n(\cos\psi)\, d\Sigma - \frac{1}{4\pi}\int a\frac{\partial \Delta g}{\partial z}\sum_2^\infty \times$$

$$\times \frac{a^n P_n(\cos\psi)}{(n-1)\, z^{n+1}}\, d\Sigma = \frac{f\Delta M}{z} + \sum_2^\infty \frac{a^n}{z^{n+1}}\left(\frac{1}{4\pi}\int \Delta g P_n d\Sigma - \frac{1}{n-1}\times\right.$$

$$\left.\times\frac{1}{4\pi}\int a\frac{\partial\Delta g}{\partial z}P_n d\Sigma\right) = \frac{f\Delta M}{z} + \sum_2^\infty \frac{a^n}{z^{n+1}}\left\{\frac{a^2}{2n+1}\Delta g_n - \frac{a}{n-1}\times\right.$$

$$\left.\times\frac{a^2}{2n+1}\left(\frac{\partial\Delta g}{\partial z}\right)_n\right\}.$$

When $n > 1$

$$a\left(\frac{\partial \Delta g}{\partial z}\right)_n = -(n+2)\,\Delta g_n,$$

so that

$$T = \frac{f\Delta M}{z} + \sum_2^\infty \frac{a^{n+2}}{z^{n+1}}\cdot\frac{\Delta g_n}{n-1} =$$

$$= \frac{f\Delta M}{z} + \frac{1}{4\pi}\int \Delta g \sum_2^\infty \frac{a^n}{z^{n+1}}\cdot\frac{2n+1}{n-1}P_n(\cos\psi)\, d\Sigma$$

Formula (7) is, indeed, our generalized Stokes formula.

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