Defence Research and Development Canada
Recherche et développement pour la défense Canada

DEFENCE DÉFENSE

Theory of the FFT Filter Bank-Based Majority and Median CFAR Detectors

Sichun Wang, Robert Inkol, Sreeraman Rajan
and François Patenaude

Defence R&D Canada – Ottawa
TECHNICAL REPORT
DRDC Ottawa TR 2007-088
May 2007

Canada

Theory of the FFT Filter Bank-Based Majority and Median CFAR Detectors

Sichun Wang
Defence R&D Canada – Ottawa

Robert Inkol
Defence R&D Canada – Ottawa

Sreeraman Rajan
Defence R&D Canada – Ottawa

François Patenaude
Communications Research Centre Canada

Defence R&D Canada – Ottawa

Technical Report

DRDC Ottawa TR 2007-088

May 2007

Author

Original signed by Sichun Wang

Sichun Wang

Approved by

Original signed by Bill Katsube

Bill Katsube
Head/Communications and Navigation Electronic Warfare Section

Approved for release by

Original signed by Cam Boulet

Cam Boulet
Head/Document Review Panel

Abstract

The FFT filter bank-based majority and median constant false alarm rate (CFAR) detectors are very useful alternatives to the well-known FFT filter bank-based summation CFAR detector. However, for overlapped input data blocks, the theoretical performance analysis of the majority and median CFAR detectors is considerably more difficult than that of the summation CFAR detector, the main reason being that formulas for computing the probability of false alarm for a given threshold can not be easily derived for them. This technical report presents new results on threshold computation for the FFT filter bank-based majority and median CFAR detectors for both overlapped and non-overlapped input data blocks. The most important contribution of this report is the successful computation of the joint probability density function of the output power levels of the FFT filter bank for overlapped white Gaussian noise input blocks. As a corollary, it is shown that the statistical correlation of the output power levels of the FFT filter bank can be neglected for several commonly used windows, such as the Blackman and Blackman-Harris windows. Complete mathematical derivations are provided for all the major theorems presented in the report.

Résumé

Les bancs de filtres à base de Transformée de Fourier Rapide (TFR) avec détection majoritaire et détection médiane représentent une alternative au populaire banc de filtres à base de TFR à détection par sommation dans le contexte de la détection avec un taux de fausse alarme constant. Cependant, pour le cas où les segments de données se chevauchent, le calcul du seuil de décision pour la détection majoritaire et médiane est considérablement plus complexe que pour la détection par sommation en raison de l'absence de formulation analytique. Ce rapport présente les résultats de calcul du seuil des détecteurs majoritaire et médian pour les cas des segments de données avec et sans chevauchement. La contribution principale de ce rapport est l'obtention de la fonction de densité de probabilité conjointe des puissances de sortie des bancs de filtres pour un canal additif Gaussien. Comme corollaire, il est également démontré que la corrélation des puissances de sortie des bancs de filtres peut être considérée négligeable pour plusieurs fenêtres communes comme les fenêtres de Blackman et Blackman-Harris. Les principaux théorèmes sont présentés accompagnés des preuves complètes.

This page intentionally left blank.

Executive summary

Theory of the FFT Filter Bank-Based Majority and Median CFAR Detectors

Sichun Wang, Robert Inkol, Sreeraman Rajan, François Patenaude;
DRDC Ottawa TR 2007-088; Defence R&D Canada – Ottawa; May 2007.

The FFT filter bank can be used for efficient detection of narrowband signals distributed in wideband noise. A basic form of constant false alarm rate (CFAR) detection processing can be implemented by comparing the power measured in each FFT bin to a suitable threshold T that is dependent on the noise spectral density. The performance of this detector can be improved by incorporating the following three steps: (a) increasing the data record size, (b) dividing the data record into non-overlapping blocks and (c) estimating the signal power through averaging the spectral power estimates computed from the individual windowed data blocks. The windowing process introduces processing loss, which can be mitigated through the use of overlapped data blocks, an idea originally proposed for power spectral estimation by Welch [1].

Various extensions of the basic FFT CFAR detector have been investigated in recent years [3]-[12]. Of these, two particularly important extensions, the FFT filter bank-based majority and median CFAR detectors, have been proposed to provide robust performance in impulsive noise. When there is data overlap, the performance analyses of the FFT filter bank-based majority and median CFAR detectors have eluded any theoretical derivation such as the computation of the probability of false alarm, P_{fa}, for a given threshold T, because of the statistical correlation among the output power levels of the FFT filter bank. As a result, there has been no systematic theoretical treatment for the FFT majority and median CFAR detectors.

This report derives for the first time in the open literature the joint probability density function of the output power levels of the FFT filter bank and presents formulas for threshold computation for the FFT filter bank-based majority and median CFAR detectors in an AWGN channel. This enables one to study the theoretical performance of the FFT majority and median CFAR detectors. An immediate inference that follows from this derivation is that, when the overlap ratio is less than or equal to $1/2$, the correlation among the output power levels of the FFT filter bank introduced by data overlap can be essentially ignored for many commonly used windows, such as the Blackman and Blackman-Harris windows, even when more than one FFT bin is assigned to a channel. This implies that formulas for computing P_{fa} for a given T derived for non-overlapped input data blocks can be used effectively to compute the threshold T for a given P_{fa} for overlapped input data blocks.

This page intentionally left blank.

Sommaire

Theory of the FFT Filter Bank-Based Majority and Median CFAR Detectors

Sichun Wang, Robert Inkol, Sreeraman Rajan, François Patenaude;
DRDC Ottawa TR 2007-088; R & D pour la défense Canada – Ottawa; mai 2007.

Le banc de filtres à TFR est une technique efficace de transformation des signaux à bande étroite présent dans du bruit à large bande. Un détecteur simple ayant un taux de fausse alarme constant peut être mis en œuvre en comparant la puissance mesurée sur chaque base de la TFR à un seuil fonction de la densité spectrale du bruit additif. La performance de ce détecteur peut être améliorée en ajoutant les trois étapes suivantes : (a) augmenter le nombre d'échantillons, (b) diviser le bloc d'échantillons en segment sans chevauchement, (c) faire une moyenne en puissance dans le temps des segments transformés. Le fenêtrage introduit une perte de traitement qui peut être amoindrit en permettant le chevauchement des segments, une idée proposée originalement par Welch [1] dans le cadre de l'estimation de la densité spectrale de puissance moyenne.

Plusieurs variantes du détecteur de TFR à taux de fausse alarme constant ont été étudiées au cours des dernières années [3]-[12]. Parmi ceux-ci, deux variantes notables sont le détecteur à majorité et le détecteur à médiane qui présentent une immunité contre le bruit impulsif. Lorsque les segments se chevauchent, les analyses de ces détecteurs ont évité toute dérivation comme par exemple la dérivation de la probabilité de fausse alarme, P_{fa}, pour un seuil donné T, dû à la corrélation entre les niveaux de la sortie du banc de filtre. En conséquence, il n'y a aucune analyse théorique systématique des détecteurs de TRF à majorité et à médiane avec un taux de fausse alarme constant.

Ce rapport présente pour la première fois dans la littérature publique la dérivation de la fonction de densité de probabilité conjointe des niveaux de puissance à la sortie du banc de filtres à TRF et présente les équations pour le calcul du seuil pour les détecteurs à majorité et à médiane ayant un taux de fausse alarme constant. Ceci permet d'étudier les performances théoriques des ces détecteurs. Un résultat corollaire à cette dérivation est l'observation que la corrélation entre les niveaux de la sortie du banc de filtres est négligeable lorsque les segments se chevauchent pour plusieurs fenêtres communes, comme la fenêtre de Blackman et Blackman-Harris. Ce résultat prévaut même lorsque la puissance de sortie de plusieurs filtres est cumulé pour mesurer la puissance d'un même canal. Ce constat suggère que les equations pour le calcul de P_{fa} pour un seuil donné T dans le cas de segments sans chevauchements peuvent être utilisées pour le calcul du seuil T pour une P_{fa} donnée avec chevauchement de

données.

Table of contents

Acknowledgements

The very helpful comments and suggestions of Dr. Y.T. Chan, Professor Emeritus at the Royal Military College, Kingston, Ontario, Canada, are gratefully acknowledged.

1 Introduction

The FFT filter bank is an efficient approach for detecting multiple narrowband signals distributed over a large bandwidth. A basic form of constant false alarm rate (CFAR) detection processing can be implemented by comparing the power measured in each FFT bin to a suitable threshold, T, that is dependent on the noise spectral density. The performance of this detector can be improved by incorporating the following three steps: (a) increasing the data record size, (b) dividing the data record into non-overlapping blocks and (c) estimating the signal power through averaging the spectral power estimates computed from the individual windowed data blocks. The windowing process introduces processing loss which can be mitigated through the use of overlapped data blocks. This idea was originally proposed for power spectral estimation by Welch [1].

Various extensions of the basic FFT CFAR detector have been investigated in recent years [3]-[12]. The majority and median CFAR detectors have been proposed to provide robust detection performance in the presence of impulsive noise. The published performance analyses of these detectors have been limited to cases involving non-overlapped data blocks. The performance analysis is much more difficult for overlapped data blocks. In particular, the statistical correlation between succesive spectral power measurements complicates the computation of the joint probability density function for the spectral power measurements, a fundamental step in the computation of the probability of false alarm, P_{fa}, for a given threshold T, for a white Gaussian noise input. This report derives theoretical results for the joint probability density function and the detection threshold for an arbitrary P_{fa}.

In this report, the joint probability density function is derived for the case when a single FFT bin is assigned to a channel and the input data overlap ratio is less than or equal to $1/2$. An immediate inference that follows is that the statistical correlations among the output power measurements from the FFT filter bank can be neglected for several commonly used windows, such as the Blackman and Blackman-Harris windows for data overlap ratios up to 50%. This result can be extended to the case where multiple FFT bins are assigned to a channel. Consequently, equations for the computation of P_{fa} for a given T derived for non-overlapped input data blocks can be used to compute the threshold T for a given P_{fa} for overlapped input data blocks. As the computation of P_{fa} is greatly simplified for overlapped data blocks, this is a very useful result.

This report consists of eight sections. Section II defines the FFT filter bank-based CFAR detectors of greatest practical interest in mathematical terms. Section III discusses the relationship of the various FFT filter bank-based CFAR detectors. In particular, the relationship between the majority and median CFAR detectors is clarified. Section IV derives the joint probability density function of the output power levels of the FFT filter bank for overlapped white Gaussian noise input blocks for the important special case where a single FFT bin is assigned to a channel. Two approximations to the joint probability density function are also formulated in this section. Section V derives formulas for the probability of false alarm, P_{fa}, for the majority and median CFAR detectors for non-overlapped data blocks containing only white Gaussian noise. Section VI presents formulas for P_{fa} for the majority and median CFAR detectors for overlapped data blocks containing only white Gaussian noise for two typical scenarios. Section VII presents approximate formulas for computing the detection threshold for the majority and median CFAR detectors. Simulation results are also presented to confirm the validity of these approximations. Finally, the proofs of the main theorems are given in Section VIII. Due to space constraints, proofs which require only routine, but tedious, verifications are omitted and some proofs are only sketched.

2 The FFT filter bank-based CFAR detectors

In this section, the FFT filter bank-based CFAR detectors of most practical interest are introduced.

Assume a band-limited signal that is uniformly sampled. Let there be M channels uniformly distributed across the frequency range contained within the Nyquist bandwidth. Assume that K FFT bins are assigned to each channel and that N FFT bins ($N \leq K$) centered within each channel are used to estimate the power contained within the channel. Consequently, an FFT of length MK is used to compute the power levels for the M channels. Without loss of generality, assume $K - N$ is an even integer. Let $\mathbf{w} = [w_0, \cdots, w_{MK-1}]^t$ be a symmetric window of length MK, where the superscript t denotes matrix transposition. Let $L \geq 1$ be a positive integer and consider L consecutive overlapped sample vectors $\mathbf{S}_l$ constructed as follows ($0 \leq l \leq L-1$):

$$\mathbf{S}_l = [r_{l(1-\gamma)MK+MK-1},\ \cdots,\ r_{l(1-\gamma)MK}]^t \tag{1}$$

Here, r_k is the k-th sample of the input data stream, γ is the overlap ratio and γMK is an integer. In practice, γ is often selected to be either 0 or 1/2 and in the case of $\gamma = 0$, no data overlapping actually takes place and if $\gamma = 1/2$, there is a 50% overlap. The vectors $\mathbf{S}_l$ are windowed by the windowing sequence $\mathbf{w}$, resulting in the windowed sample vectors $\mathbf{X}_l$:

$$\mathbf{X}_l = [w_0 r_{l(1-\gamma)MK+MK-1},\ \cdots,\ w_{MK-1} r_{l(1-\gamma)MK}]^t$$

The vectors $\mathbf{X}_l$ are then transformed by the inverse discrete Fourier transform matrix $\mathbf{F}$ of dimensions $MK \times MK$ to yield the FFT filter bank output sample vectors $\mathbf{Y}_l$:

$$\mathbf{Y}_l = \mathbf{F}\mathbf{X}_l = [s_{l,0},\ s_{l,1},\ \cdots,\ s_{l,MK-1}]^t$$

where

$$\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi jm}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)m}{MK}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi j(MK-1)}{MK}} & \cdots & e^{\frac{2\pi j(MK-1)(MK-1)}{MK}} \end{bmatrix} \tag{2}$$

From each vector $\mathbf{Y}_l$, a vector $\mathbf{z}_l = [z_{l,0}, \cdots, z_{l,M-1}]^t$ of length M is formed by summing the power from the N center FFT bins with indices $h_n,\ h_n+1,\ \cdots,\ h_n+(N-1)$, $h_n = nK + \frac{K-N}{2}$, in the n-th channel:

$$\begin{aligned} z_{l,n} &= \sum_{m=0}^{N-1} |s_{l,h_n+m}|^2 , \\ & \quad 0 \leq l \leq L-1 ,\ 0 \leq n \leq M-1 . \end{aligned} \tag{3}$$

For a given threshold, T, the detection criteria for some of the most useful FFT filter bank-based CFAR detectors are defined mathematically as follows:

1. **The L-block summation CFAR detector.** A signal is declared to exist in the n-th channel if $\sum_{l=0}^{L-1} z_{l,n} \geq T$. For brevity, this CFAR detector shall simply be called the summation CFAR detector.

2. **The J-out-of-L CFAR detector.** Let $1 \leq J \leq L$. A signal is declared to exist in the n-th channel if at least J of the L power estimates $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ exceed the threshold T.

3. **The L-block majority CFAR detector.** The L-block majority CFAR detector is the J-out-of-L CFAR detector with $J = \lfloor \frac{L}{2} \rfloor + 1$. Here $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For brevity, this detector shall simply be called the majority CFAR detector.

4. **The L-block k-th OS (order statistic-based) CFAR detector.** Let $y_1 \leq y_2 \leq \cdots \leq y_L$ be the power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ rearranged in increasing order. y_k is called the k-th order statistic of the power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. A signal is declared to exist in the n-th channel if $y_k \geq T$. For brevity, this detector shall simply called the k-th OS CFAR detector.

5. **The L-block median CFAR detector.** The statistical median of the power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$, denoted by z_{med}, is defined by $z_{med} = y_{\lfloor \frac{L}{2} \rfloor + 1}$ if L is odd and $z_{med} = \frac{y_J + y_{J+1}}{2}$ if L is even, where $J = \frac{L}{2}$. A signal is declared to exist in the n-th channel if $z_{med} \geq T$. For brevity, this CFAR detector shall simply be called the median CFAR detector.

The following section discusses the relationships between the FFT filter bank-based CFAR detectors defined in this section.

3 Relationship among the FFT Filter Bank-Based CFAR Detectors

The decision statistic for the summation CFAR detector is defined by the sum of the output power levels, namely, $\sum_{l=0}^{L-1} z_{l,n}$. If one replaces the sum of the output power levels by the mean power level, which is computed by $\frac{1}{L}\sum_{l=0}^{L-1} z_{l,n}$, a CFAR detector with identical performance to the summation CFAR detector is obtained. Hence the summation CFAR detector, with its detection decision statistic completely characterized by the mean power level, may also be called the mean CFAR detector. As the number of data blocks, L, increases, it is expected that the statistical mean and median values of the L output power levels from the FFT filter bank will be close to each other and hence the summation and median CFAR detectors should have similar performance for large L. However, the median CFAR detector is inherently insensitive to outliers in the output power level sequence $z_{l,n}$, $0 \le l \le L-1$, and therefore should have more robust probability of false alarm performance in impulsive noise.

The majority and median CFAR detectors are based on different ideas. The majority CFAR detector arrives at a detction decision by counting the number of detections from the individual data blocks while the median CFAR detector uses a soft-decision approach based on the median power level. Note that the majority and median CFAR detectors are identical if L is odd. This can be verified by showing that these detectors make identical decisions for the same given threshold T. Assume L is odd and $L = 2J + 1$. If the majority CFAR detector detects a signal in the n-th channel, then we claim that $y_{J+1} \ge T$, for otherwise we would have $y_{J+1} < T$ and this implies that $y_1 \le y_2 \le \cdots \le y_{J+1} < T$ and hence at most $L-J-1 = 2J+1-J-1 = J < J+1$ of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ would exceed the threshold T, a contradiction of the initial assumption. Hence if the majority CFAR detector detects a signal in the n-th channel, so will the median CFAR detector. It can be similarly argued that if the median CFAR detector detects a signal in the n-th channel, so will the majority CFAR detector. Therefore the majority and median CFAR detectors are identical if the number of blocks, L, is odd.

Using similar arguments, the J-out-of-L CFAR detector can be shown to be an order-statistic-based CFAR detector in the sense that it is identical to the k-th OS CFAR detector with $k = L - J + 1$, no matter whether L is odd or even. In particular, if L is even, the majority CFAR detector is identical to the $L/2$-th OS CFAR detector. Since $y_{L/2} \le z_{med} \le y_{L/2+1}$, the majority and median CFAR detectors may produce different detection decisions. Hence the majority and median CFAR detectors are different if L is even. However, for large even values of L, one can expect that the statistical median z_{med} will be close to the order statistic $y_{L/2}$ and the performance of the majority and median CFAR detectors will be nearly identical.

The majority and median CFAR detectors are based on order statistics of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. To derive formulas for computing the probability of false alarm, P_{fa}, for a given threshold, T, it is necessary to compute the joint probability density function of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ for additive white Gaussian noise input. This will be treated in the following section.

4 The Joint Probability Density Function of the Output Power Levels of the FFT Filter Bank for Overlapped White Gaussian Noise Input Blocks

For non-overlapped data, closed-form expressions for computing the probability of false alarm, P_{fa}, for a given threshold, T, can be derived for the detectors discussed in this report. Readers are referred to the references for details [8], [10], [11]. Although there are potential numerical difficulties, these formulas make it relatively easy to compute the threshold T for a given P_{fa}.

For overlapped data, formulas for computing P_{fa}, for a given T have been derived for the summation CFAR detector [3], [4]. However, this is not the case for the majority and median CFAR detectors, the reason being that the detection decision variables for the majority and median CFAR detectors are based on order statistics of the FFT filter bank output power levels. More specifically, to derive formulas for computing P_{fa} for a given T for the majority and median CFAR detectors, it is necessary to compute the joint probability density function of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. This is a non-trivial problem for overlapped data blocks containing white Gaussian noise. In this report, this problem is solved for the important special case where a single FFT bin is assigned to a channel ($N = 1$) (i.e., only one FFT bin per channel used for channel power estimation). This result, although derived only for the case $N = 1$, also yields insights into the statistical properties of the output power levels for the case $N > 1$ when the data blocks are overlapped.

Before proceeding further, we will define the notation and introduce several assumptions, which will help simplify the presentation of the main results. From now on, unless explicitly specified otherwise, it shall always be assumed, without loss of generality, that the window sequence $\mathbf{w} = [w_0, w_1, \cdots, w_{MK-1}]$ satisfies the constraint $\sum_{l=0}^{MK-1} w_l^2 = 1$ (we sometimes say that the window sequence $\mathbf{w} = [w_0, w_1, \cdots, w_{MK-1}]$ is normalized if $\sum_{l=0}^{MK-1} w_l^2 = 1$). Also it will always be assumed that the input data stream r_k is a zero-mean complex white Gaussian noise sequence with

$$E(r_p r_q^*) = \sigma^2 \delta_{pq} \tag{4}$$

where $\sigma^2 > 0$ is the noise variance (noise floor) and $\delta_{pq} = 1$ if $p = q$ and $\delta_{pq} = 0$ if $p \neq q$. Let $\mathbf{P} = \frac{1}{\sigma^2}[z_{0,n}, \cdots, z_{L-1,n}]^t$, where $z_{l,n}$ are defined by (3), $0 \leq n \leq M - 1$. The joint probability density function of the components of $\mathbf{P}$ will be denoted by $p(x_1, x_2, \cdots, x_L)$.

In theory the P_{fa} for a given T for the FFT filter bank-based CFAR detectors can be computed if the joint probability density function, $p(x_1, x_2, \cdots, x_L)$, is available. If there is no data overlap ($\gamma = 0$), the components of $\mathbf{P}$ are statistically independent and identically distributed (i.i.d.), their joint probability density function is easily computed and a closed form formula for P_{fa} is relatively easily obtained for each of the detectors. However, if $0 < \gamma < 1$, the joint probability density function $p(x_1, x_2, \cdots, x_L)$ is, in general, very difficult to compute. For the important special case of $N = 1$, the following theorem holds:

Theorem 1. Let $0 < \gamma \leq 1/2$ and $N = 1$. Let $\beta = \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK}$. Define the infinite positive sequence $\{a_m\}_{m=0}^{+\infty}$ by setting $a_0 = 1$, $a_1 = 1$, and $a_{m+1} = a_m - \beta^2 a_{m-1}$, $m \geq 1$, and define the length-L positive sequence $\{\lambda_m\}_{m=1}^{L}$ by setting $\lambda_m = \frac{a_{m-1}a_{L-m}}{a_L}$, $1 \leq m \leq L$. Then

$$p(x_1, \cdots, x_L) = e^{-\sum_{m=1}^{L} \lambda_m x_m} \times \sum_{m_1 \geq 0, \cdots, m_L \geq 0} \frac{h_{m_1 \cdots m_L}}{m_1! \cdots m_L!} x_1^{m_1} \cdots x_L^{m_L} \tag{5}$$

if $x_m \geq 0$, $1 \leq m \leq L$, and $p(x_1, \cdots, x_L) = 0$ if at least one of the variables $x_1, \cdots, x_L$ is less than 0. The coefficients $h_{m_1 \cdots m_L}$ are defined by the multidimensional power series expansion:

$$\frac{1}{q(z_1, \cdots, z_L)} = \sum_{m_1 \geq 0, \cdots, m_L \geq 0} h_{m_1 \cdots m_L} z_1^{m_1} \cdots z_L^{m_L} \tag{6}$$

where the polynomial $q(z_1, \cdots, z_L)$ is the determinant of an $L \times L$ matrix $[\alpha_{p,q}]_{L \times L}$:

$$q(z_1, \cdots, z_L) = \det [\alpha_{p,q}]_{L \times L} \tag{7}$$

In (7), the entries $\alpha_{p,q}$ are defined by

$$\begin{cases} \alpha_{m,m} & = & (1 - \lambda_m) z_m + 1, \\ \alpha_{m,m-1} & = & -\beta (\lambda_{m-1} z_{m-1} - 1), \\ \alpha_{m,m+1} & = & -\beta (\lambda_{m+1} z_{m+1} - 1), \\ \alpha_{p,q} & = & 0, \quad |p - q| \geq 2. \end{cases} \tag{8}$$

Using standard techniques from the theory of difference equations, it can be shown that a_m is computed by:

$$\begin{aligned} a_m \quad = \quad & \frac{1 - 2\beta^2 + \sqrt{1 - 4\beta^2}}{\sqrt{1 - 4\beta^2}(1 + \sqrt{1 - 4\beta^2})} \left[\frac{1 + \sqrt{1 - 4\beta^2}}{2} \right]^m \\ & - \frac{1 - 2\beta^2 - \sqrt{1 - 4\beta^2}}{\sqrt{1 - 4\beta^2}\ (1 - \sqrt{1 - 4\beta^2})} \left[\frac{1 - \sqrt{1 - 4\beta^2}}{2} \right]^m \end{aligned} \tag{9}$$

It can be verified that a_m is positive and strictly decreasing while a_m/a_{m+1} is strictly increasing. If L is small, relatively simple expressions for $p(x_1, \cdots, x_L)$ can be obtained. In fact, for $L = 1, 2$, the probability density functions $p(x_1)$ and $p(x_1, x_2)$ have very simple forms:

Theorem 2. Let $0 < \gamma < 1$ and $N = 1$. Then

$$p(x_1) = e^{-x_1} \tag{10}$$

and

$$p(x_1, x_2) = \frac{1}{1-\beta^2}\, e^{-\frac{x_2+x_1}{1-\beta^2}} \sum_{t=0}^{+\infty} \frac{1}{(t!)^2} \left[\frac{x_1 x_2 \beta^2}{(1-\beta^2)^2}\right]^t \tag{11}$$

Theorem 2 follows directly from Theorem 1 if $0 < \gamma \leq 1/2$; a proof is presented in Section VIII to demonstrate the main ideas and techniques applicable to the proof of Theorem 1. Using (11), the correlation coefficient of consecutive output power levels of the FFT filter bank can be easily computed. In fact, the following theorem holds:

Theorem 3. Let $0 < \gamma < 1$ and $N = 1$. The correlation coefficient, ρ, between the output power levels of any two consecutive overlapping data blocks $\mathbf{S}_l$, $\mathbf{S}_{l+1}$, $1 \leq l \leq L-1$, is given by:

$$\rho = \beta^2 = \left[\sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK}\right]^2 \tag{12}$$

In general, simple expressions for the coefficients $h_{m_1 \, \cdots \, m_L}$ in (5) are hard to come by. However, as the following theorem shows, the first-order terms in (5) disappear and the coefficients of all the second-order terms in (5) are negative:

Theorem 4. Let $0 < \gamma \leq 1/2$ and $N = 1$. The polynomial $q(z_1, \cdots, z_L)$ can be written as

$$q(z_1, \cdots, z_L) = a_L \left[1 + \sum_{1 \leq k < l \leq L} \gamma_{kl}\; z_k z_l + r(z_1, \cdots, z_L)\right] \tag{13}$$

where $a_L r(z_1, \cdots, z_L)$ is the sum of all the terms in the power series expansion of $q(z_1, \cdots, z_L)$ which are of order greater than or equal to 3 and for $1 \leq k < l \leq L$,

$$\begin{aligned} a_L\ \gamma_{kl} &= a_{k-1}a_{l-k-1}a_{L-l} - \lambda_k a_{l-1}a_{L-l} \\ &\quad -\lambda_l a_{k-1}a_{L-k} + \lambda_k\lambda_l a_L \\ &= a_{L-l}a_{k-1}a_{l-1}\left[\frac{a_{l-1-k}}{a_{l-1}} - \frac{a_{L-k}}{a_L}\right] < 0 \end{aligned} \tag{14}$$

If terms of order greater than 2 are ignored in the power series expansion of $1/q(z_1, \cdots, z_L)$, the following second order approximation to $p(x_1, \cdots, x_L)$ is obtained:

Theorem 5. Let $0 < \gamma \leq 1/2$ and $N = 1$. The following second order approximation holds:

$$p(x_1, \cdots, x_L) \approx \frac{\prod_{m=1}^{L} \lambda_m \, e^{-\lambda_m x_m}}{1 + \sum_{1 \leq k < l \leq L} \frac{a_L a_{L-1}}{a_{L-k} a_{L-l}} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}} \right]} \times \left[1 + \sum_{1 \leq k < l \leq L} \beta_{kl} \, x_k x_l \right], \; x_m > 0, \; 1 \leq m \leq L, \tag{15}$$

where

$$\beta_{kl} = \frac{a_{L-1} a_{k-1} a_{l-1}}{a_L} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}} \right] > 0$$

If $\rho = \beta^2$ is very small, say less than 0.01, the following simpler approximation can be derived, which implies that the output power levels of the FFT filter bank from consecutive overlapping data blocks can be considered as statistically independent:

Theorem 6. Let $0 < \gamma \leq 1/2$ and $N = 1$. If $\rho = \beta^2$ is very small, the following approximation holds:

$$p(x_1, \cdots, x_L) \approx \prod_{m=1}^{L} e^{-x_m} = \prod_{m=1}^{L} p(x_m), \; x_m \geq 0, \; 1 \leq m \leq L. \tag{16}$$

The value of ρ can be easily tabulated for windows implemented in MATLAB as demonstrated by Table 1. It is clear that most of these windows have very small ρ if $\gamma = 1/4$. If $\gamma = 1/2$, the Blackman, Blackman-Harris, Nuttal and the Flat Top windows have very small ρ. Hence for these commonly used windows, it is reasonable to assume statistical independence for the output power levels of the FFT filter bank for overlapped white Gaussian noise input blocks. This leads to substantial simplifications in the threshold computation for the FFT filter bank-based majority and median CFAR detectors, since one can use formulas derived for non-overlapped white Gaussian noise input blocks to compute T when there is data overlap [12]. The simulations described in more detail later in this report have confirmed this observation.

Table 1 ($N = 1$)

Window	ρ ($\gamma = 1/4$)	ρ ($\gamma = 1/2$)
Blackman-Harris	6.0×10^{-8}	0.00139
Nuttall	1.2×10^{-7}	0.00172
Flat Top	1.6×10^{-7}	0.00019
Blackman	3.7×10^{-6}	0.00795
Hanning	5.8×10^{-5}	0.02800
Kaiser ($\beta = 5$)	6.4×10^{-4}	0.05544
Hamming	7.1×10^{-4}	0.05441
Rectangular	6.3×10^{-2}	0.25000

5 Formulas for Computing the Probability of False Alarm P_{fa} for the Majority and Median CFAR Detectors for Non-Overlapped White Gaussian Noise Input Blocks ($\gamma = 0$)

Let μ_m, $1 \leq m \leq N$, be the N distinct positive eigenvalues of the positive definite Hermitian matrix $\mathbf{A}$:

$$\mathbf{A} = \begin{bmatrix} \tau_{11} & \tau_{12} & \cdots & \tau_{1q} & \cdots & \tau_{1N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{p1} & \tau_{p2} & \cdots & \tau_{pq} & \cdots & \tau_{pN} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{N1} & \tau_{N2} & \cdots & \tau_{Nq} & \cdots & \tau_{NN} \end{bmatrix} \tag{17}$$

where

$$\tau_{pq} \quad = \quad \sum_{l=0}^{MK-1} w_l^2 \exp \frac{2\pi j l(p-q)}{MK} \tag{18}$$

For a given T, the probability of false alarm P_{fa} for the k-th OS CFAR detector is defined by

$$P_{fa} = \Pr\{y_k \geq T\} \tag{19}$$

where y_k is the k-th order statistic of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. For the order statistic-based CFAR detectors, the following theorem holds:

Theorem 7. Assume $\gamma = 0$, $N \geq 1$ and $1 \leq k \leq L$. For a given threshold T, the probability of false alarm P_{fa} for the L-block k-th OS CFAR detector is given by:

$$P_{fa} = \frac{L!}{(k-1)!(L-k)!} \int_0^p y^{L-k} \, (1-y)^{k-1} \, dy \tag{20}$$

where, if $N = 1$, $p = \exp\left[-T/(\sigma^2 \sum_{l=0}^{MK-1} w_l^2)\right] = \exp(-T/\sigma^2)$, and, if $N > 1$,

$$\begin{cases} p & = \sum_{m=1}^{N} A_m \, e^{-\frac{T}{\sigma^2 \mu_m}} \\ A_m & = \frac{\mu_m^{N-1}}{\prod_{1 \leq l \leq N, l \neq m} (\mu_m - \mu_l)} \end{cases} \tag{21}$$

Theorem 7 follows directly from the formulas (118) and (121) in [10] using the fact that the output power levels $z_{0,n}$, $\cdots$, $z_{L-1,n}$ of the FFT filter bank are i.i.d.. It is perhaps helpful to reiterate here that it is always assumed in this report that the windows are normalized in the sense that $\sum_{l=0}^{MK-1} w_l^2 = 1$. A sketch of the proof will be given in Section VIII. Note that Theorem 7 applies directly to the L-block majority CFAR detector, since it is identical to the L-block k-th OS CFAR detector with $k = \lfloor \frac{L}{2} \rfloor + 1$.

For odd L, (20) also applies to the median CFAR detector as in this case it is identical to the majority CFAR detector. In fact, by using the Bernoulli probability distribution, we can immediately obtain the following result:

Theorem 8. Assume $\gamma = 0$ and L is odd. For a given threshold T, the probability of false alarm P_{fa} for the L-block majority and median CFAR detectors is given by (20) or by (22):

$$P_{fa} = \sum_{l=k}^{L} \frac{L!}{l!(L-l)!} \, p^l \, (1-p)^{L-l} \tag{22}$$

where in (22) p is defined in Theorem 7 and $k = \lfloor \frac{L}{2} \rfloor + 1$.

When L is even, a similar but more complex formula for computing P_{fa} for a given threshold T can be obtained for the median CFAR detector:

Theorem 9. Assume $\gamma = 0$, $N > 1$ and $L \geq 2$ is even. For a given threshold T, the probability of false alarm P_{fa} for the L-block median CFAR detector is given by:

$$P_{fa} = \frac{2(L!)}{\left[(\frac{L}{2}-1)!\right]^2} \int_T^{+\infty} du \int_0^u \left[F(v)(1-F(2u-v))\right]^{\frac{L}{2}-1} f(2u-v) f(v) dv \tag{23}$$

where $f(x)$ is the probability density function of $z_{l,n}$, $0 \leq l \leq L-1$, given by

$$f(x) = \frac{1}{\sigma^2} \sum_{m=1}^{N} \frac{A_m}{\mu_m} \, e^{-\frac{x}{\sigma^2 \mu_m}}, x > 0, \tag{24}$$

and $F(x) = \int_0^x f(t) \, dt$ is given by

$$F(x) = 1 - \sum_{m=1}^{N} A_m \, e^{-\frac{x}{\sigma^2 \mu_m}}, \; x > 0. \tag{25}$$

The integral in (23), which is derived in Section VIII, can be further simplified and expressed as a linear combination of exponential functions in T. Detailed formulations are omitted, except for the simpler case $N = 1$, for which the following result can be obtained:

Theorem 10. Assume $\gamma = 0$, $N = 1$, $L \geq 4$ is even and $\sigma^2 = 1$. For a given threshold T, the probability of false alarm P_{fa} for the L-block median CFAR detector is given by:

$$
\begin{aligned}
P_{fa} &= C \sum_{t=0}^{\frac{L}{2}-2} \frac{(-1)^t \left(\frac{L}{2}-1\right)!\ e^{-\left(\frac{L}{2}+1+t\right)T}}{t!\ \left(\frac{L}{2}-1-t\right)! \left[\left(\frac{L}{2}\right)^2 - (1+t)^2\right]} \\
&- \frac{C}{L} \sum_{t=0}^{\frac{L}{2}-2} \frac{(-1)^t \left(\frac{L}{2}-1\right)!\ e^{-LT}}{t!\ \left(\frac{L}{2}-1-t\right)! \left(\frac{L}{2}-1-t\right)} \\
&+ (-1)^{\frac{L}{2}-1} C\ e^{-LT} \left[\frac{T+\frac{1}{L}}{L}\right] \qquad (26)
\end{aligned}
$$

where $C = \frac{2\ L!}{\left[\left(\frac{L}{2}-1\right)!\right]^2}$.

6 Formulas for Computing the Probability of False Alarm P_{fa} for the Majority and Median CFAR Detectors for Overlapped White Gaussian Noise Input Blocks ($0 < \gamma \leq 1/2$)

If $0 < \gamma \leq 1/2$, the derivation of a formula for computing P_{fa} for any $T > 0$ is an open problem for the majority and median CFAR detectors, except for $N = 1$, in which case formulas can be derived by using Theorem 1 (c.f. (5)). However, the resulting expressions are complex and difficult to apply in practice. Here we only present two typical results which are relatively easy to formulate and understand:

Theorem 11. Let $N = 1$, $0 < \gamma \leq 1/2$ and $L > 2$. Assume L is odd. For a given threshold, $T > 0$, the corresponding probability of false alarm P_{fa} for the L-block majority and median CFAR detectors is given by

$$\begin{aligned} P_{fa} = &\int_{T}^{+\infty} dx_{\lfloor \frac{L}{2} \rfloor+1} \int_{x_{\lfloor \frac{L}{2} \rfloor+1}}^{+\infty} dx_{\lfloor \frac{L}{2} \rfloor+2} \cdots \\ &\cdots \int_{x_{L-1}}^{+\infty} dx_L \int_{0}^{x_{\lfloor \frac{L}{2} \rfloor+1}} dx_1 \int_{x_1}^{x_{\lfloor \frac{L}{2} \rfloor+1}} dx_2 \\ &\cdots \int_{x_{\lfloor \frac{L}{2} \rfloor-1}}^{x_{\lfloor \frac{L}{2} \rfloor+1}} \overline{p}(x_1, x_2, \cdots, x_L)\ dx_{\lfloor \frac{L}{2} \rfloor} \end{aligned} \tag{27}$$

where

$$\begin{aligned} \overline{p}(x_1, x_2, \cdots, x_L) = &\sum_{(k_1,\ k_2,\ \cdots,\ k_L)\ \in\ \mathbf{P_L}} p(x_{k_1}, x_{k_2}, \cdots, x_{k_L}), \\ &0 \leq x_1 \leq x_2 \cdots \leq x_L. \end{aligned} \tag{28}$$

In (28), $\mathbf{P_L}$ denotes the set of all the permutations of the L integers, $1, 2, \cdots, L$, and $p(x_1, \cdots, x_L)$ is defined by (5).

Theorem 12. Let $N = 1$, $0 < \gamma \leq 1/2$ and $L \geq 4$. Assume L is even. For a given threshold $T > 0$, the corresponding probability of false alarm, P_{fa}, for the L-block majority CFAR detector is given by

$$\begin{aligned} P_{fa} = &\int_{T}^{+\infty} dx_{\frac{L}{2}} \int_{x_{\frac{L}{2}}}^{+\infty} dx_{\frac{L}{2}+1} \int_{x_{\frac{L}{2}+1}}^{+\infty} dx_{\frac{L}{2}+2} \\ &\cdots \int_{x_{L-1}}^{+\infty} dx_L \int_{0}^{x_{\frac{L}{2}}} dx_1 \int_{x_1}^{x_{\frac{L}{2}}} dx_2 \\ &\cdots \int_{x_{\frac{L}{2}-2}}^{x_{\frac{L}{2}}} \overline{p}(x_1, x_2, \cdots, x_L)\ dx_{\frac{L}{2}-1} \end{aligned} \tag{29}$$

where $\overline{p}(x_1, \cdots, x_L)$ is defined by (28).

When L is even, a similar formula can be derived for the L-block median CFAR detector using (5). The details are omitted here.

7 Approximate Formulas for Computing the Threshold for the Majority and Median CFAR Detectors ($0 < \gamma \leq 1/2$)

Although (27) and (29) and other similar formulas can be used to compute T for a given P_{fa} for $0 < \gamma \leq 1/2$, their practical utility is limited by their complexity. As discussed in section IV, for windows with a very small correlation coefficient, $\rho = \beta^2$, the output power levels of the FFT filter bank can be considered as statistically independent for overlapped data blocks containing white Gaussian noise. Hence, one can compute T for a given P_{fa} for the majority and median CFAR detectors by using the corresponding formulas derived for non-overlapped data blocks (c.f. (20), (22), (23), (26)). To confirm the valdidity of these approximations, for $\gamma = 1/2$ and different combinations of L and N, simulations were performed for the Blackman and Blackman-Harris windows with the window size selected to be $MK = 1024$. Given a combination of L and N, for values of $P_{fa} = 10^{-k}$, $1 \leq k \leq 5$, the corresponding normalized detection thresholds, defined by T/σ^2, were computed for the majority and median CFAR detectors using one of the formulas (20), (22), (23) or (26). Ten million simulations were run for each value of the computed normalized detection threshold to obtain the corresponding true probability of false alarm in an AWGN channel. Typical simulation results are plotted in Figures 1-2 for the Blackman window. We observe that the differences between the true probabilities of false alarm obtained from simulations and those computed from approximate formulas are minimal. Very similar results were obtained for the Blackman-Harris window as well. Note that in Figures 1-2, T actually represents the normalized detection threshold. Hence, the formulas (20), (22), (23) or (26) can be used to compute the normalized detection threshold, T/σ^2, for the majority and median CFAR detectors for overlapped white Gaussian noise. It should be noted that further simulations are required to obtain a good understanding of the approximations for very small values of P_{fa} in the range $P_{fa} < 10^{-5}$.

It should be emphasized that, the Hanning window, one of the most common windows in practice, has a relatively large correlation coefficient ρ for $\gamma = 1/2$. This implies that the statistical correlations among the output power levels of the FFT filter bank for overlapped white Gaussian noise input blocks cannot be ignored for the Hanning window. Hence it is necessary to derive practical approximations to the theoretical expressions in (27) and (29). The second order approximation in (15) or higher order approximations should be helpful in this case. This is a research topic of considerable practical interest that will be dealt with elsewhere.

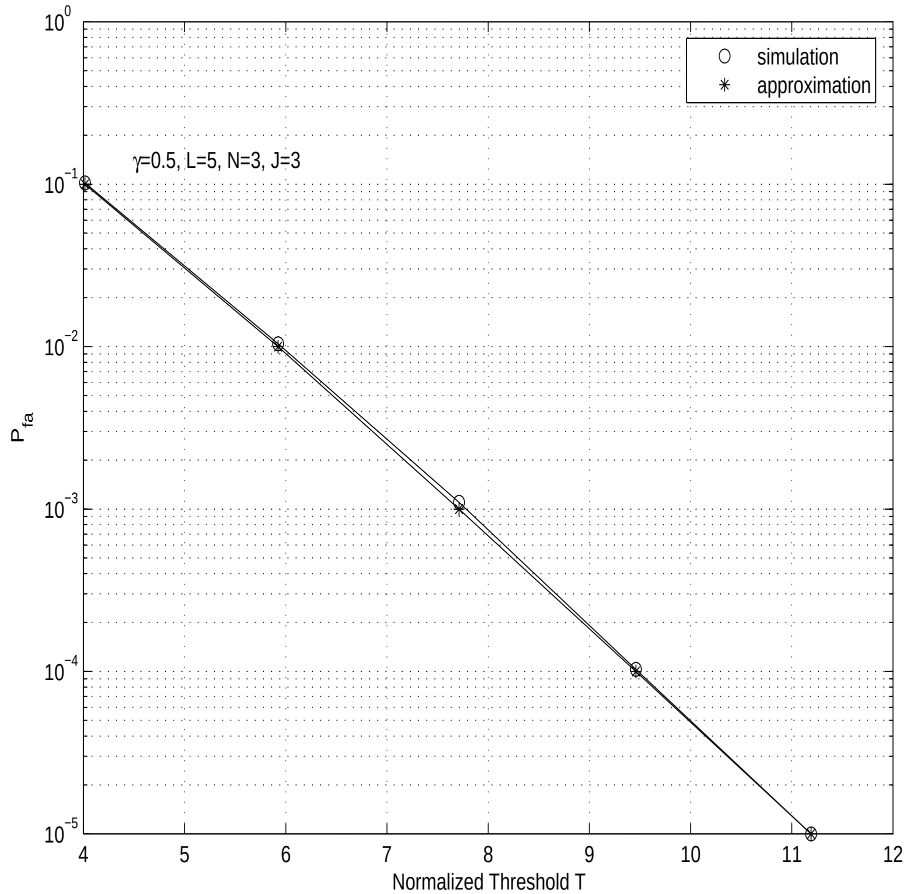

Figure 1: *P_{fa} as a function of the normalized detection threshold for the majority CFAR detector obtained through simulations and from approximations, Blackman window, $L = 5$, $N = 3$, $MK = 1024$, $\gamma = 1/2$.*

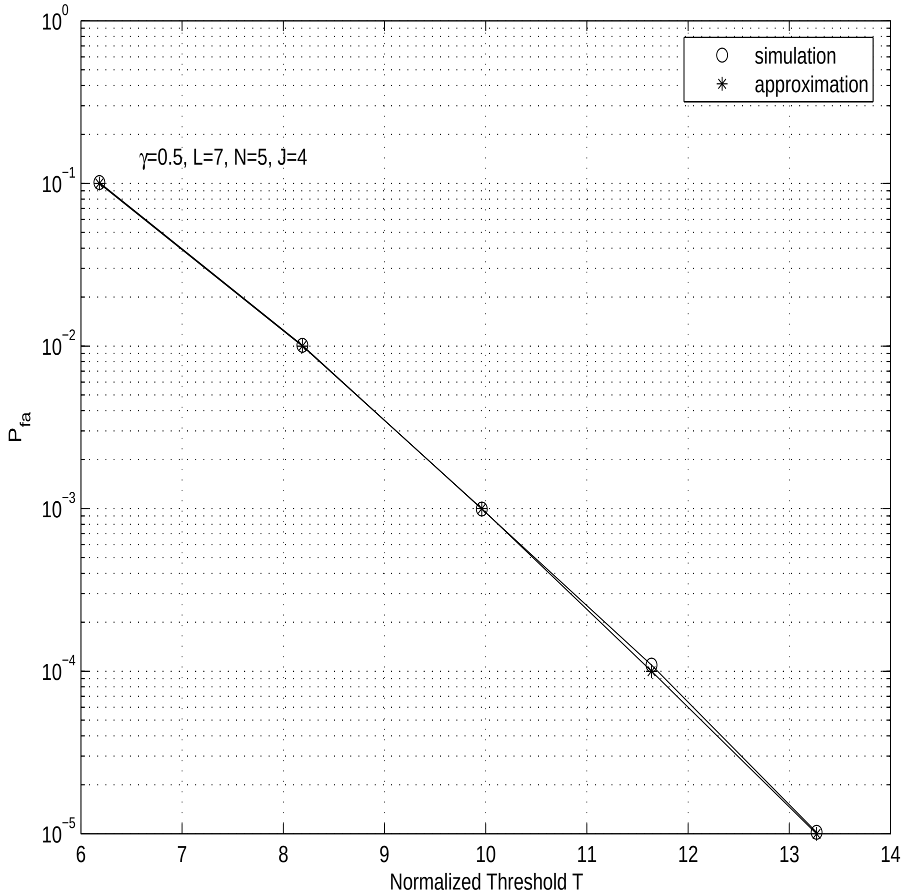

Figure 2: *P_{fa} as a function of the normalized detection threshold for the majority CFAR detector obtained through simulations and from approximations, Blackman window, $L = 7$, $N = 5$, $MK = 1024$, $\gamma = 1/2$.*

8 Proofs

The boldface letters **I** and **0** shall denote, respectively, the identity and zero matrices of appropriate dimensions. In this section it shall be assumed that $0 < \gamma \leq 1/2$, except for $L = 2$, in which case it is assumed that $0 < \gamma < 1$.

We start with the proof of Theorem 1, which will follow from a number of lemmas. Let

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_0 \\ \mathbf{Z}_1 \\ \cdots \\ \mathbf{Z}_l \\ \cdots \\ \mathbf{Z}_{L-1} \end{bmatrix} \tag{30}$$

where $\mathbf{Z}_l$ is defined by

$$\mathbf{Z}_l \quad = \quad \mathbf{F}_n \mathbf{W} \mathbf{S}_l \tag{31}$$

$\mathbf{S}_l$ is the noise sample vector defined by (1), $\mathbf{W}$ is the $MK \times MK$ diagonal matrix defined by:

$$\mathbf{W} = \begin{bmatrix} w_0 & 0 & \cdots & \cdots & 0 \\ 0 & w_1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & w_{MK-1} \end{bmatrix} \tag{32}$$

and $\mathbf{F}_n$ is the $N \times MK$ matrix consisting of the N rows of the inverse discrete Fourier transform matrix $\mathbf{F}$ of dimensions $MK \times MK$ with row indices h_n, h_n, $\cdots$, $h_n + N - 1$:

$$\mathbf{F}_n = \tag{33}$$

$$\begin{bmatrix} 1 & e^{\frac{2\pi j h_n}{MK}} & \cdots & e^{\frac{2\pi j (MK-1) h_n}{MK}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi j [h_n + l]}{MK}} & \cdots & e^{\frac{2\pi j (MK-1)[h_n + l]}{MK}} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & e^{\frac{2\pi j [h_n + N - 1]}{MK}} & \cdots & e^{\frac{2\pi j (MK-1)[h_n + N - 1]}{MK}} \end{bmatrix}$$

The signal power $z_{l,n}$ for the l-th data block is a quadratic form in Gaussian random variables defined by:

$$z_{l,n} \quad = \quad \mathbf{Z}_l^H \mathbf{Z}_l = \mathbf{Z}^H \mathbf{Q}_l \mathbf{Z} \tag{34}$$

where

$$\mathbf{Q}_l = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \end{bmatrix} \tag{35}$$

is a $L \times L$ block matrix of dimensions $LN \times LN$ which has all its $N \times N$ submatrices equal to the $N \times N$ zero matrix except for the $N \times N$ submatrix at the (l, l) position, which is the $N \times N$ identity matrix $\mathbf{I}$.

The covariance matrix $\mathbf{H} = E\left(\mathbf{Z}\mathbf{Z}^H\right)$ of the Gaussian random vector $\mathbf{Z}$ is a $L \times L$ block matrix which can be shown to be

$$\mathbf{H} = E\left(\mathbf{Z}\mathbf{Z}^H\right) = \tag{36}$$

$$\sigma^2 \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}^H & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^H & \mathbf{A} \end{bmatrix}$$

In (36), $\mathbf{A}$ is defined by (17) and (18) and

$$\mathbf{B} = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,q} & \cdots & \gamma_{1,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p,1} & \gamma_{p,2} & \cdots & \gamma_{p,q} & \cdots & \gamma_{p,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N,1} & \gamma_{N,2} & \cdots & \gamma_{N,q} & \cdots & \gamma_{N,N} \end{bmatrix} \tag{37}$$

where for $1 \leq p,\ q \leq N$,

$$\gamma_{p,q} = e^{-2\pi j(1-\gamma)(h_n+q-1)} \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} e^{\frac{2\pi j l(p-q)}{MK}} \tag{38}$$

Let $s_1, s_2, \cdots, s_L$ be L real-valued variables and define

$$\mathbf{Q} = \begin{bmatrix} s_1\mathbf{I} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & s_2\mathbf{I} & \cdots & \cdots & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & s_l\mathbf{I} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & s_L\mathbf{I} \end{bmatrix}$$

$$= s_1\mathbf{Q}_1 + s_2\mathbf{Q}_2 + \cdots + s_L\mathbf{Q}_L \tag{39}$$

Using results of [13] (cf. Eq. 11 of [13]), it can be shown that the characteristic function $\phi(s_1, s_2, \cdots, s_L)$ of the random vector $\frac{1}{\sigma^2}\mathbf{P} = \frac{1}{\sigma^2}[z_{0,n}, z_{1,n}, \cdots, z_{L-1,n}]^t$ is given by

$$\begin{aligned} & \phi(s_1, s_2, \cdots, s_L) \\ = \; & E\left(e^{j\frac{s_1 z_{0,n} + s_2 z_{1,n} + \cdots + s_L z_{L-1,n}}{\sigma^2}}\right) \\ = \; & E\left(e^{j\frac{s_1 \mathbf{Z}^H \mathbf{Q}_1 \mathbf{Z} + s_2 \mathbf{Z}^H \mathbf{Q}_2 \mathbf{Z} + \cdots + s_L \mathbf{Z}^H \mathbf{Q}_L \mathbf{Z}}{\sigma^2}}\right) \\ = \; & E\left(e^{j\frac{\mathbf{Z}^H \mathbf{Q} \mathbf{Z}}{\sigma^2}}\right) = \left|\mathbf{I} - j\frac{1}{\sigma^2}\mathbf{H}\mathbf{Q}\right|^{-1} \end{aligned} \tag{40}$$

Since

$$\frac{1}{\sigma^2}\mathbf{H}\mathbf{Q} = \tag{41}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}^H & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^H & \mathbf{A} \end{bmatrix} \times$$

$$\begin{bmatrix} s_1\mathbf{I} & \mathbf{0} & \cdots & \cdots & \cdots & \mathbf{0} \\ \mathbf{0} & s_2\mathbf{I} & \cdots & \cdots & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & s_l\mathbf{I} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & \cdots & s_L\mathbf{I} \end{bmatrix} =$$

$$\begin{bmatrix} s_1\mathbf{A} & s_2\mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ s_1\mathbf{B}^H & s_2\mathbf{A} & s_3\mathbf{B} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & s_2\mathbf{B}^H & s_3\mathbf{A} & \cdots & \mathbf{0} & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & \cdots & s_{L-1}\mathbf{A} & s_L\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & s_{L-1}\mathbf{B}^H & s_L\mathbf{A} \end{bmatrix}$$

it follows that

$$\phi(s_1, s_2, \cdots, s_L) = \psi(s_1, \cdots, s_L)^{-1} \tag{42}$$

where

$$\psi(s_1,\cdots,s_L)=\begin{vmatrix} \mathbf{I}-js_1\mathbf{A} & -js_2\mathbf{B} & \mathbf{0} & \cdots \\ -js_1\mathbf{B}^H & \mathbf{I}-js_2\mathbf{A} & -js_3\mathbf{B} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \mathbf{0} & \cdots & -js_L\mathbf{B} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}-js_L\mathbf{A} \end{vmatrix} \tag{43}$$

The joint probability density function $p(x_1,\cdots,x_L)$ of the components of the random vector $\frac{1}{\sigma^2}[z_{0,n},\cdots,z_{L-1,n}]^t$ is then computed by

$$p(x_1,\cdots,x_L)=\frac{1}{(2\pi)^L}\int_{\mathbf{R}^L}\phi(s_1,\cdots,s_L)e^{-j\sum_{k=1}^{L}s_kx_k}\,ds_1\cdots ds_L \tag{44}$$

In general, it is very difficult to use the formula (44) to compute $p(x_1,\cdots,x_L)$ explicitly. However, if $N=1$, a procedure to compute $p(x_1,\cdots,x_L)$ from $\phi(s_1,\cdots,s_L)$ by evaluating the integral in (44) has been discovered. In the sequel, unless explicitly specified otherwise, we shall always assume $N=1$. Under this assumption, $\mathbf{A}$ and $\mathbf{B}$ are reduced to complex numbers. In fact, $\mathbf{A}=\sum_{l=0}^{MK-1}w_l^2=1$ and

$$\begin{aligned}\mathbf{B} &= e^{-2\pi j(1-\gamma)h_n}\sum_{l=0}^{\gamma MK-1}w_lw_{l+(1-\gamma)MK}\\ &= C\beta \end{aligned}\tag{45}$$

where $C=e^{-2\pi j(1-\gamma)h_n}$ and $\beta=\sum_{l=0}^{\gamma MK-1}w_lw_{l+(1-\gamma)MK}$. It can be verified by mathematical induction that h_n can be set to 0 without having any impact on $\phi(s_1,\cdots,s_L)$ or $\psi(s_1,\cdots,s_L)$. In fact, if $L=2$, we obtain

$$\begin{aligned} &\psi(s_1,s_2)=\\ &\begin{vmatrix} 1-js_1 & -js_2C\beta \\ -js_1\overline{C}\beta & 1-js_2 \end{vmatrix}\\ =\;& [1-js_1][1-js_2]-\left[-js_1\overline{C}\beta\right][-js_2C\beta]\\ =\;& [1-js_1][1-js_2]-[-js_1\beta][-js_2\beta]\\ =\;& \begin{vmatrix} 1-js_1 & -js_2\beta \\ -js_1\beta & 1-js_2 \end{vmatrix}\end{aligned}$$

Assuming for all integers $k\le L$,

$$\begin{aligned}&\psi(s_1,\cdots,s_k)=\\ &\begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_k\beta \\ 0 & 0 & \cdots & 1-js_k \end{vmatrix}\end{aligned}$$

we will show that

$$\psi(s_1, \cdots, s_L, s_{L+1}) = \begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_{L+1}\beta \\ 0 & 0 & \cdots & 1-js_{L+1} \end{vmatrix}$$

In fact,

$$\psi(s_1, \cdots, s_L, s_{L+1}) = \begin{vmatrix} 1-js_1 & -js_2C\beta & 0 & \cdots \\ -js_1\overline{C}\beta & 1-js_2 & -js_3C\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -jCs_{L+1}\beta \\ 0 & 0 & \cdots & 1-js_{L+1} \end{vmatrix}$$

$$= (1-js_{L+1}) \begin{vmatrix} 1-js_1 & -js_2C\beta & 0 & \cdots \\ -js_1\overline{C}\beta & 1-js_2 & -js_3C\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -jCs_L\beta \\ 0 & 0 & \cdots & 1-js_L \end{vmatrix}$$

$$- \left[-js_{L+1}C\beta\right] \left[-js_{L+1}\overline{C}\beta\right] \begin{vmatrix} 1-js_1 & -js_2C\beta & 0 & \cdots \\ -js_1\overline{C}\beta & 1-js_2 & -js_3C\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -jCs_{L-1}\beta \\ 0 & 0 & \cdots & 1-js_{L-1} \end{vmatrix}$$

$$= (1-js_{L+1}) \begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_L\beta \\ 0 & 0 & \cdots & 1-js_L \end{vmatrix}$$

$$- \left[-js_{L+1}\beta\right]\left[-js_{L+1}\beta\right] \begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_{L-1}\beta \\ 0 & 0 & \cdots & 1-js_{L-1} \end{vmatrix}$$

$$= \begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_{L+1}\beta \\ 0 & 0 & \cdots & 1-js_{L+1} \end{vmatrix}$$

This completes the proof that the polynomial $\psi(s_1, \cdots, s_L)$ is independent of the value of h_n. From now on, we shall always assume that $h_n = 0$. In summary, we can write

$$\phi(s_1, \cdots, s_L) = \psi(s_1, \cdots, s_L)^{-1} \tag{46}$$

where

$$\psi(s_1, \cdots, s_L) = \tag{47}$$
$$\begin{vmatrix} 1-js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1-js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_L\beta \\ 0 & 0 & \cdots & 1-js_L \end{vmatrix}$$

It can be directly verified that

$$\begin{aligned} \psi(s_1, s_2) &= \begin{vmatrix} 1-js_1 & -js_2\beta \\ -js_1\beta & 1-js_2 \end{vmatrix} \\ &= (1-js_1)(1-js_2) - (-js_1\beta)(-js_2\beta) \\ &= 1 - j(s_1+s_2) - s_1 s_2 + s_1 s_2 \beta^2 \\ &= 1 - (1-\beta^2) s_1 s_2 - j(s_1+s_2) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \psi(s_1, s_2, s_3) &= \begin{vmatrix} 1-js_1 & -js_2\beta & 0 \\ -js_1\beta & 1-js_2 & -js_3\beta \\ 0 & -js_2\beta & 1-js_3 \end{vmatrix} \\ &= (1-js_1)\psi(s_2, s_3) + js_1\beta \begin{vmatrix} -js_2\beta & 0 \\ -js_2\beta & 1-js_3 \end{vmatrix} \end{aligned}$$

$$
\begin{aligned}
&= (1 - js_1)\{1 - (1-\beta^2)s_2 s_3 - j(s_2 + s_3)\} \\
&\quad + s_1 s_2 \beta^2 (1 - js_3) \\
&= 1 - js_1 - (1 - js_1)(1-\beta^2)s_2 s_3 \\
&\quad -(1 - js_1) j(s_2 + s_3) + s_1 s_2 \beta^2 - js_1 s_2 s_3 \beta^2 \\
&= 1 - s_1 s_2 - s_2 s_3 - s_3 s_1 + \beta^2 (s_1 + s_3) s_2 + \\
&\quad j\{(1 - 2\beta^2) s_1 s_2 s_3 - s_1 - s_2 - s_3\} \qquad (49)
\end{aligned}
$$

For $L \geq 3$, the following recursions can be used to compute $\psi(s_1, \cdots, s_L)$:

$$
\begin{cases}
\psi(s_1) &= 1 - js_1 \\
\psi(s_1, s_2) &= (1 - js_1)\psi(s_2) + s_1 s_2 \beta^2 \\
\psi(s_1, \cdots, s_L) &= (1 - js_1)\psi(s_2, \cdots, s_L) \\
& \quad + s_1 s_2 \beta^2 \psi(s_3, \cdots, s_L)
\end{cases} \qquad (50)
$$

It is clear that the parameter β is of fundamental importance in the computation of $\phi(s_1, \cdots, s_L)$ and $\psi(s_1, \cdots, s_L)$ and it is helpful to gain an idea of its range of values. In fact, if $0 < \gamma \leq 1/2$, the Cauchy-Schwartz inequality implies that

$$
\begin{aligned}
|\beta| &= \left| \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} \right| \\
&\leq \left[\sum_{l=0}^{\gamma MK-1} w_l^2 \right]^{1/2} \left[\sum_{l=0}^{\gamma MK-1} w_{l+(1-\gamma)MK}^2 \right]^{1/2} \\
&= \left[\sum_{l=0}^{\gamma MK-1} w_l^2 \right]^{1/2} \left[\sum_{l=(1-\gamma)MK}^{MK-1} w_l^2 \right]^{1/2} \\
&= \left[\sum_{l=0}^{\gamma MK-1} w_l^2 \right]^{1/2} \left[1 - \sum_{l=0}^{(1-\gamma)MK-1} w_l^2 \right]^{1/2} \\
&\leq \left[\sum_{l=0}^{(1-\gamma)MK-1} w_l^2 \right]^{1/2} \left[1 - \sum_{l=0}^{(1-\gamma)MK-1} w_l^2 \right]^{1/2} \\
&\leq \max_{0 \leq x \leq 1} \sqrt{x(1-x)} = 1/2
\end{aligned}
$$

with the equality achieved by the rectangular window.

The polynomial $\psi(s_1, \cdots, s_L)$ admits useful factorizations which make it possible to compute the probability density function $p(x_1, \cdots, x_L)$ explicitly by performing iterated integration in (44). Before proceeding further, we digress here to introduce two sequences $\{a_k\}_{k=0}^{+\infty}$ and $\{b_k\}_{k=1}^{+\infty}$, defined recursively by:

$$\left\{\begin{array}{lcl} a_0 & = & 1 \\ a_1 & = & 1 \\ b_1 & = & \beta^2 \\ a_{k+1} & = & a_k - b_k \\ b_{k+1} & = & \beta^2 a_k \end{array}\right. \tag{51}$$

These two sequences are instrumental in deriving the aforementioned factorizations for the polynomial $\psi(s_1, \cdots, s_L)$. It can be shown that a_k and b_k are positive for all $k \geq 1$. In fact, from the fourth and fifth equations of (51) it follows that

$$a_{k+1} = a_k - \beta^2 a_{k-1} \tag{52}$$

The recursive difference equation (52) has the characteristic equation

$$r^2 = r - \beta^2 \tag{53}$$

which has two distinct positive roots $r_1 = \frac{1+\sqrt{1-4\beta^2}}{2}$ and $r_2 = \frac{1-\sqrt{1-4\beta^2}}{2}$ if $|\beta| < 1/2$ and a single positive root $r_0 = 1/2$ if $|\beta| = 1/2$. We first assume $|\beta| < 1/2$. It then follows that there exist two real constant numbers c_1 and c_2 such that

$$a_k = c_1 r_1^k + c_2 r_2^k \ , \ k \geq 1 \ . \tag{54}$$

Setting $k = 1, 2$ in (54) yields the following simple linear equation system

$$\left\{\begin{array}{lcl} c_1 r_1 + c_2 r_2 & = & 1 \\ c_1 r_1^2 + c_2 r_2^2 & = & 1 - \beta^2 \end{array}\right. \tag{55}$$

with the solution:

$$\left\{\begin{array}{lcl} c_1 & = & \frac{1-2\beta^2+\sqrt{1-4\beta^2}}{\sqrt{1-4\beta^2}\ (1+\sqrt{1-4\beta^2})} \\ c_2 & = & -\frac{1-2\beta^2-\sqrt{1-4\beta^2}}{\sqrt{1-4\beta^2}\ (1-\sqrt{1-4\beta^2})} \end{array}\right. \tag{56}$$

It follows that, if $|\beta| < 1/2$, then for $k \geq 0$,

$$\begin{aligned} a_k = & \frac{1 - 2\beta^2 + \sqrt{1 - 4\beta^2}}{\sqrt{1 - 4\beta^2}\ (1 + \sqrt{1 - 4\beta^2})} \left[\frac{1 + \sqrt{1 - 4\beta^2}}{2}\right]^k \\ & - \frac{1 - 2\beta^2 - \sqrt{1 - 4\beta^2}}{\sqrt{1 - 4\beta^2}\ (1 - \sqrt{1 - 4\beta^2})} \left[\frac{1 - \sqrt{1 - 4\beta^2}}{2}\right]^k \\ = & \frac{\left[\frac{1+\sqrt{1-4\beta^2}}{2}\right]^{k+1} - \left[\frac{1-\sqrt{1-4\beta^2}}{2}\right]^{k+1}}{\sqrt{1 - 4\beta^2}} > 0 \end{aligned} \tag{57}$$

and for all $k \geq 1$, $b_k > 0$, since $b_k = \beta^2 a_{k-1}$. Next assume $|\beta| = 1/2$. In this case it can be shown that

$$a_k = \frac{k+1}{2^k} > 0, \; k \geq 0, \tag{58}$$

and $b_k = \beta^2 a_{k-1} > 0$, for $k \geq 1$. This completes the proof that $a_k > 0$, $b_k > 0$, $k \geq 1$.

Similar arguments can be used to show that a_k/a_{k+1} is strictly increasing while a_k is strictly decreasing.

It is very easy to show that $p(x_1) = e^{-x_1}, x_1 \geq 0$. In the sequel, it will always be assumed that $L \geq 2$. We need to prove several propositions which play an important role in the computation of the probability density function $p(x_1, \cdots, x_L)$. The following polynomials shall be used repeatedly:

$$\begin{cases} \tau_0(s_1, s_2, \cdots, s_L) = \psi(s_1, s_2, \cdots, s_L), \\ \tau_k(s_1, s_2, \cdots, s_{L-k}) = \\ -a_k j \; \psi(s_1, s_2, \cdots, s_{L-k}) + \\ \quad b_k s_1 \psi(s_2, s_3, \cdots, s_{L-k}), \\ \quad 1 \leq k \leq L-2, \\ \tau_{L-1}(s) = -a_{L-1} j \; \psi(s) + b_{L-1} s, \\ \tau_L = -j a_L, \end{cases} \tag{59}$$

where s_1, $\cdots$, s_L will always be assumed to be real-valued. Note that the above group of polynomials with complex-valued coefficients is strongly dependent on the integer L. As L changes, the constituent polynomials in this group will change as well. For example, if $L = 3$,

$$\tau_1(s_1, s_2) = -a_1 j \; \psi(s_1, s_2) + b_1 s_1 \psi(s_2),$$

and if $L = 4$,

$$\tau_1(s_1, s_2, s_3) = -a_1 j \; \psi(s_1, s_2, s_3) + b_1 s_1 \psi(s_2, s_3).$$

Lemma 1. For $k = 0$,

$$\tau_0(s_1, \cdots, s_L) = \tau_1(s_2, \cdots, s_L) \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)} \right] \tag{60}$$

and for $1 \leq k \leq L-2$,

$$\begin{aligned} \tau_k(s_{k+1}, s_{k+2}, \cdots, s_L) \;\; = \;\; & -j \; \left[s_{k+1} + \frac{a_k \psi(s_{k+2}, s_{k+3}, \cdots, s_L)}{\tau_{k+1}(s_{k+2}, s_{k+3}, \cdots, s_L)} \right] \times \\ & \tau_{k+1}(s_{k+2}, s_{k+3}, \cdots, s_L) \end{aligned} \tag{61}$$

Moreover,

$$\begin{aligned}\tau_{L-1}(s_L) &= -a_{L-1}j\ \psi(s_L) + b_{L-1}s_L \\ &= -j\left[s_L + \frac{a_{L-1}}{-a_L j}\right](-ja_L) \\ &= -j\left[s_L + \frac{a_{L-1}}{\tau_L}\right]\tau_L \end{aligned} \tag{62}$$

Proof. We first prove (60). From the third equation of (50), it follows that

$$\begin{aligned} & \psi(s_1,\cdots,s_L) = (1-js_1)\psi(s_2,\cdots,s_L) + s_1 s_2\beta^2\psi(s_3,\cdots,s_L) \\ = \ & \psi(s_2,\cdots,s_L) + s_1\left[-j\psi(s_2,\cdots,s_L) + \beta^2 s_2\psi(s_3,\cdots,s_L)\right] \\ = \ & \psi(s_2,\cdots,s_L) + s_1\left[-a_1 j\psi(s_2,\cdots,s_L) + b_1 s_2\psi(s_3,\cdots,s_L)\right] \\ = \ & \psi(s_2,\cdots,s_L) + s_1\tau_1(s_2,\cdots,s_L) \\ = \ & \tau_1(s_2,\cdots,s_L)\left[s_1 + \frac{a_0\psi(s_2,\cdots,s_L)}{\tau_1(s_2,\cdots,s_L)}\right] \end{aligned}$$

This completes the proof of (60). To prove (61), note that from (50), (59), one can write

$$\begin{aligned} & \tau_k(s_{k+1},s_{k+2},\cdots,s_L) \\ = \ & -a_k j\psi(s_{k+1},s_{k+2},\cdots,s_L) + b_k s_{k+1}\psi(s_{k+2},\cdots,s_L) \\ = \ & -a_k j(1-js_{k+1})\psi(s_{k+2},\cdots,s_L) - a_k j s_{k+1}s_{k+2}\beta^2\psi(s_{k+3},\cdots,s_L) \\ & +b_k s_{k+1}\psi(s_{k+2},\cdots,s_L) \\ = \ & -a_k j\psi(s_{k+2},\cdots,s_L) + (b_k - a_k)s_{k+1}\psi(s_{k+2},\cdots,s_L) \\ & -a_k j s_{k+1}s_{k+2}\beta^2\psi(s_{k+3},\cdots,s_L) \end{aligned}$$

Using the fact that $a_{k+1} = a_k - b_k$ and $b_{k+1} = \beta^2 a_k$, one can rewrite

$$\begin{aligned} & \tau_k(s_{k+1},s_{k+2},\cdots,s_L) = \\ & s_{k+1}\left[-a_{k+1}\psi(s_{k+2},\cdots,s_L) - b_{k+1}js_{k+2}\psi(s_{k+3},\cdots,s_L)\right] \\ & -a_k j\psi(s_{k+2},\cdots,s_L) \\ = \ & -js_{k+1}\left[-a_{k+1}j\psi(s_{k+2},\cdots,s_L) + b_{k+1}s_{k+2}\psi(s_{k+3},\cdots,s_L)\right] \\ & -a_k j\psi(s_{k+3},\cdots,s_L) \\ = \ & -js_{k+1}\tau_{k+1}(s_{k+2},\cdots,s_L) - a_k j\psi(s_{k+3},\cdots,s_L) \\ = \ & -j\tau_{k+1}(s_{k+2},\cdots,s_L)\left[s_{k+1} + \frac{a_k\psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}\right] \end{aligned}$$

This completes the proof of (61). The verification of (62) is trivial.

The following two lemmas follow from Lemma 1.

Lemma 2. For $L \geq 1$,

$$\begin{aligned}
&\psi(s_1, s_2, \cdots, s_L) = \\
&(-j)^L a_L \left[s_1 + \frac{a_0 \psi(s_2, s_3, \cdots, s_L)}{\tau_1(s_2, s_3, \cdots, s_L)}\right] \times \\
&\left[s_2 + \frac{a_1 \psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times \\
&\left[s_3 + \frac{a_2 \psi(s_4, s_5, \cdots, s_L)}{\tau_3(s_4, s_5, \cdots, s_L)}\right] \times \cdots \\
&\left[s_{L-2} + \frac{a_{L-3}\ \psi(s_{L-1}, s_L)}{\tau_{L-2}(s_{L-1}, s_L)}\right] \times \\
&\left[s_{L-1} + \frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)}\right] \times \left[s_L + \frac{a_{L-1}}{\tau_L}\right]
\end{aligned} \tag{63}$$

Proof. From Lemma 1 it follows that

$$\begin{aligned}
&\psi(s_1, s_2, \cdots, s_L) = \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}\right] \tau_1(s_2, \cdots, s_L) \\
&= -j \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}\right] \left[s_2 + \frac{a_1 \psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times \\
&\quad \tau_2(s_3, s_4, \cdots, s_L) \\
&= (-j)^{L-2} \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}\right] \left[s_2 + \frac{a_1 \psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times \cdots \\
&\quad \left[s_{L-1} + \frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)}\right] \times \tau_{L-1}(s_L) \\
&= (-j)^{L-2} \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}\right] \left[s_2 + \frac{a_1 \psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times \cdots \\
&\quad \left[s_{L-1} + \frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)}\right] \times (-j) \left[s_L + \frac{a_{L-1}}{\tau_L}\right] (-j a_L) \\
&= (-j)^L a_L \left[s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}\right] \left[s_2 + \frac{a_1 \psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times \cdots \\
&\quad \left[s_{L-1} + \frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)}\right] \times \left[s_L + \frac{a_{L-1}}{\tau_L}\right]
\end{aligned}$$

Lemma 3. For $1 \leq k \leq L - 1$,

$$\begin{aligned}
&\psi(s_1, s_2, \cdots, s_L) = (-j)^{k-1} \times \\
&\left[s_1 + \frac{a_0 \psi(s_2, s_3, \cdots, s_L)}{\tau_1(s_2, s_3, \cdots, s_L)}\right] \times
\end{aligned}$$

$$\left[s_2 + \frac{a_1\psi(s_3, s_4, \cdots, s_L)}{\tau_2(s_3, s_4, \cdots, s_L)}\right] \times$$
$$\left[s_3 + \frac{a_2\psi(s_4, s_5, \cdots, s_L)}{\tau_3(s_4, s_5, \cdots, s_L)}\right] \times \cdots$$
$$\left[s_{k-1} + \frac{a_{k-2}\psi(s_k, s_{k+1}, \cdots, s_L)}{\tau_{k-1}(s_k, s_{k+1}, \cdots, s_L)}\right] \times$$
$$\left[s_k + \frac{a_{k-1}\psi(s_{k+1}, s_{k+2}, \cdots, s_L)}{\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L)}\right] \times$$
$$\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L) \tag{64}$$

and

$$\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L) = (-j)^{L-k+1} a_L \times$$
$$\left[s_{k+1} + \frac{a_k\psi(s_{k+2}, s_{k+3}, \cdots, s_L)}{\tau_{k+1}(s_{k+2}, s_{k+3}, \cdots, s_L)}\right] \times$$
$$\left[s_{k+2} + \frac{a_{k+1}\psi(s_{k+3}, s_{k+4}, \cdots, s_L)}{\tau_{k+2}(s_{k+3}, s_{k+4}, \cdots, s_L)}\right] \times \cdots$$
$$\left[s_{L-2} + \frac{a_{L-3}\ \psi(s_{L-1}, s_L)}{\tau_{L-2}(s_{L-1}, s_L)}\right] \times$$
$$\left[s_{L-1} + \frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)}\right] \times \left[s_L + \frac{a_{L-1}}{\tau_L}\right] \tag{65}$$

The proof of Lemma 3, as it is completely analogous to that of Lemma 2, is omitted here.

It must be emphasized that the identities (60)-(65) in Lemmas 1-3 are algebraic ones that hold only for real numbers s_1, s_2, $\cdots$, s_L where $\tau_0(s_1, \cdots, s_L)$, $\tau_1(s_2, \cdots, s_L)$,$\cdots$, $\tau_k(s_{k+1}, \cdots, s_L)$, $\cdots$, τ_L are all non-zero. However, it can be proved that the identities (60)-(65) in Lemmas 1-3 actually hold for all real numbers s_1, s_2, $\cdots$, s_L as the following lemma holds:

Lemma 4. If s_1, s_2, $\cdots$, s_L are real-valued, then for all k, $0 \le k \le L-1$,

$$\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L) \neq 0 \tag{66}$$

Proof. From (46) it follows that for all k, $0 \le k \le L-1$, $\psi(s_{k+1}, s_{k+2}, \cdots, s_L) \neq 0$. In particular, $\tau_0(s_1, s_2, \cdots, s_L) = \psi(s_1, s_2, \cdots, s_L) \neq 0$. Using (63), we claim that for all k, $1 \le k \le L-1$, $\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L) \neq 0$. We shall prove this statement by contradiction. Assuming there existed a real-valued vector $(s_1^0, \cdots, s_L^0) \in R^L$ for which there exists at least one integer k, $1 \le k \le L-1$, such that $\tau_k(s_{k+1}^0, s_{k+2}^0, \cdots, s_L^0) = 0$. Since $\tau_k(s_{k+1}, \cdots, s_L)$, $1 \le k \le L-1$, are non-zero polynomials satisfying the constraints (60) and (61), one can find an infinite sequence of real-valued vectors in R^L, denoted by $\left\{\left(s_1^{(m)}, s_2^{(m)}, \cdots, s_L^{(m)}\right)\right\}_{m=1}^{+\infty}$, such that

$$\lim_{m\to+\infty}\left(s_1^{(m)}, s_2^{(m)}, \cdots, s_L^{(m)}\right) = \left(s_1^0, s_2^0, \cdots, s_L^0\right)$$

and

$$\tau_k\left(s_{k+1}^{(m)}, s_{k+2}^{(m)}, \cdots, s_L^{(m)}\right) \neq 0,\ m \geq 1,\ 1 \leq k \leq L-1.$$

It follows from (63) that

$$\begin{aligned}
&\psi\left(s_1^{(m)}, \cdots, s_L^{(m)}\right) = \\
&(-j)^L a_L \left[s_1^{(m)} + \frac{a_0\psi(s_2^{(m)}, \cdots, s_L^{(m)})}{\tau_1(s_2^{(m)}, \cdots, s_L^{(m)})}\right] \times \\
&\left[s_2^{(m)} + \frac{a_1\psi(s_3^{(m)}, \cdots, s_L^{(m)})}{\tau_2(s_3^{(m)}, \cdots, s_L^{(m)})}\right] \times \\
&\left[s_3^{(m)} + \frac{a_2\psi(s_4^{(m)}, \cdots, s_L^{(m)})}{\tau_3(s_4^{(m)}, \cdots, s_L^{(m)})}\right] \times \cdots \\
&\left[s_{L-2}^{(m)} + \frac{a_{L-3}\ \psi(s_{L-1}^{(m)}, s_L^{(m)})}{\tau_{L-2}(s_{L-1}^{(m)}, s_L^{(m)})}\right] \times \\
&\left[s_{L-1}^{(m)} + \frac{a_{L-2}\ \psi(s_L^{(m)})}{\tau_{L-1}(s_L^{(m)})}\right] \times \left[s_L^{(m)} + \frac{a_{L-1}}{\tau_L}\right]
\end{aligned} \tag{67}$$

Taking the limit on both sides of (67) as $m \to \infty$, we can see that the left side of (67) approaches a non-zero constant while the right side of (67) approaches infinity, which is a contradiction. This completes the proof of Lemma 4.

Lemma 5. If s_1, s_2, $\cdots$, s_L are real-valued, then for $1 \leq k \leq L-1$,

$$\mathrm{Im}\left(\frac{\psi(s_{k+1}, s_{k+2}, \cdots, s_L)}{\tau_k(s_{k+1}, s_{k+2}, \cdots, s_L)}\right) > 0 \tag{68}$$

Moreover

$$\mathrm{Im}\left(\frac{a_{L-1}}{\tau_L}\right) > 0 \tag{69}$$

Here $\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

Proof. We first show that the continuous function $\mathrm{Im}\left(\frac{\psi(s_{k+1},s_{k+2},\cdots,s_L)}{\tau_k(s_{k+1},s_{k+2},\cdots,s_L)}\right)$ defined on the Euclidean space $\mathbf{R}^{L-k}$ can never take the zero value. We shall prove this by contradiction. If for some $1 \le k \le L-1$ there existed real numbers s_{k+1}, s_{k+2}, $\cdots$, s_L such that $\mathrm{Im}\left(\frac{\psi(s_{k+1},s_{k+2},\cdots,s_L)}{\tau_k(s_{k+1},s_{k+2},\cdots,s_L)}\right) = 0$, then from (63) one would find a real number s_k such that the right hand side of (63) becomes zero, which is a contradiction, as the left-hand side of (63) is never zero. Hence for real numbers s_{k+1}, $\cdots$, s_L, $\mathrm{Im}\left(\frac{\psi(s_{k+1},s_{k+2},\cdots,s_L)}{\tau_k(s_{k+1},s_{k+2},\cdots,s_L)}\right)$ can never be zero-valued. Since $\mathrm{Im}\left(\frac{\psi(0,s_{k+2},\cdots,s_L)}{\tau_k(0,s_{k+2},\cdots,s_L)}\right) = \frac{1}{a_k} > 0$, by the Intermediate Value Theorem from multivariate calculus, the continuous function $\mathrm{Im}\left(\frac{\psi(s_{k+1},s_{k+2},\cdots,s_L)}{\tau_k(s_{k+1},s_{k+2},\cdots,s_L)}\right)$ can only take positive values and hence (68) holds. It is easy to verify (69) since $\tau_L = -ja_L$.

Lemma 6. Let $c = a + jb$ with $\mathrm{Im}(c) = b > 0$. Then for $x > 0$ and positive integers $k \ge 1$,

$$\int_{-\infty}^{+\infty} \frac{e^{-jsx}}{(s+c)^k}\, ds = 2\pi(-j)^k \frac{x^{k-1}}{(k-1)!} e^{jcx} \tag{70}$$

This lemma can be proved using standard contour integral techniques from the theory of functions of one complex variable. The assumption that $\mathrm{Im}(c) = b > 0$ is essential in the calculations. Details are omitted.

We are now in a position to compute explicitly the probability density function $p(x_1, \cdots, x_L)$ defined by (44). Before proceeding further, note that Fubini's theorem does not directly apply to the integral in (44) as the complex-valued function $\phi(s_1, s_2, \cdots, s_L)$ is not absolutely integrable. However, this difficulty can be circumvented by first computing the inverse Fourier transform of the function $\phi(s_1, s_2, \cdots, s_L)e^{-\alpha_1 s_1^2 - \alpha_2 s_2^2 - \cdots - \alpha_L s_L^2}$ where $\alpha_1 > 0, \alpha_2 > 0, \cdots, \alpha_L > 0$ and then take its limit as $\alpha_1 \to 0^+$, $\alpha_2 \to 0^+$, $\cdots$, $\alpha_L \to 0^+$. Specifically, with elaboration (details omitted), it can be rigorously shown that, if $x_1 > 0$, $\cdots$, $x_L > 0$, then

$$p(x_1, \cdots, x_L) = \lim_{\substack{\alpha_1 \to 0^+ \\ \cdots \\ \alpha_L \to 0^+}} \frac{1}{(2\pi)^L} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \phi(s_1, \cdots, s_L) \times \tag{71}$$

$$e^{-\sum_{k=1}^{L} \alpha_k s_k^2}\; e^{-j\sum_{k=1}^{L} s_k x_k}\; ds_1 \cdots ds_L$$

$$= \frac{1}{(2\pi)^L} \int_{-\infty}^{+\infty} ds_L \left[\int_{-\infty}^{+\infty} ds_{L-1} \cdots \left[\int_{-\infty}^{+\infty} ds_1 \phi(s_1, \cdots, s_L) e^{-j\sum_{k=1}^{L} s_k x_k} \right] \cdots \right]$$

In other words, $p(x_1, \cdots, x_L)$ can be computed via iterated integration.

We first consider the simple case $L = 2$ and prove Theorem 2, which is reproduced here for easy reference:

Theorem 2. Let $L = 2$ and $0 < \gamma < 1$. We have

$$p(x_1, x_2) = \frac{e^{-\frac{x_2+x_1}{1-\beta^2}}}{1-\beta^2} \sum_{k=0}^{+\infty} \frac{1}{(k!)^2} \left(\frac{x_1 x_2 \beta^2}{(1-\beta^2)^2} \right)^k, \; x_1 > 0, \; x_2 > 0, \tag{72}$$

where $\beta = \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK}$.

Proof. From (71), (60) and (70), it follows that

$$\begin{aligned}
& (2\pi)^2 p(x_1, x_2) = \\
& \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \phi(s_1, s_2) \; e^{-j(s_1 x_1 + s_2 x_2)} \; ds_1 \right] \; ds_2 \\
= & \int_{-\infty}^{+\infty} \left[\frac{e^{-js_2 x_2}}{\tau_1(s_2)} \int_{-\infty}^{+\infty} \frac{e^{-js_1 x_1}}{s_1 + \frac{a_0 \psi(s_2)}{\tau_1(s_2)}} \; ds_1 \right] \; ds_2 \\
= & \int_{-\infty}^{+\infty} \frac{e^{-js_2 x_2}}{\tau_1(s_2)} \left[-2\pi j e^{j x_1 \frac{a_0 \psi(s_2)}{\tau_1(s_2)}} \right] ds_2
\end{aligned} \tag{73}$$

Note that by (68), for any real-valued s_2, $\mathrm{Im}\left(\frac{a_0 \psi(s_2)}{\tau_1(s_2)}\right) > 0$ and hence to evaluate the inner integral, the formula (70) can be applied. From (62), $\tau_1(s_2)$ can be written as

$$\tau_1(s_2) = -j \left[s_2 + \frac{a_1}{\tau_2} \right] \tau_2 = -a_2 \left[s_2 + \frac{a_1}{a_2} j \right] \tag{74}$$

Hence

$$\frac{a_0 \psi(s_2)}{\tau_1(s_2)} = \frac{1 - js_2}{-a_2 \left[s_2 + \frac{a_1}{a_2} j \right]} = \frac{j}{a_2} + \frac{\frac{b_1}{a_2^2}}{s_2 + \frac{a_1}{a_2} j} \tag{75}$$

and it follows that

$$\begin{aligned}
& \frac{-2\pi j}{\tau_1(s_2)} \; e^{j x_1 \frac{a_0 \psi(s_2)}{\tau_1(s_2)}} = \frac{2\pi j}{a_2 \left[s_2 + \frac{a_1}{a_2} j \right]} e^{-\frac{x_1}{a_2}} \; e^{j x_1 \frac{\frac{b_1}{a_2^2}}{s_2 + \frac{a_1}{a_2} j}} \\
= & \frac{2\pi j}{a_2 \left[s_2 + \frac{a_1}{a_2} j \right]} e^{-\frac{x_1}{a_2}} \sum_{k=0}^{+\infty} \frac{1}{k!} \left[j x_1 \frac{\frac{b_1}{a_2^2}}{s_2 + \frac{a_1}{a_2} j} \right]^k \\
= & \frac{2\pi j}{a_2} e^{-\frac{x_1}{a_2}} \sum_{k=0}^{+\infty} \frac{\left[j x_1 \frac{b_1}{a_2^2} \right]^k}{k!} \left[s_2 + \frac{a_1}{a_2} j \right]^{-k-1}
\end{aligned} \tag{76}$$

Substituting (76) into (73) yields

$$p(x_1, x_2) = \frac{j}{2\pi a_2} e^{-\frac{x_1}{a_2}} \sum_{k=0}^{+\infty} \frac{\left[j x_1 \frac{b_1}{a_2^2} \right]^k}{k!} \int_{-\infty}^{+\infty} e^{-js_2 x_2} \left[s_2 + \frac{a_1}{a_2} j \right]^{-k-1} ds_2 \quad (77)$$

Since $\mathrm{Im}\left(\frac{a_1}{a_2} j\right) = \frac{a_1}{a_2} > 0$, the formula (70) applies and it follows that

$$\begin{aligned} p(x_1, x_2) &= \frac{j}{2\pi a_2} e^{-\frac{x_1}{a_2}} \sum_{k=0}^{+\infty} \frac{\left[j x_1 \frac{b_1}{a_2^2} \right]^k}{k!} \times 2\pi (-j)^{k+1} \frac{x_2^k}{k!} e^{j \frac{a_1}{a_2} j x_2} \\ &= \frac{1}{a_2} \; e^{-\frac{x_1}{a_2}} \; e^{-\frac{a_1 x_2}{a_2}} \; \sum_{k=0}^{+\infty} \frac{\left[x_1 x_2 \frac{b_1}{a_2^2} \right]^k}{(k!)^2} \end{aligned} \quad (78)$$

Since $a_1 = 1$, $b_1 = \beta^2$ and $a_2 = a_1 - b_1 = 1 - \beta^2$, (72) holds.

To further demonstrate the general ideas used in the computation of $p(x_1, \cdots, x_L)$, we next consider the case $L = 3$.

Theorem 13. For $L = 3$ and $0 < \gamma \leq 1/2$,

$$\begin{aligned} & p(x_1, x_2, x_3) = \\ & \frac{1}{1 - 2\beta^2} \; e^{-\frac{1-\beta^2}{1-2\beta^2} x_1} \; e^{-\frac{1}{1-2\beta^2} x_2} \; e^{-\frac{1-\beta^2}{1-2\beta^2} x_3} \sum_{k_1 \geq 0, k_2 \geq 0, k_3 \geq 0} C_{k_1 k_2 k_3} x_1^{k_1} x_2^{k_2} x_3^{k_3}, \\ & \quad x_1 > 0, \; x_2 > 0, \; x_3 > 0, \end{aligned} \quad (79)$$

where the coefficients $C_{k_1 k_2 k_3}$ are real-valued.

Proof. Applying (71), we can write

$$\begin{aligned} & (2\pi)^3 p(x_1, x_2, x_3) = \\ & \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \phi(s_1, s_2, s_3) \; e^{-j \sum_{k=1}^{3} s_k x_k} \; ds_1 \right] ds_2 \right] \; ds_3 \end{aligned} \quad (80)$$

with

$$\begin{aligned} & \int_{-\infty}^{+\infty} \phi(s_1, s_2, s_3) \; e^{-j \sum_{k=1}^{3} s_k x_k} \; ds_1 = \\ & e^{-j(s_2 x_2 + s_3 x_3)} \int_{-\infty}^{+\infty} \phi(s_1, s_2, s_3) e^{-j s_1 x_1} ds_1 \end{aligned} \quad (81)$$

From (60), it follows that

$$\phi(s_1, s_2, s_3) = \frac{1}{\left[s_1 + \frac{a_0 \psi(s_2, s_3)}{\tau_1(s_2, s_3)} \right] \tau_1(s_2, s_3)} \quad (82)$$

Substituting (82) into (81) and applying (68) and (70) yields:

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \phi(s_1, s_2, s_3)\ e^{-j\sum_{k=1}^{3} s_k x_k}\ ds_1 \qquad (83)\\
&= \frac{e^{-j(s_2x_2+s_3x_3)}}{\tau_1(s_2,s_3)} \int_{-\infty}^{+\infty} \frac{e^{-js_1x_1}}{s_1 + \frac{a_0\psi(s_2,s_3)}{\tau_1(s_2,s_3)}} ds_1\\
&= \frac{e^{-j(s_2x_2+s_3x_3)}}{\tau_1(s_2,s_3)} \left[-2\pi j\ e^{jx_1\frac{a_0\psi(s_2,s_3)}{\tau_1(s_2,s_3)}}\right]
\end{aligned}$$

Applying (60) with $L = 2$ and applying (61) with $L = 3$ and $k = 1$, we obtain

$$\begin{cases} \psi(s_2,s_3) &= \left[s_2 + \frac{a_0\psi(s_3)}{\tau_1(s_3)}\right]\tau_1(s_3)\\ \tau_1(s_2,s_3) &= -j\left[s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}\right]\tau_2(s_3) \end{cases} \qquad (84)$$

and therefore

$$\begin{aligned}
&\frac{a_0\psi(s_2,s_3)}{\tau_1(s_2,s_3)} = \frac{a_0\left[s_2 + \frac{a_0\psi(s_3)}{\tau_1(s_3)}\right]\tau_1(s_3)}{-j\left[s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}\right]\tau_2(s_3)} \qquad (85)\\
&= a_0 j\left[1 + \frac{\frac{a_0\psi(s_3)}{\tau_1(s_3)} - \frac{a_1\psi(s_3)}{\tau_2(s_3)}}{s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}}\right]\frac{\tau_1(s_3)}{\tau_2(s_3)}\\
&= a_0 j\frac{\tau_1(s_3)}{\tau_2(s_3)} + \frac{a_0\psi(s_3)\frac{[\tau_2(s_3)-\tau_1(s_3)]}{\tau_2^2(s_3)}}{s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}} = a_0 j\frac{\tau_1(s_3)}{\tau_2(s_3)} + \frac{\frac{-a_0b_1\psi^2(s_3)}{\tau_2^2(s_3)}}{s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}}
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{e^{-j(s_2x_2+s_3x_3)}}{\tau_1(s_2,s_3)} \left[-2\pi j\ e^{jx_1\frac{a_0\psi(s_2,s_3)}{\tau_1(s_2,s_3)}}\right] \qquad (86)\\
&= \frac{-2\pi j e^{-j(s_2x_2+s_3x_3)}}{-j\left[s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}\right]\tau_2(s_3)} \times e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}}\ e^{\frac{\frac{-jx_1a_0b_1\psi^2(s_3)}{\tau_2^2(s_3)}}{s_2+\frac{a_1\psi(s_3)}{\tau_2(s_3)}}}\\
&= \frac{2\pi e^{-js_3x_3}e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}}}{\tau_2(s_3)} \times e^{-js_2x_2} \times\\
&\sum_{k=0}^{+\infty}\left[s_2 + \frac{a_1\psi(s_3)}{\tau_2(s_3)}\right]^{-k-1} \frac{\left[\frac{-jx_1a_0b_1\psi^2(s_3)}{\tau_2^2(s_3)}\right]^k}{k!}
\end{aligned}$$

Applying (68) and (70), it follows from (80)-(81), (83) and (86) that

$$(2\pi)^3 p(x_1,x_2,x_3) = \int_{-\infty}^{+\infty} \frac{2\pi e^{-js_3x_3}e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}}}{\tau_2(s_3)} \times \qquad (87)$$

$$\left[\sum_{k=0}^{+\infty}\frac{\left[\frac{-jx_1a_0b_1\psi^2(s_3)}{\tau_2^2(s_3)}\right]^k}{k!}\int_{-\infty}^{+\infty}\left[s_2+\frac{a_1\psi(s_3)}{\tau_2(s_3)}\right]^{-k-1}e^{-js_2x_2}\ ds_2\right]ds_3$$

$$= \int_{-\infty}^{+\infty} ds_3 \frac{2\pi e^{-js_3x_3}e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}}}{\tau_2(s_3)}\sum_{k=0}^{+\infty}\frac{\left[\frac{-jx_1a_0b_1\psi^2(s_3)}{\tau_2^2(s_3)}\right]^k}{k!}\times \tag{88}$$

$$2\pi(-j)^{k+1}\frac{x_2^k}{k!}\ e^{jx_2\frac{a_1\psi(s_3)}{\tau_2(s_3)}}$$

$$= (2\pi)^2\sum_{k=0}^{+\infty}\frac{-j[-a_0b_1x_1x_2]^k}{(k!)^2}\int_{-\infty}^{+\infty}e^{-js_3x_3}\frac{\psi^{2k}(s_3)}{\tau_2^{2k+1}(s_3)}\ e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}}e^{ja_1x_2\frac{\psi(s_3)}{\tau_2(s_3)}}\ ds_3$$

From (62),

$$\tau_1(s_3) = -j\left(s_3+\frac{a_1}{a_2}j\right)(-a_2j)$$
$$\tau_2(s_3) = -j\left(s_3+\frac{a_2}{a_3}j\right)(-a_3j)$$
$$\psi(s_3) = -j(s_3+j)$$

hence

$$\left\{\begin{array}{lcl}\frac{\tau_1(s_3)}{\tau_2(s_3)} & = & \frac{a_2}{a_3}+\frac{\frac{a_1a_3-a_2^2}{a_3^2}j}{s_3+\frac{a_2}{a_3}j}\\ \frac{\psi(s_3)}{\tau_2(s_3)} & = & \frac{j}{a_3}+\frac{\frac{b_2}{a_3}}{s_3+\frac{a_2}{a_3}j}\end{array}\right. \tag{89}$$

and

$$e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}+ja_1x_2\frac{\psi(s_3)}{\tau_2(s_3)}} = e^{\frac{-a_0a_2}{a_3}x_1+\frac{-\frac{a_1a_3-a_2^2}{a_3^2}a_0x_1j}{s_3+\frac{a_2}{a_3}j}}\ e^{\frac{-a_1x_2}{a_3}+\frac{\frac{b_2}{a_3}a_1x_2j}{s_3+\frac{a_2}{a_3}j}} \tag{90}$$

$$= e^{-\frac{a_0a_2}{a_3}x_1-\frac{a_1x_2}{a_3}}\ e^{\frac{\left[-\frac{a_1a_3-a_2^2}{a_3^2}a_0x_1+\frac{b_2}{a_3}a_1x_2\right]j}{s_3+\frac{a_2}{a_3}j}}$$

$$= e^{-\frac{a_2x_1+a_1x_2}{a_3}}\sum_{l=0}^{+\infty}\frac{j^l}{l!}\left[-\frac{a_1a_3-a_2^2}{a_3^2}a_0x_1+\frac{b_2}{a_3}a_1x_2\right]^l\left[s_3+\frac{a_2}{a_3}j\right]^{-l}$$

Also

$$\frac{\psi^{2k}(s_3)}{\tau_2^{2k+1}(s_3)} = \frac{-a_3^{-1}}{(s_3+\frac{a_2}{a_3}j)}\left[\frac{j}{a_3}+\frac{\frac{b_2}{a_3}}{s_3+\frac{a_2}{a_3}j}\right]^{2k}$$

$$= \frac{-a_3^{-1}}{(s_3+\frac{a_2}{a_3}j)}\sum_{m=0}^{2k}\binom{2k}{m}\left[\frac{j}{a_3}\right]^m\left[\frac{\frac{b_2}{a_3}}{s_3+\frac{a_2}{a_3}j}\right]^{2k-m} \tag{91}$$

Substituting (90) and (91) into the integral in the last equation of (87) yields

$$\int_{-\infty}^{+\infty} e^{-js_3x_3} \frac{\psi^{2k}(s_3)}{\tau_2^{2k+1}(s_3)} \; e^{-a_0x_1\frac{\tau_1(s_3)}{\tau_2(s_3)}} e^{ja_1x_2\frac{\psi(s_3)}{\tau_2(s_3)}} ds_3 \tag{92}$$

$$= e^{-\frac{a_2x_1+a_1x_2}{a_3}} \sum_{l=0}^{+\infty} \frac{j^l}{l!} \left[-\frac{a_1a_3-a_2^2}{a_3^2} a_0x_1 + \frac{b_2}{a_3}a_1x_2 \right]^l$$

$$\times(-a_3^{-1}) \sum_{m=0}^{2k} \left(\begin{array}{c} 2k \\ m \end{array} \right) \left[\frac{j}{a_3}\right]^m \left[\frac{b_2}{a_3}\right]^{2k-m} \int_{-\infty}^{+\infty} \frac{e^{-js_3x_3}}{(s_3+\frac{a_2}{a_3}j)^{2k-m+l+1}} \; ds_3$$

$$= e^{-\frac{a_2x_1+a_1x_2}{a_3}} \sum_{l=0}^{+\infty} \frac{j^l}{l!} \left[-\frac{a_1a_3-a_2^2}{a_3^2} a_0x_1 + \frac{b_2}{a_3}a_1x_2 \right]^l$$

$$\times(-a_3^{-1}) \sum_{m=0}^{2k} \left(\begin{array}{c} 2k \\ m \end{array} \right) \left[\frac{j}{a_3}\right]^m \left[\frac{b_2}{a_3}\right]^{2k-m} \times$$

$$2\pi(-j)^{2k-m+l+1} \frac{x_3^{2k-m+l}}{(2k-m+l)!} \; e^{j\frac{a_2}{a_3}jx_3}$$

$$= \frac{(-1)^k 2\pi j}{a_3} \; e^{-\frac{a_2x_1+a_1x_2+a_2x_3}{a_3}} \times \sum_{l=0}^{+\infty} \frac{1}{l!} \left[-\frac{a_1a_3-a_2^2}{a_3^2} a_0x_1 + \frac{b_2}{a_3}a_1x_2 \right]^l \times$$

$$\sum_{m=0}^{2k} (-1)^m \frac{\left(\begin{array}{c} 2k \\ m \end{array} \right)}{(2k-m+l)!} \; b_2^{2k-m} x_3^{2k-m+l}$$

Combining (87) and (92) finally yields

$$p(x_1,x_2,x_3) = \frac{1}{a_3} \; e^{-\frac{a_2x_1+a_1x_2+a_2x_3}{a_3}} \sum_{k=0}^{+\infty} \frac{[a_0b_1x_1x_2]^k}{(k!)^2} \times$$

$$\sum_{l=0}^{+\infty} \frac{1}{l!} \left[-\frac{a_1a_3-a_2^2}{a_3^2} a_0x_1 + \frac{b_2}{a_3}a_1x_2 \right]^l \times$$

$$\sum_{m=0}^{2k} (-1)^m \frac{\left(\begin{array}{c} 2k \\ m \end{array} \right)}{(2k-m+l)!} \; b_2^{2k-m} x_3^{2k-m+l} \tag{93}$$

Since $a_3 = 1-2\beta^2$, $a_2 = 1-\beta^2$ and $a_1 = 1$, it can be verified that the above expression can be written in the form of (79). This completes the proof.

The ideas and procedures in the proof of Theorem 2 and Theorem 13 generalize in the computation of $p(x_1, \cdots, x_L)$ for any $L \geq 4$. The identities in Lemma 7 are required in the computation of $p(x_1, \cdots, x_L)$:

Lemma 7. For $L \geq 4$, the following identities hold:

$$jx_1 \frac{a_0\psi(s_2,s_3,\cdots,s_L)}{\tau_1(s_2,s_3,\cdots,s_L)} = -\frac{ja_0x_1b_1\frac{\psi^2(s_3,s_4,\cdots,s_L)}{\tau_2^2(s_3,s_4,\cdots,s_L)}}{s_2+\frac{a_1\psi(s_3,s_4,\cdots,s_L)}{\tau_2(s_3,s_4,\cdots,s_L)}} - \frac{ja_0x_1b_1\beta^2\frac{\psi^2(s_4,s_5,\cdots,s_L)}{\tau_3^2(s_4,s_5,\cdots,s_L)}}{s_3+\frac{a_2\psi(s_4,s_5,\cdots,s_L)}{\tau_3(s_4,s_5,\cdots,s_L)}}$$
$$-\frac{ja_0x_1b_1\beta^4\frac{\psi^2(s_5,s_6,\cdots,s_L)}{\tau_4^2(s_5,s_6,\cdots,s_L)}}{s_4+\frac{a_3\psi(s_5,s_6,\cdots,s_L)}{\tau_4(s_5,s_6,\cdots,s_L)}} - \cdots - \frac{ja_0x_1b_1\frac{\beta^{2(L-2)}}{\tau_L^2}}{s_L+\frac{a_{L-1}}{a_L}j} - \frac{a_0a_{L-1}}{a_L}x_1 \tag{94}$$

$$jx_2 \frac{a_1\psi(s_3,s_4,\cdots,s_L)}{\tau_2(s_3,s_4,\cdots,s_L)} = -\frac{ja_1x_2b_2\frac{\psi^2(s_4,s_5,\cdots,s_L)}{\tau_3^2(s_4,s_5,\cdots,s_L)}}{s_3+\frac{a_2\psi(s_4,s_5,\cdots,s_L)}{\tau_3(s_4,s_5,\cdots,s_L)}} - \frac{ja_1x_2b_2\beta^2\frac{\psi^2(s_5,s_6,\cdots,s_L)}{\tau_4^2(s_5,s_6,\cdots,s_L)}}{s_4+\frac{a_3\psi(s_5,s_6,\cdots,s_L)}{\tau_4(s_5,s_6,\cdots,s_L)}}$$
$$-\frac{ja_1x_2b_2\beta^4\frac{\psi^2(s_6,s_7,\cdots,s_L)}{\tau_5^2(s_6,s_7,\cdots,s_L)}}{s_5+\frac{a_4\psi(s_6,s_7,\cdots,s_L)}{\tau_5(s_6,s_7,\cdots,s_L)}} - \cdots - \frac{ja_1x_2b_2\frac{\beta^{2(L-3)}}{\tau_L^2}}{s_L+\frac{a_{L-1}}{a_L}j} - \frac{a_1a_{L-2}}{a_L}x_2 \tag{95}$$

$$jx_k \frac{a_{k-1}\psi(s_{k+1},\cdots,s_L)}{\tau_k(s_{k+1},\cdots,s_L)} = \tag{96}$$
$$-\frac{ja_{k-1}x_kb_k\frac{\psi^2(s_{k+2},\cdots,s_L)}{\tau_{k+1}^2(s_{k+2},\cdots,s_L)}}{s_{k+1}+\frac{a_k\psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}} - \frac{ja_{k-1}x_kb_k\beta^2\frac{\psi^2(s_{k+3},\cdots,s_L)}{\tau_{k+2}^2(s_{k+3},\cdots,s_L)}}{s_{k+2}+\frac{a_{k+1}\psi(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)}}$$
$$-\frac{ja_{k-1}x_kb_k\beta^4\frac{\psi^2(s_{k+4},\cdots,s_L)}{\tau_{k+3}^2(s_{k+4},\cdots,s_L)}}{s_{k+3}+\frac{a_{k+2}\psi(s_{k+4},\cdots,s_L)}{\tau_{k+3}(s_{k+4},\cdots,s_L)}} - \cdots - \frac{ja_{k-1}x_kb_k\frac{\beta^{2(L-k-1)}}{\tau_L^2}}{s_L+\frac{a_{L-1}}{a_L}j} - \frac{a_{k-1}a_{L-k}}{a_L}x_k,$$
$$3 \leq k \leq L-2.$$

$$jx_{L-1}\frac{a_{L-2}\ \psi(s_L)}{\tau_{L-1}(s_L)} = -\frac{ja_{L-2}x_{l-1}b_{L-1}\frac{1}{\tau_L^2}}{s_L+\frac{a_{L-1}}{a_L}j} - \frac{a_{L-2}\ a_1}{a_L}x_{L-1} \tag{97}$$

$$jx_L\frac{a_{L-1}}{\tau_L} = -\frac{a_{L-1}a_0}{a_L}x_L \tag{98}$$

We will prove the identity (96) here. The proofs of the other identities are very similar. In fact, we can write

$$\frac{\psi(s_{k+1},\cdots,s_L)}{\tau_k(s_{k+1},\cdots,s_L)} = \tag{99}$$
$$= \frac{\left[s_{k+1}+\frac{a_0\psi(s_{k+2},\cdots,s_L)}{\tau_1(s_{k+2},\cdots,s_L)}\right]\tau_1(s_{k+2},\cdots,s_L)}{(-j)\left[s_{k+1}+\frac{a_k\psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}\right]\tau_{k+1}(s_{k+2},\cdots,s_L)}$$

$$= j \frac{s_{k+1} + \frac{a_k \psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)} + \frac{\psi(s_{k+2},\cdots,s_L)[\tau_{k+1}(s_{k+2},\cdots,s_L) - a_k \tau_1(s_{k+2},\cdots,s_L)]}{\tau_1(s_{k+2},\cdots,s_L)\tau_{k+1}(s_{k+2},\cdots,s_L)}}{\left[s_{k+1} + \frac{a_k \psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}\right]} \times$$

$$\frac{\tau_1(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}$$

$$= j \left\{ \frac{\tau_1(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)} + \frac{\frac{\psi(s_{k+2},\cdots,s_L)[\tau_{k+1}(s_{k+2},\cdots,s_L) - a_k \tau_1(s_{k+2},\cdots,s_L)]}{\tau_{k+1}^2(s_{k+2},\cdots,s_L)}}{s_{k+1} + \frac{a_k \psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}} \right\}$$

where

$$\begin{aligned} &\tau_{k+1}(s_{k+2},\cdots,s_L) - a_k \tau_1(s_{k+2},\cdots,s_L) = \\ &= -a_{k+1} j \psi(s_{k+2},\cdots,s_L) + b_{k+1} s_{k+2} \psi(s_{k+3},\cdots,s_L) \\ &\quad - a_k \left[-j\psi(s_{k+2},\cdots,s_L) + \beta^2 s_{k+2} \psi(s_{k+3},\cdots,s_L)\right] \\ &= (a_k - a_{k+1}) j \psi(s_{k+2},\cdots,s_L) + (b_{k+1} - a_k \beta^2) \psi(s_{k+3},\cdots,s_L) \\ &= b_k j \psi(s_{k+2},\cdots,s_L) \end{aligned} \quad (100)$$

Combining (99) and (100) yields the identity

$$\frac{\psi(s_{k+1},\cdots,s_L)}{\tau_k(s_{k+1},\cdots,s_L)} = j \frac{\tau_1(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)} + \frac{-b_k \frac{\psi^2(s_{k+2},\cdots,s_L)}{\tau_{k+1}^2(s_{k+2},\cdots,s_L)}}{s_{k+1} + \frac{a_k \psi(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)}} \quad (101)$$

Similarly, we can write

$$\frac{\tau_1(s_{k+2},\cdots,s_L)}{\tau_{k+1}(s_{k+2},\cdots,s_L)} = \quad (102)$$

$$= \frac{(-j)\left[s_{k+2} + \frac{a_1 \psi(s_{k+3},\cdots,s_L)}{\tau_2(s_{k+3},\cdots,s_L)}\right] \tau_2(s_{k+3},\cdots,s_L)}{(-j)\left[s_{k+2} + \frac{a_{k+1} \psi(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)}\right] \tau_{k+2}(s_{k+3},\cdots,s_L)}$$

$$= \frac{s_{k+2} + \frac{a_{k+1} \psi(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)} + \frac{\psi(s_{k+3},\cdots,s_L)[a_1 \tau_{k+2}(s_{k+3},\cdots,s_L) - a_{k+1} \tau_2(s_{k+3},\cdots,s_L)]}{\tau_2(s_{k+3},\cdots,s_L)\tau_{k+2}(s_{k+3},\cdots,s_L)}}{\left[s_{k+2} + \frac{a_{k+1} \psi(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)}\right]} \times$$

$$\frac{\tau_2(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)}$$

$$= \frac{\tau_2(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)} + \frac{\frac{\psi(s_{k+3},\cdots,s_L)[a_1 \tau_{k+2}(s_{k+3},\cdots,s_L) - a_{k+1} \tau_2(s_{k+3},\cdots,s_L)]}{\tau_{k+2}^2(s_{k+3},\cdots,s_L)}}{s_{k+2} + \frac{a_{k+1} \psi(s_{k+3},\cdots,s_L)}{\tau_{k+2}(s_{k+3},\cdots,s_L)}}$$

where

$$a_1 \tau_{k+2}(s_{k+3},\cdots,s_L) - a_{k+1} \tau_2(s_{k+3},\cdots,s_L) \quad (103)$$

$$
\begin{aligned}
&= a_1 \left[-a_{k+2} j\psi(s_{k+3}, \cdots, s_L) + b_{k+2} s_{k+3} \psi(s_{k+4}, \cdots, s_L)\right] \\
&- a_{k+1} \left[-a_2 j\psi(s_{k+3}, \cdots, s_L) + b_2 s_{k+3} \psi(s_{k+4}, \cdots, s_L)\right] \\
&= (-a_1 a_{k+2} + a_2 a_{k+1}) j\psi(s_{k+3}, \cdots, s_L) + \\
&\quad (a_1 b_{k+2} - b_2 a_{k+1}) s_{k+3} \psi(s_{k+4}, \cdots, s_L) \\
&= \beta^2 b_k j\psi(s_{k+3}, \cdots, s_L)
\end{aligned}
$$

since

$$
\begin{aligned}
&-a_1 a_{k+2} + a_2 a_{k+1} = -a_{k+2} + (1-\beta^2) a_{k+1} = -a_{k+2} + a_{k+1} - \beta^2 a_{k+1} \\
&= b_{k+1} - \beta^2 a_{k+1} = \beta^2 a_k - \beta^2 a_{k+1} = \beta^2 b_k
\end{aligned}
$$

and

$$
-a_1 b_{k+2} - b_2 a_{k+1} = -\beta^2 a_{k+1} - \beta^3 a_{k+1} = 0
$$

Combining (102) and (103) yields

$$
j\frac{\tau_1(s_{k+2}, \cdots, s_L)}{\tau_{k+1}(s_{k+2}, \cdots, s_L)} = j\frac{\tau_2(s_{k+3}, \cdots, s_L)}{\tau_{k+2}(s_{k+3}, \cdots, s_L)} + \frac{\frac{-\beta^2 b_k \psi^2(s_{k+3}, \cdots, s_L)}{\tau_{k+2}^2(s_{k+3}, \cdots, s_L)}}{s_{k+2} + \frac{a_{k+1}\psi(s_{k+3}, \cdots, s_L)}{\tau_{k+2}(s_{k+3}, \cdots, s_L)}} \tag{104}
$$

In general, it can be shown that

$$
\begin{aligned}
&\frac{\tau_m(s_{k+m+1}, \cdots, s_L)}{\tau_{k+m}(s_{k+m+1}, \cdots, s_L)} = \frac{\tau_{m+1}(s_{k+m+2}, \cdots, s_L)}{\tau_{k+m+1}(s_{k+m+2}, \cdots, s_L)} \\
&+ \frac{\frac{\psi(s_{k+m+2}[a_m \tau_{k+m+1}(s_{k+m+2}, \cdots, s_L) - a_{k+m}\tau_{m+1}(s_{k+m+2}, \cdots, s_L)]}{\tau_{k+m+1}^2(s_{k+m+2}, \cdots, s_L)}}{s_{k+m+1} + \frac{a_{k+m}\psi(s_{k+m+2}, \cdots, s_L)}{\tau_{k+m+1}(s_{k+m+2}, \cdots, s_L)}}
\end{aligned} \tag{105}
$$

where

$$
\begin{aligned}
&a_m \tau_{k+m+1}(s_{k+m+2}, \cdots, s_L) - a_{k+m}\tau_{m+1}(s_{k+m+2}, \cdots, s_L) \\
&= a_m \left\{-a_{k+m+1} j\psi(s_{k+m+2}, \cdots, s_L) + b_{k+m+1} s_{k+m+2} \psi(s_{k+m+3}, \cdots, s_L)\right\} \\
&- a_{k+m} \left\{-a_{m+1} j\psi(s_{k+m+2}, \cdots, s_L) + b_{m+1} s_{k+m+2} \psi(s_{k+m+3}, \cdots, s_L)\right\} \\
&= \left[a_{k+m} a_{m+1} - a_{k+m+1} a_m\right] j\psi(s_{k+m+2}, \cdots, s_L) + \\
&\quad \left[a_m b_{k+m+1} - a_{k+m} b_{m+1}\right] s_{k+m+2} \psi(s_{k+m+3}, \cdots, s_L) \\
&= \beta^{2m} b_k j\psi(s_{k+m+2}, \cdots, s_L)
\end{aligned} \tag{106}
$$

since

$$
\begin{aligned}
&a_{k+m} a_{m+1} - a_{k+m+1} a_m = a_{k+m}(a_m - \beta^2 a_{m-1}) - (a_{k+m} - b_{k+m}) a_m \\
&= a_{k+m} a_m - \beta^2 a_{k+m} a_{m-1} - a_{k+m} a_m + b_{k+m} a_m \\
&= -\beta^2 a_{k+m} a_{m-1} + \beta^2 a_{k+m-1} a_m = \beta^2 (a_{k+m-1} a_m - a_{k+m} a_{m-1}) \\
&= \beta^4 (a_{k+m-2} a_{m-1} - a_{k+m-1} a_{m-2}) = \cdots \\
&= \beta^{2m} (a_k a_1 - a_{k+1} a_0) = \beta^{2m} b_k
\end{aligned} \tag{107}
$$

and

$$a_m b_{k+m+1} - a_{k+m} b_{m+1} = a_m \beta^2 a_{k+m} - a_{k+m} \beta^2 a_m = 0 \tag{108}$$

Combining (105) and (106) yields

$$j\frac{\tau_m(s_{k+m+1}, \cdots, s_L)}{\tau_{k+m}(s_{k+m+1}, \cdots, s_L)} = j\frac{\tau_{m+1}(s_{k+m+2}, \cdots, s_L)}{\tau_{k+m+1}(s_{k+m+2}, \cdots, s_L)}$$
$$+\frac{\frac{-\beta^{2m} b_k \psi^2(s_{k+m+2}, \cdots, s_L)}{\tau^2_{k+m+1}(s_{k+m+2}, \cdots, s_L)}}{s_{k+m+1} + \frac{a_{k+m}\psi(s_{k+m+2}, \cdots, s_L)}{\tau_{k+m+1}(s_{k+m+2}, \cdots, s_L)}} \tag{109}$$

Applying (109) repeatedly, we obtain

$$\frac{\psi(s_{k+1}, \cdots, s_L)}{\tau_k(s_{k+1}, \cdots, s_L)} = \tag{110}$$
$$-\frac{b_k \frac{\psi^2(s_{k+2}, \cdots, s_L)}{\tau^2_{k+1}(s_{k+2}, \cdots, s_L)}}{s_{k+1} + \frac{a_k \psi(s_{k+2}, \cdots, s_L)}{\tau_{k+1}(s_{k+2}, \cdots, s_L)}} - \frac{b_k \beta^2 \frac{\psi^2(s_{k+3}, \cdots, s_L)}{\tau^2_{k+2}(s_{k+3}, \cdots, s_L)}}{s_{k+2} + \frac{a_{k+1} \psi(s_{k+3}, \cdots, s_L)}{\tau_{k+2}(s_{k+3}, \cdots, s_L)}}$$
$$-\frac{b_k \beta^4 \frac{\psi^2(s_{k+4}, \cdots, s_L)}{\tau^2_{k+3}(s_{k+4}, \cdots, s_L)}}{s_{k+3} + \frac{a_{k+2} \psi(s_{k+4}, \cdots, s_L)}{\tau_{k+3}(s_{k+4}, \cdots, s_L)}} - \cdots - \frac{b_k \beta^{2(L-k-2)} \frac{\psi^2(s_L)}{\tau^2_{L-1}(s_L)}}{s_{L-1} + \frac{a_{L-2}\psi(s_L)}{\tau_{L-1}(s_L)}}$$
$$+j\frac{\tau_{L-k-1}(s_L)}{\tau_{L-1}(s_L)}$$

where

$$j\frac{\tau_{L-k-1}(s_L)}{\tau_{L-1}(s_L)} = j\frac{-a_{L-k}\left[s_L + j\frac{a_{L-k-1}}{a_{L-k}}\right]}{-a_L\left[s_L + j\frac{a_{L-1}}{a_L}\right]}$$
$$= j\frac{a_{L-k}}{a_L} - \frac{a_{L-k}}{a_L}\frac{\frac{a_{L-k-1}}{a_{L-k}} - \frac{a_{L-1}}{a_L}}{s_L + j\frac{a_{L-1}}{a_L}}$$
$$= j\frac{a_{L-k}}{a_L} - \frac{\frac{a_L a_{L-k-1} - a_{L-1} a_{L-k}}{a_L^2}}{s_L + j\frac{a_{L-1}}{a_L}}$$
$$= j\frac{a_{L-k}}{a_L} - \frac{\frac{\beta^{2(L-k-1)} b_k}{\tau_L^2}}{s_L + j\frac{a_{L-1}}{a_L}} \tag{111}$$

Note that here we used the following identity which follows from (107)

$$a_L a_{L-k-1} - a_{L-1} a_{L-k} = -\beta^{2(L-k-1)} b_k \tag{112}$$

This completes the proof of (96).

Theorem 14. For $L \geq 1$,

$$p(x_1, \cdots, x_L) = e^{-\sum_{m=1}^{L} \lambda_m x_m} \\ \times \sum_{m_1, \cdots, m_L \geq 0} C_{m_1 m_2 \cdots m_L} x_1^{m_1} x_2^{m_2} \cdots x_L^{m_L}, \\ x_1 > 0, x_2 > 0, \cdots, x_L > 0, \tag{113}$$

where $\lambda_m = a_{m-1} a_{L-m}/a_L$, $1 \leq m \leq L$, and $C_{m_1 m_2 \cdots m_L}$ are real-valued constants.

Proof. It suffices to prove this theorem for $L \geq 4$, since it is already proved for $1 \leq L \leq 3$. Following exactly the same procedures as in the proof of Theorem 13 and applying Lemmas 5-7 repeatedly, it can be verified rigorously that the iterated integral in (71) can be computed and expressed in the form of (113) for any $L \geq 4$. The details of the computations are completely routine but very lengthy and tedious. Here we will only describe the first steps of these calculations. In fact, from (71) we can write

$$p(x_1, \cdots, x_L) = \\ = \frac{1}{(2\pi)^L} \int_{-\infty}^{+\infty} ds_L \left[\int_{-\infty}^{+\infty} ds_{L-1} \cdots \left[\int_{-\infty}^{+\infty} ds_1 \phi(s_1, \cdots, s_L) e^{-j \sum_{k=1}^{L} s_k x_k} \right] \cdots \right] \tag{114}$$

where

$$\int_{-\infty}^{+\infty} ds_1 \phi(s_1, \cdots, s_L) e^{-j \sum_{k=1}^{L} s_k x_k} = \\ e^{-j \sum_{k=2}^{L} s_k x_k} \int_{-\infty}^{+\infty} \frac{e^{-j s_1 x_1}}{\tau_0(s_1, \cdots, s_L)} ds_1 \\ = \frac{e^{-j \sum_{k=2}^{L} s_k x_k}}{\tau_1(s_2, \cdots, s_L)} \int_{-\infty}^{+\infty} \frac{e^{-j s_1 x_1}}{s_1 + \frac{a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}} ds_1 \\ = \frac{e^{-j \sum_{k=2}^{L} s_k x_k}}{\tau_1(s_2, \cdots, s_L)} \times (2\pi)(-j) e^{\frac{j x_1 a_0 \psi(s_2, \cdots, s_L)}{\tau_1(s_2, \cdots, s_L)}} \\ = \frac{2\pi e^{-j \sum_{k=3}^{L} s_k x_k}}{\tau_2(s_3, \cdots, s_L)} \frac{e^{-j x_2 s_2}}{s_2 + \frac{a_1 \psi(s_3, \cdots, s_L)}{\tau_2(s_3, \cdots, s_L)}} \times e^{-j \sum_{m=2}^{L} \frac{a_0 b_1 x_1 \frac{\psi^2(s_{m+1}, \cdots, s_L)}{\tau_m^2(s_{m+1}, \cdots, s_L))}}{s_m + \frac{a_{m-1} \psi(s_{m+1}, \cdots, s_L)}{\tau_m(s_{m+1}, \cdots, s_L)}}} \times \\ e^{-\frac{a_0 a_{L-1}}{a_L} x_1} \\ = 2\pi e^{-\frac{a_0 a_{L-1}}{a_L} x_1} \frac{e^{-j \sum_{k=3}^{L} s_k x_k}}{\tau_2(s_3, \cdots, s_L)} e^{-j \sum_{m=3}^{L} \frac{a_0 b_1 x_1 \frac{\psi^2(s_{m+1}, \cdots, s_L)}{\tau_m^2(s_{m+1}, \cdots, s_L))}}{s_m + \frac{a_{m-1} \psi(s_{m+1}, \cdots, s_L)}{\tau_m(s_{m+1}, \cdots, s_L)}}} \times \\ \frac{e^{-j x_2 s_2}}{s_2 + \frac{a_1 \psi(s_3, \cdots, s_L)}{\tau_2(s_3, \cdots, s_L)}} \sum_{m_1=0}^{+\infty} \left[\frac{-j a_0 b_1 x_1 \frac{\psi^2(s_3, \cdots, s_L)}{\tau_2^2(s_3, \cdots, s_L))}}{s_2 + \frac{a_1 \psi(s_3, \cdots, s_L)}{\tau_2(s_3, \cdots, s_L)}} \right]^{m_1} \tag{115}$$

Substituting (115) into (114) and continuing the same type of manipulations, it can be shown that the joint probability density function $p(x_1, \cdots, x_L)$ can be written in the form of (112). This completes the proof of Theorem 14.

To complete the proof of Theorem 1, it remains to compute the coefficients $C_{m_1 m_2 \cdots m_L}$ in (113). To do that, the following lemma is needed:

Lemma 8. Assume $\mathrm{Re}(c) > 0$. For all integers $k \geq 1$,

$$\int_0^{+\infty} x^k e^{-cx} dx = \frac{k!}{c^{k+1}} \tag{116}$$

The proof of this lemma is easily completed by an argument using integration by parts.

We now compute the coefficients $C_{m_1 m_2 \cdots m_L}$ in (113). In fact,

$$\begin{aligned} & \phi(s_1, \cdots, s_L) = \psi^{-1}(s_1, s_2, \cdots, s_L) \qquad (117) \\ = & \int_{\mathbf{R}^L} p(x_1, \cdots, x_L)\ e^{j \sum_{k=1}^L s_k x_k}\ dx_1 \cdots dx_L \\ = & \sum_{m_1, \cdots, m_L \ \geq\ 0} C_{m_1 m_2 \cdots m_L} \prod_{k=1}^L \int_0^{+\infty} x_k^{m_k} e^{-\lambda_k x_k} e^{j x_k s_k} dx_k \\ = & \sum_{m_1, \cdots, m_L \ \geq\ 0} C_{m_1 m_2 \cdots m_L} \prod_{k=1}^L \frac{m_k!}{(\lambda_k - j s_k)^{m_k+1}} \\ = & \begin{vmatrix} 1 - js_1 & -js_2\beta & 0 & \cdots \\ -js_1\beta & 1 - js_2 & -js_3\beta & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -js_L\beta \\ 0 & 0 & \cdots & 1 - js_L \end{vmatrix}^{-1} \end{aligned}$$

Setting

$$h_{m_1 m_2 \cdots m_L} = C_{m_1 m_2 \cdots m_L}\ m_1! m_2! \cdots m_L! \tag{118}$$

and making the change of variables $\lambda_k - js_k = z_k^{-1}$, $1 \leq k \leq L$, it can be verified that (5)-(8) hold. This finally completes the proof of Theorem 1.

We next prove Theorem 4. It can be easily verified that Theorem 4 holds for $L = 1, 2$. Hence we only need to consider $L \geq 3$. Since $q(z_1, \cdots, z_L)$ is a polynomial, to prove that the first order terms of $q(z_1, \cdots, z_L)$ disappear, it suffices to show that the partial derivative $\frac{\partial q}{\partial z_k}(0, 0. \cdots, 0) = 0, 1 \leq k \leq L$. For clarity, we shall write $q(z_1, \cdots, z_L)$

as $q_L(z_1, \cdots, z_L)$ in the proof of Theorem 4. In the definitions (7) and (8), setting $z_k = 0$, $1 \le k \le L$, yields

$$q_L(0, 0, \cdots, 0) = \tag{119}$$
$$\begin{bmatrix} 1 & \beta & 0 & \cdots & \cdots & 0 & 0 \\ \beta & 1 & \beta & 0 & \cdots & 0 & 0 \\ 0 & \beta & 1 & \beta & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \beta & 1 & \beta \\ 0 & 0 & 0 & \cdots & 0 & \beta & 1 \end{bmatrix}_{L\times L}$$

and it can be verified by mathematical induction that

$$a_k = q_k(0, \cdots, 0), \; k \ge 1. \tag{120}$$

By expanding the determinant that defines $q(z_1, \cdots, z_L)$ along the k-th column (c.f. (7) and (8)) and applying the identity (120) appropriately, it can be shown that, for $2 \le k \le L$,

$$\begin{aligned} & \frac{\partial q}{\partial z_k}(0, 0, \cdots, 0) \qquad (121) \\ = \;& \beta^2 \lambda_k a_{k-2} a_{L-k} + (1 - \lambda_k) a_{k-1} a_{L-k} + \beta^2 \lambda_k a_{k-1} a_{L-k-1} \\ = \;& \beta^2 \frac{a_{k-1} a_{L-k}}{a_L} a_{k-2} a_{L-k} + \left[\frac{a_L - a_{k-1} a_{L-k}}{a_L}\right] a_{k-1} a_{L-k} + \beta^2 \frac{a_{k-1} a_{L-k}}{a_L} a_{k-1} a_{L-k-1} \\ = \;& \frac{a_{k-1} a_{L-k}}{a_L} \left[\beta^2 a_{k-2} a_{L-k} + a_L - a_{k-1} a_{L-k} + \beta^2 a_{k-1} a_{L-k-1}\right] \\ = \;& \lambda_k \left[a_L + \beta^2 a_{k-2} a_{L-k} + a_{k-1}(-a_{L-k} + \beta^2 a_{L-k-1})\right] \\ = \;& \lambda_k \left[a_L + \beta^2 a_{k-2} a_{L-k} - a_{k-1} a_{L-k+1}\right] \end{aligned}$$

By mathematical induction, it can be shown that, for $2 \le k \le L$,

$$a_L + \beta^2 a_{k-2} a_{L-k} - a_{k-1} a_{L-k+1} = 0 \tag{122}$$

Hence

$$\frac{\partial q}{\partial z_k}(0, 0, \cdots, 0) = 0, \; 2 \le k \le L. \tag{123}$$

It can also be verified that

$$\begin{aligned} & \frac{\partial q}{\partial z_1}(0, 0, \cdots, 0) = (1 - \lambda_1) a_{L-1} + \beta^2 \lambda_1 a_{L-2} \qquad (124) \\ = \;& \left[1 - \frac{a_{L-1}}{a_L}\right] a_{L-1} + \beta^2 \frac{a_{L-1}}{a_L} a_{L-2} \\ = \;& \frac{1}{a_L} \left[(a_L - a_{L-1}) a_{L-1} + \beta^2 a_{L-1} a_{L-2}\right] \\ = \;& \frac{1}{a_L} \left[-\beta^2 a_{L-2} a_{L-1} + \beta^2 a_{L-1} a_{L-2}\right] = 0 \end{aligned}$$

(123) and (124) combined complete the proof that the first order terms of $q(z_1, \cdots, z_L)$ are all zero. Using similar arguments and performing lengthy algebraic manipulations which are omitted here, it can be shown that

$$\begin{aligned} a_L \gamma_{kl} &= a_{k-1} a_{l-k-1} a_{L-l} - \lambda_k a_{l-1} a_{L-l} - \lambda_l a_{k-1} a_{L-k} + \lambda_k \lambda_l a_L \\ &= a_{L-l} a_{k-1} a_{l-1} \left[\frac{a_{l-1-k}}{a_{l-1}} - \frac{a_{L-k}}{a_L} \right] \end{aligned} \tag{125}$$

Since $\frac{a_k}{a_{k+1}}$ is a strictly increasing positive sequence and $l - 1 < L$, it follows that $\frac{a_{l-1-k}}{a_{l-k}} < \frac{a_{L-k}}{a_{L-k+1}}$, $\frac{a_{l-k}}{a_{l-k+1}} < \frac{a_{L-k+1}}{a_{L-k+2}}$, $\cdots$, $\frac{a_{l-3}}{a_{l-2}} < \frac{a_{L-3}}{a_{L-2}}$ and $\frac{a_{l-2}}{a_{l-1}} < \frac{a_{L-2}}{a_{L-1}}$ and hence

$$\begin{aligned} \frac{a_{l-1-k}}{a_{l-1}} &= \frac{a_{l-1-k}}{a_{l-k}} \frac{a_{l-k}}{a_{l-k+1}} \cdots \frac{a_{l-3}}{a_{l-2}} \frac{a_{l-2}}{a_{l-1}} \\ &< \frac{a_{L-k}}{a_{L-k+1}} \frac{a_{L-k+1}}{a_{L-k+2}} \cdots \frac{a_{L-2}}{a_{L-1}} \frac{a_{L-1}}{a_L} = \frac{a_{L-k}}{a_L} \end{aligned} \tag{126}$$

This, together with (125), implies that $\gamma_{kl} < 0$. The proof of Theorem 4 is therefore completed.

Next we prove Theorem 5. From Theorem 4 (c.f. (13)), we obtain

$$\begin{aligned} &\frac{1}{q(z_1, \cdots, z_L)} = \\ &\frac{1}{a_L} \left[1 + \sum_{1 \le k < l \le L} \gamma_{kl}\ z_k z_l + r(z_1, \cdots, z_L) \right]^{-1} \\ = &\sum_{m=0}^{+\infty} \frac{(-1)^m}{a_L} \left[\sum_{1 \le k < l \le L} \gamma_{kl}\ z_k z_l + r(z_1, \cdots, z_L) \right]^m \end{aligned}$$

Ignoring all terms of order greater than or equal to 3, we obtain

$$\frac{1}{q(z_1, \cdots, z_L)} \approx \frac{1}{a_L} \left[1 - \sum_{1 \le k < l \le L} \gamma_{kl}\ z_k z_l \right]$$

It follows from Theorem 1 that the joint probability density function $p(x_1, \cdots, x_L)$ can be approximated by

$$p(x_1, \cdots, x_L) \approx p_1(x_1, \cdots, x_L) = \frac{e^{-\sum_{m=1}^{L} \lambda_m x_m}}{a_L} \left[1 - \sum_{1 \le k < l \le L} \gamma_{kl}\ x_k x_l \right],$$

$$x_1 > 0, \cdots, x_L > 0.$$

However, $p_1(x_1, \cdots, x_L)$ is not a true probability density function and it is necessary to multiply it by a positive constant C so that

$$\int_0^{+\infty} \cdots \int_0^{+\infty} C p_1(x_1, \cdots, x_L) = 1$$

The value of C can be easily computed and it can be verified that

$$Cp_1(x_1, \cdots, x_L) = \frac{\prod_{m=1}^{L} \; \lambda_m \; e^{-\lambda_m x_m}}{1 + \sum_{1 \le k < l \le L} \; \frac{a_L a_{L-1}}{a_{L-k} a_{L-l}} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}} \right]} \times$$

$$\left[1 + \sum_{1 \le k < l \le L} \beta_{kl} \; x_k x_l \right], \; x_m > 0, \; 1 \le m \le L,$$

where

$$\beta_{kl} = \frac{a_{L-1} a_{k-1} a_{l-1}}{a_L} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}} \right] > 0 \tag{127}$$

This completes the proof of Theorem 5.

We next turn to the proof of Theorem 6. Since $\rho = \beta^2$ is very small, from (57), it can be verified that

$$a_k = \frac{\left[\frac{1+\sqrt{1-4\beta^2}}{2} \right]^{k+1} - \left[\frac{1-\sqrt{1-4\beta^2}}{2} \right]^{k+1}}{\sqrt{1-4\beta^2}} \approx (1-\beta^2)^k$$

and

$$\lambda_m = \frac{a_{m-1} a_{L-m}}{a_L} \approx \frac{(1-\beta^2)^{m-1} (1-\beta^2)^{L-m}}{(1-\beta^2)^L} = \frac{1}{1-\beta^2} \approx 1 \tag{128}$$

From (122) we have

$$a_{L-k} a_k - a_L = \beta^2 a_{k-1} a_{L-k-1}$$

and it follows from (127) that

$$\beta_{k(k+1)} = \frac{a_{L-1} a_{k-1} a_k}{a_L} \left[\frac{a_{L-k}}{a_L} - \frac{a_0}{a_k} \right] = \frac{a_{L-1} a_{k-1}}{a_L^2} \left[a_{L-k} a_k - a_L \right]$$

$$= \frac{a_{L-1} a_{k-1}}{a_L^2} \beta^2 a_{k-1} a_{L-k-1} \approx (1-\beta^2)^{k-4} \beta^2 \approx \beta^2 \approx 0 \tag{129}$$

Using (51) and (122) appropriately, it can be shown by mathematical induction that, in general, for $1 \le k < l \le L$,

$$a_{L-k} a_{l-1} - a_L a_{l-1-k} = \beta^{2(l-k)} a_{k-1} a_{L-l} \tag{130}$$

and hence

$$\beta_{kl} = \frac{a_{L-1} a_{k-1} a_{l-1}}{a_L} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}} \right] = \frac{a_{L-1} a_{k-1}}{a_L^2} \left[a_{L-k} a_{l-1} - a_L a_{l-1-k} \right]$$

$$= \frac{a_{L-1} a_{k-1}}{a_L^2} \beta^{2(l-k)} a_{k-1} a_{L-l} \approx (1-\beta^2)^{2k-3-l} \beta^{2(l-k)} \approx \beta^{2(l-k)} \approx 0 \tag{131}$$

The estimates (130) and (131) show that the coefficients β_{kl} in the approximation (15) approach 0 and hence can be ignored when $\rho = \beta^2$ is very small. Similarly, it can be shown that

$$1 + \sum_{1\le k<l\le L} \frac{a_L a_{L-1}}{a_{L-k}a_{L-l}} \left[\frac{a_{L-k}}{a_L} - \frac{a_{l-1-k}}{a_{l-1}}\right] \approx 1 \tag{132}$$

The estimates (128), (129), (131) and (132) imply that the approximation in (16) holds. The proof of Theorem 6 is therefore completed.

To prove Theorem 7, note that the probability density function of the k-th order statistic y_k of the output power levels $z_{0,n}, z_{1,n}, \cdots, z_{L-1,n}$, denoted by $p_k(x)$ here, is given, for $x > 0$, by

$$p_k(x) = \frac{L!}{(k-1)!(L-k)!} F(x)^{k-1}(1-F(x))^{L-k} f(x) \tag{133}$$

where $f(x)$ and $F(x)$ are, respectively, the probability density function and the cumulative distribution function of the output power levels $z_{0,n}, z_{1,n}, \cdots, z_{L-1,n}$. For a given threshold $T > 0$, the corresponding probability of false alarm P_{fa} for the k-th OS CFAR detector is given by

$$\begin{aligned} P_{fa} &= \int_T^{+\infty} p_k(x)dx \qquad (134) \\ &= C_{Lk}\int_T^{+\infty} F(x)^{k-1}[1-F(x)]^{L-k} f(x)dx \\ &= C_{Lk}\int_{F(T)}^{1} y^{k-1}(1-y)^{L-k}dy \\ &= C_{Lk}\int_0^{1-F(T)} (1-y)^{k-1}y^{L-k}dy \end{aligned}$$

where $C_{Lk} = \frac{L!}{(k-1)!(L-k)!}$ and

$$1 - F(T) = 1 - \int_0^T f(x)dx = \int_T^{\infty} f(x)dx = p \tag{135}$$

is the probability of false alarm of the single-block FFT summation CFAR detector for the threshold T, computed by $p = \exp(-T/\sigma^2)$ if $N = 1$ and by

$$\left\{ \begin{array}{lcl} p &=& \sum_{m=1}^{N} A_m \, e^{-\frac{T}{\sigma^2 \mu_m}} \\ A_m &=& \frac{\mu_m^{N-1}}{\prod_{1\le l\le N, l\ne m}(\mu_m - \mu_l)} \end{array} \right. \tag{136}$$

if $N > 1$ (c.f. [4], [10]). Here μ_k, $k = 1, 2, \cdots, N$, are the N distinct positive eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (17). This completes the proof of Theorem 7.

The proof of Theorem 8 is straightforward. It suffices to note that the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ are independent and identically distributed (i.i.d.) and therefore an argument based on the Bernoulli distribution directly yields the proof.

We next move onto the proof of Theorem 9. Since the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$ are i.i.d., the joint probability density function of the two order statistics $y_{L/2}$ and $y_{L/2+1}$, denoted by $g(y,z)$, is given, for $0 < y < z$, by

$$g(y,z) = \frac{L!}{[(L/2-1)!]^2} F(y)^{L/2-1} f(y) f(z) [1 - F(z)]^{L/2-1} \tag{137}$$

Define a 2-dimensional linear transformation by setting:

$$u = (z+y)/2, \ v = (z-y)/2$$

This linear transformation maps the region $0 < y < z$ in the y-z plane into the first quadrant $u > 0$, $v > 0$ in the u-v plane. It follows that the probability density function of the random variable $(y_{L/2}+y_{L/2+1})/2$, denoted by $h(u)$, is given, for $u > 0$, by

$$\begin{aligned} h(u) &= \frac{2(L!)}{[(L/2-1)!]^2} \times \\ &\int_0^\infty [F(u-v)[1-F(u+v)]]^{L/2-1} f(u-v) f(u+v) dv \\ &= \frac{2(L!)}{[(L/2-1)!]^2} \int_0^u [F(v)[1-F(2u-v)]]^{L/2-1} f(v) f(2u-v) dv \end{aligned} \tag{138}$$

where $f(x)$ and $F(x)$ are, respectively, the probability density and cumulative distribution functions of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. Hence, for a given threshold T, the probability of false alarm P_{fa} of the median CFAR detector with detection statistic $(y_{L/2} + y_{L/2+1})/2$ is given by

$$\begin{aligned} P_{fa} &= \frac{2(L!)}{[(L/2-1)!]^2} \times \\ &\int_T^\infty du \int_0^u [F(v)[1-F(2u-v)]]^{L/2-1} f(v) f(2u-v) dv \end{aligned} \tag{139}$$

The identities (24) and (25) follow directly from (21). This completes the proof of Theorem 9. The proof of Theorem 10 is very similar and omitted here.

It remains to prove Theorems 11-12. We shall only give a sketch of the proof of Theorem 11. The proof of Theorem 12 is almost identical. To prove Theorem 11, first it is necessary to compute the probability density function of the statistical median $y_{\lfloor \frac{L}{2} \rfloor + 1}$ of the output power levels $z_{0,n}$, $z_{1,n}$, $\cdots$, $z_{L-1,n}$. For any point $X = (x_1, \cdots, x_L)$

in the L-dimensional Euclidean space R^L, reorder $x_1, \cdots, x_L$ in increasing order to obtain the new sequence $u_1 \leq u_2 \leq \cdots \leq u_k \leq \cdots \leq u_L$ and define the map $\mathcal{T}(x_1, x_2, \cdots, x_L)$ by setting

$$\mathcal{T}(x_1, x_2, \cdots, x_L) = (u_1, u_2, \cdots, u_L) \tag{140}$$

The map $\mathcal{T}$ is not one-to-one from R^L to R^L but it is one-to-one onto on each of the regions $R_{(k_1,k_2,\cdots,k_L)}$

$$R_{(k_1,k_2,\cdots,k_L)} = \left\{(x_1, \cdots, x_L) \mid 0 < x_{k_1} \leq x_{k_2} \leq \cdots \leq x_{k_L}\right\} \tag{141}$$

where $(k_1, k_2, \cdots, k_L)$ is a permutation of the integers $1, 2, \cdots, L$. In fact, for all $(x_1, \cdots, x_L) \in R_{(k_1,k_2,\cdots,k_L)}$, we can write

$$\mathcal{T}(x_1, \cdots, x_L) = (x_{k_1}, x_{k_2}, \cdots, x_{k_L}) \tag{142}$$

and the absolute value of the Jacobian of the map $\mathcal{T}$ on $R_{(k_1,k_2,\cdots,k_L)}$ is equal to 1. It can then be verified that the joint probability density function of the order statistics $y_1, y_2, \cdots, y_L$ of the output power levels $z_{0,n}, z_{1,n}, \cdots, z_{L-1,n}$, denoted by

$\overline{p}(x_1, x_2, \cdots, x_L)$, is given by

$$\overline{p}(x_1, x_2, \cdots, x_L) = \sum_{(k_1,\ k_2,\ \cdots,\ k_L)\ \in\ \mathbf{P_L}} p_0(x_{k_1}, x_{k_2}, \cdots, x_{k_L}), \tag{143}$$
$$0 \leq x_1 \leq x_2 \cdots \leq x_L.$$

In (143), $\mathbf{P_L}$ denotes the set of all the permutations of the L integers, $1, 2, \cdots, L$, and $p_0(x_1, \cdots, x_L)$, defined by:

$$p_0(x_1, \cdots, x_L) = \frac{1}{\sigma^{2L}} p\left(\frac{x_1}{\sigma^2}, \cdots, \frac{x_L}{\sigma^2}\right) \tag{144}$$

is the joint probability density function of the output power levels $z_{0,n}, z_{1,n}, \cdots, z_{L-1,n}$, with $p(x_1, \cdots, x_L)$ defined by (5). It can then be verified that, for a given threshold T, the probability of false alarm P_{fa} of the majority and median CFAR detectors, defined by:

$$P_{fa} = \left\{y_{\lfloor \frac{L}{2} \rfloor + 1} \geq T\right\},$$

is given by

$$P_{fa} =$$
$$\int_T^{+\infty} dx_{\lfloor \frac{L}{2} \rfloor + 1} \int_{x_{\lfloor \frac{L}{2} \rfloor + 1}}^{+\infty} dx_{\lfloor \frac{L}{2} \rfloor + 2} \cdots$$
$$\cdots \int_{x_{L-1}}^{+\infty} dx_L \int_0^{x_{\lfloor \frac{L}{2} \rfloor + 1}} dx_1 \int_{x_1}^{x_{\lfloor \frac{L}{2} \rfloor + 1}} dx_2$$
$$\cdots \int_{x_{\lfloor \frac{L}{2} \rfloor - 1}}^{x_{\lfloor \frac{L}{2} \rfloor + 1}} \overline{p}(x_1, x_2, \cdots, x_L)\ dx_{\lfloor \frac{L}{2} \rfloor}$$

This completes the proof of Theorem 11.

9 Conclusions

This report presents a comprehensive treatment of the FFT filter bank-based majority and median CFAR detectors. One of the most important contributions is the explicit computation of the joint probability density function of the output power levels of the FFT filter bank for overlapped data blocks containing white Gaussian noise with an overlap ratio of $\gamma \leq 1/2$. A very useful corollary is that the statistical correlation among the output power levels of the FFT filter bank for overlapped data blocks containing only white Gaussian noise can be neglected for many common window functions if $\gamma \leq 1/2$. Mathematical details are provided for the main results. The mathematical techniques described in this report should be very useful for similar problems in other areas of statistical signal processing.

References

[1] P. D. Welch, "The Use of Fast Fourier Transform for the Estimation of Power Spectra: A Method Based on Time Averaging Over Short, Modified Periodograms," IEEE Trans. Audio Electroacoustics, vol. AU-15, no. 2, pp. 70-73, June 1967.

[2] R. S. Walker, "The Detection Performance of FFT Processors for Narrowband Signals," DREA Technical Memorandum 82/A, February 1982.

[3] B. H. Maranda, "On the False Alarm Probability for an Overlapped FFT Processor," IEEE Transactions on Aerospace and Electronic Systems, vol. 32, no.4, pp. 1452-1456, October 1996.

[4] F. Patenaude, D. Boudreau and R. Inkol, "CFAR detection based on windowed and polyphase FFT filter banks for channel occupancy measurements," Proceedings of the 19th Biennial Symposium on Communications, Kingston, Ontario, Canada, pp. 339-343, May 1998.

[5] H.C. So, Y. T. Chan, Q. Ma, P. C. Ching, "Comparison of Various Periodograms for Sinusoid Detection and Frequency Estimation," IEEE Transactions on Aerospace and Electronic Systems, vol. 35, no. 3, pp. 945-952, July 1999.

[6] R. Inkol and S. Wang, "A Comparative Study of FFT-Summation and Polyphase-FFT CFAR Detectors," Canadian Conference on Electrical and Computer Engineering, pp. 1175-1178, May 2004.

[7] S. Wang and R. Inkol, "Operating Characteristics of the Wideband FFT Filter Bank J-out-of-L CFAR Detector," DRDC Ottawa TR 2003-235, Defence R&D – Ottawa, December 2003.

[8] S. Wang and R. Inkol, "FFT Filter Bank-Based Majority and Summation CFAR Detectors: A Comparative Study," Canadian Conference on Electrical and Computer Engineering, pp. 1039-1044, May 2004.

[9] S. Wang, R. Inkol and S. Rajan, "A Formula for the Probability of False Alarm for the FFT Filter Bank-Based J-out-of-L CFAR Detector," Proceedings of MWSCAS2005, pp. 728-731, Cincinnati, Ohio, USA, August 7-10, 2005.

[10] S. Wang and R. Inkol, "Theoretical Performance of the FFT Filter Bank Based Summation Detector," DRDC Ottawa TR 2005-153, Defence R&D – Ottawa, December 2005.

[11] S. Wang, F. Patenaude, R. Inkol, "Upper and Lower Bounds for the Threshold of the FFT Filter Bank-Based Summation CFAR Detector," Proceedings of ICASSP 2006, Toulouse, France, May 14-17, 2006.

[12] R. Inkol, S. Wang and S. Rajan, “A Comparison of FFT Filter Bank-Based CFAR Detectors,” Proceedings of the 23rd Biennial Symposium on Communications, Kingston, Ontario, Canada, May 2006.

[13] G. L. Turin, “The Characteristic Function of Hermitian Quadratic Forms in Complex Normal Variables,” Biometrika, vol. 47, pp.199-201, June 1960.

This page intentionally left blank.

DOCUMENT CONTROL DATA

(Security classification of title, body of abstract and indexing annotation must be entered when document is classified)

1. ORIGINATOR (the name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Centre sponsoring a contractor's report, or tasking agency, are entered in section 8.) Defence R&D Canada – Ottawa 3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4	2. SECURITY CLASSIFICATION (overall security classification of the document including special warning terms if applicable). UNCLASSIFIED

3. TITLE (the complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S,C,R or U) in parentheses after the title).

Theory of the FFT Filter Bank-Based Majority and Median CFAR Detectors

4. AUTHORS (last name, first name, middle initial)

Wang, Sichun ; Inkol, Robert ; Rajan, Sreeraman ; Patenaude, François

5. DATE OF PUBLICATION (month and year of publication of document) May 2007	6a. NO. OF PAGES (total containing information. Include Annexes, Appendices, etc). 64	6b. NO. OF REFS (total cited in document) 13

7. DESCRIPTIVE NOTES (the category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered).

Technical Report

8. SPONSORING ACTIVITY (the name of the department project office or laboratory sponsoring the research and development. Include address).

Defence R&D Canada – Ottawa
3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4

9a. PROJECT NO. (the applicable research and development project number under which the document was written. Specify whether project). 12pj11	9b. GRANT OR CONTRACT NO. (if appropriate, the applicable number under which the document was written).
10a. ORIGINATOR'S DOCUMENT NUMBER (the official document number by which the document is identified by the originating activity. This number must be unique.) DRDC Ottawa TR 2007-088	10b. OTHER DOCUMENT NOs. (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

11. DOCUMENT AVAILABILITY (any limitations on further dissemination of the document, other than those imposed by security classification)

(X) Unlimited distribution
(   ) Defence departments and defence contractors; further distribution only as approved
(   ) Defence departments and Canadian defence contractors; further distribution only as approved
(   ) Government departments and agencies; further distribution only as approved
(   ) Defence departments; further distribution only as approved
(   ) Other (please specify):

12. DOCUMENT ANNOUNCEMENT (any limitation to the bibliographic announcement of this document. This will normally correspond to the Document Availability (11). However, where further distribution beyond the audience specified in (11) is possible, a wider announcement audience may be selected).

13. ABSTRACT (a brief and factual summary of the document. It may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall begin with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (S), (C), (R), or (U). It is not necessary to include here abstracts in both official languages unless the text is bilingual).

The FFT filter bank-based majority and median constant false alarm rate (CFAR) detectors are very useful alternatives to the well-known FFT filter bank-based summation CFAR detector. However, for overlapped input data blocks, the theoretical performance analysis of the majority and median CFAR detectors is considerably more difficult than that of the summation CFAR detector, the main reason being that formulas for computing the probability of false alarm for a given threshold can not be easily derived for them. This technical report presents new results on threshold computation for the FFT filter bank-based majority and median CFAR detectors for both overlapped and non-overlapped input data blocks. The most important contribution of this report is the successful computation of the joint probability density function of the output power levels of the FFT filter bank for overlapped white Gaussian noise input blocks. As a corollary, it is shown that the statistical correlation of the output power levels of the FFT filter bank can be neglected for several commonly used windows, such as the Blackman and Blackman-Harris windows. Complete mathematical derivations are provided for all the major theorems presented in the report.

14. KEYWORDS, DESCRIPTORS or IDENTIFIERS (technically meaningful terms or short phrases that characterize a document and could be helpful in cataloguing the document. They should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location may also be included. If possible keywords should be selected from a published thesaurus. e.g. Thesaurus of Engineering and Scientific Terms (TEST) and that thesaurus-identified. If it not possible to select indexing terms which are Unclassified, the classification of each should be indicated as with the title).

Fast Fourier Transform, probability of detection, probability of false alarm, constant probability of false alarm (CFAR), majority CFAR detector, median CFAR detector, detection threshold

Defence R&D Canada

Canada's leader in Defence
and National Security
Science and Technology

R & D pour la défense Canada

Chef de file au Canada en matière
de science et de technologie pour
la défense et la sécurité nationale

DEFENCE R&D DÉFENSE

www.drdc-rddc.gc.ca