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Theoretical Performance of the FFT Filter Bank-Based Summation Detector

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Abstract

This technical report derives the probabilities of detection and false alarm for the FFT filter bank-based summation detector when the received signal is a complex pure tone embedded in additive white Gaussian noise. These results are useful for the performance analysis of the FFT filter bank-based summation detector and can be used to set up the detector for operation at a desired constant false alarm rate and predict the detection performance.

Résumé

Le présent rapport technique présente une dérivation des probabilités de détection et de fausses alarmes pour le détecteur de sommation fondé sur les bancs de filtres à TFR lorsque le signal reçu est un son pur complexe intégré à du bruit blanc gaussien additif. Les résultats obtenus sont utiles pour l'analyse de la performance d'un détecteur de sommation fondé sur les bancs de filtres à TFR et peuvent être utilisés pour régler le fonctionnement du détecteur au taux de fausses alarmes constant voulu et pour prédire la performance de détection.

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Executive summary

Theoretical Performance of the FFT Filter Bank-Based Summation Detector

Sichun Wang, Robert Inkol; DRDC Ottawa TR 2005-153; Defence R&D Canada – Ottawa; December 2005.

Channelized receivers are useful for performing the fast detection and center frequency estimation of narrowband signals. The application of FFT filter bank techniques has attracted considerable interest as a result of the computational efficiency of the FFT algorithm and the availability of function specific FFT hardware [1]-[8]. One of the practical limitations of FFT filter bank detectors is that the restriction of the FFT length to powers of two for common implementations of the algorithm limits the choice of the channel spacing, particularly when the sampling rate of the analog-to-digital converters, used to convert an analog signal from a sensor to a digital signal for subsequent processing, is fixed for technical reasons.

The FFT filter bank-based summation detector is a modification of the basic FFT filter bank detector. Whereas the basic FFT filter bank detector performs detection by comparing the power computed for each FFT bin to a detection threshold, the FFT summation detector groups the FFT bins to correspond to the desired channelization and forms an estimate of the power contained in each channel by summing the power computed for the individual bins. If the FFT length is sufficiently large, an arbitrary channelization scheme can be approximated if the number of FFT bins assigned to each channel is allowed to vary. A further idea is that the bandwidth of each detector can be adjusted by summing the power for a subset of the bins assigned to each channel. In many practical applications, the enhanced flexibility in defining channel bandwidths and center frequencies is very useful.

Although the performance of the FFT summation detector has been discussed in previous reports, a complete derivation of the performance analysis has not been previously published. This report presents a thorough treatment of the theoretical derivation of the formulas for probabilities of detection and false alarm. These results are very useful for analyzing the performance bounds of practical sensor systems and are extensible to similar problems.

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Sommaire

Theoretical Performance of the FFT Filter Bank-Based Summation Detector

Sichun Wang, Robert Inkol; DRDC Ottawa TR 2005-153; R & D pour la défense Canada – Ottawa; décembre 2005.

Les récepteurs multicanaux sont utiles pour détecter rapidement et estimer la fréquence centrale des signaux en bande étroite. L'application de techniques fondées sur les bancs de filtres à TFR (transformée de Fourier rapide) a suscité un intérêt considérable en raison de l'efficacité de calcul de l'algorithme TFR et de la disponibilité du matériel TFR à fonction spécifique [1]-[7]. Une des limites concrétes des détecteurs à bancs de filtres TFR est qu'en raison de la longueur restreinte de la TFR à des puissances de deux pour les applications communes de l'algorithme, le choix en matière d'espacement entre canaux est limité, particulièrement lorsque le taux d'échantillonnage des convertisseurs analogiques/numériques, qui servent à convertir un signal analogique provenant d'un capteur en un signal numérique pour fins de traitement ultérieur, est fixe pour des raisons d'ordre technique.

Le détecteur de sommation fondé sur les bancs de filtres à TFR est un détecteur de bancs de filtres à TFR de base auquel des modifications ont été apportées. Le détecteur de bancs de filtres à TFR de base remplit sa fonction en comparant la puissance calculée pour chaque intervalle TFR à un seuil de détection, tandis que le détecteur de sommation TFR regroupe les intervalles TFR pour qu'ils correspondent au découpage en canaux voulu et donne une estimation de la puissance de chaque canal en faisant la somme de la puissance calculée pour chacun des intervalles individuels. Si la longueur de la TFR est suffisante, on peut estimer un plan de découpage en canaux arbitraire lorsque le nombre d'intervalles TFR assignés à chaque canal peut varier. Une autre possibilité consisterait à régler la largeur de bande de chaque détecteur en totalisant la puissance d'un sous-ensemble d'intervalles assignés à chaque canal. Dans beaucoup d'applications pratiques, la souplesse accrue en matière de définition de largeur de bande des canaux et de fréquences centrales est très utile.

Bien que la performance du détecteur de sommation TFR ait fait l'objet d'une discussion dans des rapports antérieurs, la dérivation complète de l'analyse de performance n'avait pas été publiée. Le présent rapport traite en profondeur de la dérivation théorique des formules utilisées pour calculer les probabilités de détection et les fausses alarmes. Les résultats obtenus sont très utiles pour analyser les limites de performance des systèmes de détection pratiques et peuvent être appliqués à la résolution de problèmes similaires.

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1 Introduction

Channelized receivers are useful for performing the fast detection and center frequency estimation of narrowband signals. The application of FFT filter bank techniques has attracted considerable interest as a result of the computational efficiency of the FFT algorithm and the availability of function specific FFT hardware [1]-[8]. One of the practical limitations of FFT filter bank detectors is that the restriction of the FFT length to powers of two for common implementations of the algorithm limits the choice of the channel spacing, particularly when for technical reasons the sampling rate of the analog-to-digital converters, used to convert an analog signal from a sensor to a digital signal for subsequent processing, is fixed.

The FFT filter bank-based summation detector is a modification of the basic FFT filter bank detector. Whereas the basic FFT filter bank detector performs detection by comparing the power computed for each FFT bin to a detection threshold, the FFT summation detector groups the FFT bins to correspond to the desired channelization and forms an estimate of the power contained in each channel by summing the power computed for the individual bins. If the FFT length is sufficiently large, an arbitrary channelization scheme can be approximated if the number of FFT bins assigned to each channel is allowed to vary. A further idea is that the bandwidth of each detector can be adjusted by summing the power for a subset of the FFT bins assigned to each channel. In many practical applications, the enhanced flexibility in defining channel bandwidths and center frequencies is very useful.

In [11], [12], the theoretical performance of the FFT majority and summation detectors designed to provide a constant false-alarm rate (CFAR) was compared. It was found that the FFT majority detector performed within 1 dB of the performance of the FFT summation detector for CFAR operation. This conclusion was derived from closed-form algebraic formulas for the probability of false alarm P_{fa} and the probability of detection P_d for the FFT summation detector, with the received signal being a pure tone embedded in additive white Gaussian noise.

Due to space limitations, the full derivation for P_{fa} and P_d was omitted in [11], [12]. In this report, we present the full derivation for P_{fa} and P_d. The formulas for P_d and P_{fa} presented in this report provide the basis for comparing the theoretical performance of the FFT summation detector and other closely related detectors. It is noted that the formulas for P_d and some of the formulas for P_{fa} presented here have not been derived elsewhere and therefore our results extend and complement the literature on FFT filter bank-based CFAR detectors.

This report is organized as follows. In section 2, notation and definitions used in this report are introduced. In section 3, a Hermitian covariance matrix is computed, the distinct eigenvalues of which play a central role in computing P_{fa} and P_d for the

FFT summation detector. In sections 4, 5 and 6, formulas for P_d are obtained for overlapping and non-overlapping data blocks. In sections 7, 8 and 9, formulas for P_{fa} are derived as corollaries from the corresponding formulas for P_d obtained in sections 4, 5 and 6. Finally, in section 10, results obtained in this report are summarized. For readers not interested in the mathematical details, they can go directly to section 10 to locate the relevant formulas they are interested in.

2 The FFT Filter Bank-Based Summation Detector

Various aspects of the FFT filter bank-based summation detector have already been investigated (see [3], [9]-[15], and the references therein). To be consistent with the report [12], we shall use similar notation and terminology. In this section, we briefly describe how the power level for a channel in the FFT filter bank-based summation detector is computed. The reader is referred to the references at the end of this report for additional details ([3], [9]-[15]).

Assume there are M channels in the sampled bandwidth, with the channels being uniformly distributed in frequency. Assume an integer number of K FFT bins are assigned per channel. Out of the K FFT bins per channel, N center bins are used to estimate the power. Hence a FFT of length $M \times K$ is needed to compute the power for the M channels. In this report, without loss of generality, it is assumed that the integers K and N are both a power of 2 and $K - N$ is an even integer. Let $\mathbf{w} = [w_0, \cdots, w_{MK-1}]^t$ be a linear phase FIR filter of length MK, where the superscript t denotes matrix or vector transposition. Dividing the input vector $\mathbf{r} = [r_{(L-1)(1-\gamma)MK+MK-1}, \cdots, r_1, r_0]^t$ into L overlapping sample vectors $\mathbf{R}_0, \cdots, \mathbf{R}_{L-1}$ as follows :

$$\mathbf{R}_m = [r_{m(1-\gamma)MK+MK-1},\ r_{m(1-\gamma)MK+MK-2},\ \cdots,\ r_{m(1-\gamma)MK}]^t \tag{1}$$

where each vector $\mathbf{R}_m$ has γMK samples in common with the preceding vector $\mathbf{R}_{m-1}$ and $0 \leq \gamma < 1$ is the overlapping ratio. The vectors $\mathbf{R}_m$ are windowed by the windowing sequence $\mathbf{w}$, resulting in the windowed sample vectors $\mathbf{X}_m$:

$$\mathbf{X}_m \;\; = \;\; [w_0 r_{m(1-\gamma)MK+MK-1},\ w_1 r_{m(1-\gamma)MK+MK-2},\ \cdots,\ w_{MK-1} r_{m(1-\gamma)MK}]^t \tag{2}$$

The vectors $\mathbf{X}_m$ are then transformed by the inverse discrete Fourier transform matrix $\mathbf{F}$ of dimensions $MK \times MK$ to yield a corresponding sequence of FFT sample vectors $\mathbf{Y}_m$:

$$\mathbf{Y}_m \;\; = \;\; \mathbf{F}\mathbf{X}_m = [y_{0,m}, y_{1,m}, \cdots, y_{MK-1,m}]^t \tag{3}$$

where

$$
\mathbf{F} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi jl}{MK} & \cdots & \exp\frac{2\pi j(MK-1)l}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi j(MK-1)}{MK} & \cdots & \exp\frac{2\pi j(MK-1)(MK-1)}{MK} \end{bmatrix} \tag{4}
$$

and $j = \sqrt{-1}$. We remark here that even though it is the inverse FFT matrix $\mathbf{F}$ that is used in the FFT filter bank, we simply call the vectors $\mathbf{Y}_m$ FFT sample vectors and a sample in $\mathbf{Y}_m$ a FFT bin. Corresponding to each sample vector $\mathbf{Y}_m$, a decision vector $\mathbf{z}_m = [z_{0,m}, \cdots, z_{M-1,m}]^t$ of dimension M is formed by summing the power from the N center FFT bins in each channel:

$$
z_{n,m} = \sum_{l=0}^{N-1} |y_{nK+\frac{K-N}{2}+l,m}|^2, \quad n = 0, 1, 2, \cdots, M-1 \ . \tag{5}
$$

That is, the power from the N FFT bins with indices $nK + \frac{K-N}{2} + 0$, $nK + \frac{K-N}{2} + 1$, $\cdots$, $nK + \frac{K-N}{2} + (N-1)$ is summed to form the power for the n-th channel which is assigned the K FFT bins with indices $nK + 0$, $nK + 1$, $\cdots$, $nK + K - 1$. The final decision vector $\mathbf{z} = [z_0, z_1, \cdots, z_{M-1}]^t$ is obtained by summing the individual decision vectors $\mathbf{z}_m$:

$$
\mathbf{z} = \sum_{m=0}^{L-1} \mathbf{z}_m \tag{6}
$$

We have

$$
z_n = \sum_{m=0}^{L-1} z_{n,m} = \sum_{m=0}^{L-1} \sum_{l=0}^{N-1} |y_{nK+\frac{K-N}{2}+l,m}|^2 \ , \quad n = 0, 1, 2, \cdots, M-1 \ . \tag{7}
$$

Given a probability of false alarm P_{fa}, a threshold T is first computed such that $\Pr\{z_n > T\} = P_{fa}$, where the power level z_n for the n-th channel is computed from the input data record $\mathbf{r} = [r_{(L-1)(1-\gamma)MK+MK-1}, \cdots, r_1, r_0]^t$ that is assumed to contain additive white Gaussian noise (AWGN) only. The FFT filter bank-based summation detector declares the presence of a signal in the n-th channel if $z_n > T$ and declares the absence of a signal in the n-th channel if $z_n \leq T$. In the rest of this report, for brevity, this detector will simply be called the L-block FFT summation detector or simply the FFT summation detector.

The signal power z_n computed for the n-th channel (c.f. (7)) is a quadratic form in the input random variables. In fact,

$$
\begin{aligned} z_{n,m} &= \sum_{l=0}^{N-1} |y_{nK+\frac{K-N}{2}+l,m}|^2 = \sum_{l=0}^{N-1} (y_{nK+\frac{K-N}{2}+l,m})(y_{nK+\frac{K-N}{2}+l,m})^* \\ &= \mathbf{Z}_{n,m}^H \mathbf{Z}_{n,m} \end{aligned} \tag{8}
$$

where

$$\begin{aligned}\mathbf{Z}_{n,m} &= [y_{nK+\frac{K-N}{2},m},\ y_{nK+\frac{K-N}{2}+1,m},\ \cdots,\ y_{nK+\frac{K-N}{2}+N-1,m}]^t \\ &= \mathbf{F}_n\mathbf{W}\mathbf{R}_m \end{aligned} \tag{9}$$

In (9), $\mathbf{R}_m$ is the $(m+1)$-th sample vector defined by (1), $\mathbf{W}$ is the $MK \times MK$ diagonal matrix with its diagonal elements defined by the windowing sequence $\mathbf{w}$:

$$\mathbf{W} = \begin{bmatrix} w_0 & 0 & \cdots & \cdots & 0 \\ 0 & w_1 & \cdots & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & w_{MK-1} \end{bmatrix} \tag{10}$$

and $\mathbf{F}_n$ is the $N \times MK$ matrix consisting of the N rows of the $MK \times MK$ inverse discrete Fourier transform matrix $\mathbf{F}$ with row indices $nK + \frac{K-N}{2}$, $nK + \frac{K-N}{2} + 1$, $\cdots$, $nK + \frac{K-N}{2} + N - 1$:

$$\mathbf{F}_n = \begin{bmatrix} 1 & \exp\frac{2\pi j I_n}{MK} & \cdots & \exp\frac{2\pi j (MK-1) I_n}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi j (I_n+l)}{MK} & \cdots & \exp\frac{2\pi j (MK-1)(I_n+l)}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi j (I_n+N-1)}{MK} & \cdots & \exp\frac{2\pi j (MK-1)(I_n+N-1)}{MK} \end{bmatrix} \tag{11}$$

where $I_n = nK + \frac{K-N}{2}$. In matrix form, the signal power z_n for the n-th channel is a quadratic form defined by:

$$\begin{aligned} z_n &= \sum_{m=0}^{L-1} z_{n,m} = \sum_{m=0}^{L-1} \mathbf{Z}_{n,m}^H \mathbf{Z}_{n,m} = \begin{bmatrix} \mathbf{Z}_{n,0} \\ \mathbf{Z}_{n,1} \\ \cdots \\ \mathbf{Z}_{n,m} \\ \cdots \\ \mathbf{Z}_{n,L-1} \end{bmatrix}^H \begin{bmatrix} \mathbf{Z}_{n,0} \\ \mathbf{Z}_{n,1} \\ \cdots \\ \mathbf{Z}_{n,m} \\ \cdots \\ \mathbf{Z}_{n,L-1} \end{bmatrix} \\ &= \sum_{m=0}^{L-1} \mathbf{R}_m^H \mathbf{W} \mathbf{F}_n^H \mathbf{F}_n \mathbf{W} \mathbf{R}_m \end{aligned} \tag{12}$$

3 Eigenvalues of the Covariance Matrix $\mathbf{H} = E\left(\mathbf{Z}_n\mathbf{Z}_n^H\right)$

Assume the input sample vectors $\mathbf{R}_m$ (c.f.(1)) contain zero-mean additive white Gaussian noise only; specifically, assume the input samples $r_0, r_1, \cdots, r_k, \cdots$, are independent, identically distributed, zero-mean Gaussian random variables with $E(|r_k|^2) =$

σ^2. For $0 \le n \le M-1$, define the Gaussian random vector $\mathbf{Z}_n$ by

$$\mathbf{Z}_n = \begin{bmatrix} \mathbf{Z}_{n,0} \\ \mathbf{Z}_{n,1} \\ \cdots \\ \mathbf{Z}_{n,m} \\ \cdots \\ \mathbf{Z}_{n,L-1} \end{bmatrix} \tag{13}$$

where $\mathbf{Z}_{n,m}$ is defined by (9). As will be shown in the following sections, the eigenvalues of the covariance matrix $\mathbf{H} = E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right)$ play an important role in the computation of the probability of false alarm P_{fa} and the probability of detection P_d for the FFT summation detector. Here in this section, we compute the matrix $\mathbf{H}$ for the practically important case $\gamma \le 1/2$ and then investigate its eigenvalue distribution. We shall see that the eigenvalues of $\mathbf{H}$ roughly cluster into N groups and each group has L eigenvalues which are approximately evenly spaced. This property of eigenvalues of $\mathbf{H}$ is very useful in the computation of threshold T for the FFT summation detector.

The covariance matrix $\mathbf{H}$ is given by

$$\begin{aligned} \mathbf{H} &= E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right) \qquad (14) \\ &= E\left(\begin{bmatrix} \mathbf{Z}_{n,0} \\ \mathbf{Z}_{n,1} \\ \cdots \\ \mathbf{Z}_{n,m} \\ \cdots \\ \mathbf{Z}_{n,L-1} \end{bmatrix} \left[\mathbf{Z}_{n,0}^H,\ \mathbf{Z}_{n,1}^H,\ \cdots,\ \mathbf{Z}_{n,m}^H,\ \cdots,\ \mathbf{Z}_{n,L-1}^H\right] \right) \\ &= \begin{bmatrix} E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,0}^H\right) & E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,1}^H\right) & \cdots & \cdots & E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,L-1}^H\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ E\left(\mathbf{Z}_{n,m}\mathbf{Z}_{n,0}^H\right) & E\left(\mathbf{Z}_{n,m}\mathbf{Z}_{n,1}^H\right) & \cdots & \cdots & E\left(\mathbf{Z}_{n,m}\mathbf{Z}_{n,L-1}^H\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ E\left(\mathbf{Z}_{n,L-1}\mathbf{Z}_{n,0}^H\right) & E\left(\mathbf{Z}_{n,L-1}\mathbf{Z}_{n,1}^H\right) & \cdots & \cdots & E\left(\mathbf{Z}_{n,L-1}\mathbf{Z}_{n,L-1}^H\right) \end{bmatrix} \end{aligned}$$

For any $l,\ m,\ 0 \le l \le m \le L-1$,

$$\begin{aligned} E\left(\mathbf{Z}_{n,l}\mathbf{Z}_{n,m}^H\right) &= E\left(\left(\mathbf{F}_n\mathbf{W}\mathbf{R}_l\right)\left(\mathbf{F}_n\mathbf{W}\mathbf{R}_m\right)^H\right) \\ &= E\left(\mathbf{F}_n\mathbf{W}\mathbf{R}_l\mathbf{R}_m^H\mathbf{W}^H\mathbf{F}_n^H\right) = \mathbf{F}_n\mathbf{W}E\left(\mathbf{R}_l\mathbf{R}_m^H\right)\mathbf{W}\mathbf{F}_n^H \end{aligned} \tag{15}$$

where

$$E\left(\mathbf{R}_l\mathbf{R}_m^H\right) = \tag{16}$$

$$
\sigma^2 \begin{bmatrix} \delta_{(m-l)(1-\gamma)MK} & \cdots & \delta_{(m-l)(1-\gamma)MK-MK+1} \\ \cdots & \cdots & \cdots \\ \delta_{(m-l)(1-\gamma)MK+p-1} & \cdots & \delta_{(m-l)(1-\gamma)MK-MK+p} \\ \cdots & \cdots & \cdots \\ \delta_{(m-l)(1-\gamma)MK+MK-1} & \cdots & \delta_{(m-l)(1-\gamma)MK} \end{bmatrix}
$$

In (16), $\delta_k = 0$, if $k \neq 0$ and $\delta_k = 1$ if $k = 0$. Since $\gamma \leq \frac{1}{2}$, it follows that if $|m-l| \geq 2$, then

$$
|(m-l)(1-\gamma)MK| \geq 2(1-\gamma)MK \geq MK
$$

Therefore, (16) and (15) imply that if $|m-l| \geq 2$, then $E\left(\mathbf{R}_l \mathbf{R}_m^H\right) = \mathbf{0}$, $E\left(\mathbf{Z}_{n,l}\mathbf{Z}_{n,m}^H\right) = \mathbf{0}$ and the covariance matrix $\mathbf{H} = E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right)$ in (14) becomes the following tridiagonal $L \times L$ block matrix

$$
\mathbf{H} = E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right) = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}^H & \mathbf{A} & \mathbf{B} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{B}^H & \mathbf{A} \end{bmatrix} \tag{17}
$$

where $\mathbf{A} = E\left(\mathbf{Z}_{n,0}\ \mathbf{Z}_{n,0}^H\right)$ and $\mathbf{B} = E\left(\mathbf{Z}_{n,0}\ \mathbf{Z}_{n,1}^H\right)$ are $N \times N$ matrices.

We now consider the two separate cases $\gamma = 0$ and $0 < \gamma \leq \frac{1}{2}$.

1. $\gamma = 0$. In this case, $\mathbf{B} = 0$ and the covariance matrix of $\mathbf{Z}_n$ is the diagonal $L \times L$ block matrix

$$
\mathbf{H} = E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right) = \begin{bmatrix} \mathbf{A} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A} & \mathbf{0} & \cdots & \mathbf{0} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{A} \end{bmatrix} \tag{18}
$$

where

$$
\mathbf{A} \;=\; E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,0}^H\right) = \mathbf{F}_n \mathbf{W} E\left(\mathbf{R}_0 \mathbf{R}_0^H\right) \mathbf{W}\mathbf{F}_n^H = \sigma^2 \mathbf{F}_n \mathbf{W}^2 \mathbf{F}_n^H \tag{19}
$$

With $I_n = nK + \frac{K-N}{2}$, we have

$$
\mathbf{F}_n \mathbf{W}^2 =
$$

$$
\begin{bmatrix} w_0^2 & w_1^2 \exp\frac{2\pi j(I_n+0)}{MK} & \cdots & w_{MK-1}^2 \exp\frac{2\pi j(MK-1)(I_n+0)}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2 \exp\frac{2\pi j(I_n+l)}{MK} & \cdots & w_{MK-1}^2 \exp\frac{2\pi j(MK-1)(I_n+l)}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2 \exp\frac{2\pi j(I_n+N-1)}{MK} & \cdots & w_{MK-1}^2 \exp\frac{2\pi j(MK-1)(I_n+N-1)}{MK} \end{bmatrix} \tag{20}
$$

and

$$\mathbf{A} = \sigma^2 \mathbf{F}_n \mathbf{W}^2 \mathbf{F}_n^H = \sigma^2 \times \tag{21}$$

$$\begin{bmatrix} w_0^2 & w_1^2 \exp \frac{2\pi j I_n}{MK} & \cdots & w_{MK-1}^2 \exp \frac{2\pi j (MK-1) I_n}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2 \exp \frac{2\pi j (I_n + l)}{MK} & \cdots & w_{MK-1}^2 \exp \frac{2\pi j (MK-1)(I_n + l)}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ w_0^2 & w_1^2 \exp \frac{2\pi j (I_n + N - 1)}{MK} & \cdots & w_{MK-1}^2 \exp \frac{2\pi j (MK-1)(I_n + N - 1)}{MK} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \cdots & 1 \\ \cdots & \cdots & \cdots \\ \exp \frac{-2\pi j l I_n}{MK} & \cdots & \exp \frac{-2\pi j l (I_n + N - 1)}{MK} \\ \cdots & \cdots & \cdots \\ \exp \frac{-2\pi j (MK-1) I_n}{MK} & \cdots & \exp \frac{-2\pi j (MK-1)(I_n + N - 1)}{MK} \end{bmatrix}$$

$$= \sigma^2 \begin{bmatrix} \tau_{1,1} & \tau_{1,2} & \cdots & \tau_{1,q} & \cdots & \tau_{1,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{p,1} & \tau_{p,2} & \cdots & \tau_{p,q} & \cdots & \tau_{p,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \tau_{N,1} & \tau_{N,2} & \cdots & \tau_{N,q} & \cdots & \tau_{N,N} \end{bmatrix}$$

where, for $1 \leq p \leq q \leq N$,

$$\tau_{p,q} = \sum_{l=0}^{MK-1} w_l^2 \exp \frac{2\pi j l (p - q)}{MK} \tag{22}$$

2. $0 < \gamma \leq \frac{1}{2}$. In this case $\mathbf{A}$ is computed by (21) and it remains to compute $\mathbf{B}$. In fact,

$$E\left(\mathbf{R}_0 \mathbf{R}_1^H\right) = \sigma^2 \begin{bmatrix} \mathbf{0}_{\gamma MK \times (1-\gamma) MK} & \mathbf{I}_{\gamma MK \times \gamma MK} \\ \mathbf{0}_{(1-\gamma) MK \times (1-\gamma) MK} & \mathbf{0}_{(1-\gamma) MK \times \gamma MK} \end{bmatrix} \tag{23}$$

where $\mathbf{0}_{\gamma MK \times (1-\gamma) MK}$, $\mathbf{0}_{(1-\gamma) MK \times (1-\gamma) MK}$ and $\mathbf{0}_{(1-\gamma) MK \times \gamma MK}$ are zero matrices of dimensions $\gamma MK \times (1-\gamma) MK$, $(1-\gamma) MK \times (1-\gamma) MK$ and $(1-\gamma) MK \times \gamma MK$ respectively and $\mathbf{I}_{\gamma MK \times \gamma MK}$ is the identity matrix of dimensions $\gamma MK \times \gamma MK$. Let

$$\mathbf{W}_1 = \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_{\gamma MK-1} \end{bmatrix}, \mathbf{W}_2 = \begin{bmatrix} w_0 & 0 & \cdots & 0 \\ 0 & w_1 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & w_{(1-\gamma) MK-1} \end{bmatrix} \tag{24}$$

$$\mathbf{W}_3 = \begin{bmatrix} w_{\gamma MK} & 0 & \cdots & 0 & 0 \\ 0 & w_{\gamma MK+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & w_{MK-1} \end{bmatrix} \quad (25)$$

$$\mathbf{W}_4 = \begin{bmatrix} w_{(1-\gamma)MK} & 0 & \cdots & 0 & 0 \\ 0 & w_{(1-\gamma)MK+1} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & w_{MK-1} \end{bmatrix} \quad (26)$$

We have

$$\begin{aligned} & \mathbf{B} = E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,1}^H\right) = \mathbf{F}_n \mathbf{W} E\left(\mathbf{R}_0\mathbf{R}_1^H\right)\mathbf{W}\mathbf{F}_n^H \quad (27) \\ = \ & \sigma^2 \mathbf{F}_n \begin{bmatrix} \mathbf{W}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0}_{\gamma MK \times (1-\gamma)MK} & \mathbf{I}_{\gamma MK \times \gamma MK} \\ \mathbf{0}_{(1-\gamma)MK \times (1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK \times \gamma MK} \end{bmatrix} \times \\ & \begin{bmatrix} \mathbf{W}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_4 \end{bmatrix} \mathbf{F}_n^H \\ = \ & \sigma^2 \mathbf{F}_n \begin{bmatrix} \mathbf{0}_{\gamma MK \times (1-\gamma)MK} & \mathbf{W}_1 \\ \mathbf{0}_{(1-\gamma)MK \times (1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK \times \gamma MK} \end{bmatrix} \begin{bmatrix} \mathbf{W}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_4 \end{bmatrix} \mathbf{F}_n^H \\ = \ & \sigma^2 \mathbf{F}_n \begin{bmatrix} \mathbf{0}_{\gamma MK \times (1-\gamma)MK} & \mathbf{W}_1\mathbf{W}_4 \\ \mathbf{0}_{(1-\gamma)MK \times (1-\gamma)MK} & \mathbf{0}_{(1-\gamma)MK \times \gamma MK} \end{bmatrix} \mathbf{F}_n^H \end{aligned}$$

Let

$$\mathbf{F}_n^0 = \begin{bmatrix} 1 & \exp\frac{2\pi j I_n}{MK} & \cdots & \exp\frac{2\pi j(\gamma MK-1)I_n}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi j(I_n+p-1)}{MK} & \cdots & \exp\frac{2\pi j(\gamma MK-1)(I_n+p-1)}{MK} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \exp\frac{2\pi j(I_n+N-1)}{MK} & \cdots & \exp\frac{2\pi j(\gamma MK-1)(I_n+N-1)}{MK} \end{bmatrix} \quad (28)$$

and

$$\mathbf{F}_n^1 = \begin{bmatrix} \exp\frac{2\pi j(1-\gamma)MKI_n}{MK} & \cdots & \exp\frac{2\pi j(MK-1)I_n}{MK} \\ \cdots & \cdots & \cdots \\ \exp\frac{2\pi j(1-\gamma)MK(I_n+p-1)}{MK} & \cdots & \exp\frac{2\pi j(MK-1)(I_n+p-1)}{MK} \\ \cdots & \cdots & \cdots \\ \exp\frac{2\pi j(1-\gamma)MK(I_n+N-1)}{MK} & \cdots & \exp\frac{2\pi j(MK-1)(I_n+N-1)}{MK} \end{bmatrix} \quad (29)$$

(27) can be simplified to yield:

$$\begin{aligned} & \mathbf{B} = E\left(\mathbf{Z}_{n,0}\mathbf{Z}_{n,1}^H\right) = \sigma^2 \mathbf{F}_n \begin{bmatrix} \mathbf{0} & \mathbf{W}_1\mathbf{W}_4 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{F}_n^H \quad (30) \\ = \ & \sigma^2 \ \mathbf{F}_n^0 \ \mathbf{W}_1\mathbf{W}_4 \ \left(\mathbf{F}_n^1\right)^H = \sigma^2 \mathbf{C} \end{aligned}$$

where

$$\mathbf{C} = \mathbf{F}_n^0 \, \mathbf{W}_1 \mathbf{W}_4 \, \left(\mathbf{F}_n^1\right)^H = \begin{bmatrix} \gamma_{1,1} & \gamma_{1,2} & \cdots & \gamma_{1,q} & \cdots & \gamma_{1,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{p,1} & \gamma_{p,2} & \cdots & \gamma_{p,q} & \cdots & \gamma_{p,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \gamma_{N,1} & \gamma_{N,2} & \cdots & \gamma_{N,q} & \cdots & \gamma_{N,N} \end{bmatrix} \tag{31}$$

with

$$\begin{aligned} \gamma_{p,q} &= \sum_{l=0}^{\gamma MK-1} \exp \frac{2\pi j l(I_n + p - 1)}{MK} \times \\ & \quad w_l w_{l+(1-\gamma)MK} \exp \frac{-2\pi j (l + (1-\gamma)MK)(I_n + q - 1)}{MK} \\ &= e^{-2\pi j (1-\gamma)(I_n + q - 1)} \sum_{l=0}^{\gamma MK-1} w_l w_{l+(1-\gamma)MK} \exp \frac{2\pi j l (p-q)}{MK} \end{aligned} \tag{32}$$

To gain insights into the distribution of eigenvalues of the covariance matrix $E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right)$, we first compute the characteristic polynomial $P_L(\lambda) = \det\left(E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right) - \lambda \mathbf{I}\right)$, where $\mathbf{I}$ denotes the identity matrix of dimensions $LN \times LN$. The characteristic polynomial $P_L(\lambda)$ can be expressed in terms of the two $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$. We consider the two separate cases $\gamma = 0$ and $0 < \gamma \leq \frac{1}{2}$.

1. $\gamma = 0$. In this case the covariance matrix $\mathbf{H} = E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right)$ is given by (18) and the characteristic polynomial $P_L(\lambda)$ is computed by

$$P_L(\lambda) = \det\left(E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right) - \lambda \mathbf{I}\right) = \left(\det(\mathbf{A} - \lambda \mathbf{I})\right)^L \tag{33}$$

   Obviously, each eigenvalue of $\mathbf{H}$ has a multiplicity of L.

2. $0 < \gamma \leq \frac{1}{2}$. In this case we assume that $\mathbf{B}$ is an invertible matrix of dimensions $N \times N$ (c.f. (30)). As an illustration, the characteristic polynomial $P_L(\lambda)$ is computed here for the two cases $L = 2$ and $L = 3$. If $L = 2$, it follows from (17) that the covariance matrix $E\left(\mathbf{Z}_n \mathbf{Z}_n^H\right)$ is equal to

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{A} \end{bmatrix}$$

   and hence

$$P_2(\lambda) = \det\left(\begin{bmatrix} \mathbf{A} - \lambda \mathbf{I} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{A} - \lambda \mathbf{I} \end{bmatrix}\right) \tag{34}$$

   Let

$$\mathbf{X} = -\left(\mathbf{A} - \lambda \mathbf{I}\right)\left(\mathbf{B}^H\right)^{-1} \tag{35}$$

It follows that

$$
\begin{aligned}
& P_2(\lambda) = \qquad (36) \\
& \det\left(\begin{bmatrix} \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{I} & \mathbf{X} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
= \; & \det\left(\begin{bmatrix} \mathbf{0} & \mathbf{B}-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
= \; & (-1)^N \det\left(\begin{bmatrix} \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \\ \mathbf{0} & \mathbf{B}-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \end{bmatrix}\right) \\
= \; & (-1)^N \det\left(\mathbf{B}^H\right)\det\left(\mathbf{B}-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I})\right) \\
= \; & (-1)^N \det\left(\mathbf{B}^H\right) \times \\
& \det\left[\mathbf{B}\mathbf{B}^H\left(\mathbf{B}^H\right)^{-1} - (\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\left(\mathbf{B}^H\right)^{-1}\right] \\
= \; & (-1)^N \det\left(\mathbf{B}^H\right) \times \\
& \det\left[\mathbf{B}\mathbf{B}^H - (\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\right]\det\left[\left(\mathbf{B}^H\right)^{-1}\right] \\
= \; & (-1)^N \det\left[\mathbf{B}\mathbf{B}^H - (\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\right] \\
= \; & \det\left[(\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right) - \mathbf{B}\mathbf{B}^H\right]
\end{aligned}
$$

If the entries of $\mathbf{B}$ are relatively small , it should be reasonable to assume that $\mathbf{B}\mathbf{B}^H \approx \mathbf{0}$ and therefore

$$
\begin{aligned}
P_2(\lambda) &= \det\left((\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right) - \mathbf{B}\mathbf{B}^H\right) \\
&\approx \det\left((\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\right) \\
&= \det(\mathbf{A}-\lambda\mathbf{I})^2
\end{aligned}
$$

Hence we can expect the eigenvalues of $\mathbf{H}$ to be close to the roots of $\det(\mathbf{A}-\lambda\mathbf{I})^2 = 0$. In other words, the eigenvalues of $\mathbf{H}$ should cluster into N groups with each group consisting of $L = 2$ closely spaced eigenvalues.

Similarly, if $L = 3$ and let $\mathbf{X}$ be defined by (35), then

$$
P_3(\lambda) = \det\left(\begin{bmatrix} \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \qquad (37)
$$

$$
\begin{aligned}
= \; & \det\left(\begin{bmatrix} \mathbf{I} & \mathbf{X} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} & \mathbf{0} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
= \; & \det\left(\begin{bmatrix} \mathbf{0} & \mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I}) & \mathbf{X}\mathbf{B} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= (-1)^N \det\left(\begin{bmatrix} \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} & \mathbf{B} \\ \mathbf{0} & \mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I}) & \mathbf{XB} \\ \mathbf{0} & \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
&= (-1)^N \det\left(\mathbf{B}^H\right) \det\left(\begin{bmatrix} \mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I}) & \mathbf{XB} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right)
\end{aligned}
$$

Let

$$
\mathbf{Y} = -\left(\mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I})\right)\left(\mathbf{B}^H\right)^{-1} \tag{38}
$$

It follows that

$$
\begin{aligned}
&\det\left(\begin{bmatrix} \mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I}) & \mathbf{XB} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \qquad (39) \\
&= \det\left(\begin{bmatrix} \mathbf{I} & \mathbf{Y} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I}) & \mathbf{XB} \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} \mathbf{0} & \mathbf{XB}+\mathbf{Y}(\mathbf{A}-\lambda\mathbf{I}) \\ \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \end{bmatrix}\right) \\
&= (-1)^N \det\left(\begin{bmatrix} \mathbf{B}^H & \mathbf{A}-\lambda\mathbf{I} \\ \mathbf{0} & \mathbf{XB}+\mathbf{Y}(\mathbf{A}-\lambda\mathbf{I}) \end{bmatrix}\right) \\
&= (-1)^N \det\left(\mathbf{B}^H\right) \det\left(\mathbf{XB}+\mathbf{Y}(\mathbf{A}-\lambda\mathbf{I})\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbf{XB}+\mathbf{Y}(\mathbf{A}-\lambda\mathbf{I}) \qquad (40) \\
=\ & \left(-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\right)\mathbf{B} + \left(-\left(\mathbf{B}+\mathbf{X}(\mathbf{A}-\lambda\mathbf{I})\right)\left(\mathbf{B}^H\right)^{-1}\right)(\mathbf{A}-\lambda\mathbf{I}) \\
=\ & -(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B} - \mathbf{B}\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \\
& -\mathbf{X}(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \\
=\ & -(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B} - \mathbf{B}\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \\
& +(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}(\mathbf{A}-\lambda\mathbf{I}) \\
=\ & -(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B} - \mathbf{B}\mathbf{B}^H\left(\left(\mathbf{B}^H\right)^{-2}\mathbf{A}\left(\mathbf{B}^H\right)^2-\lambda\mathbf{I}\right)\left(\mathbf{B}^H\right)^{-2} \\
& +(\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\left(\left(\mathbf{B}^H\right)^{-2}\mathbf{A}\left(\mathbf{B}^H\right)^2-\lambda\mathbf{I}\right)\left(\mathbf{B}^H\right)^{-2} \\
=\ & -(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B}\mathbf{B}^H\mathbf{B}^H\left(\mathbf{B}^H\right)^{-2} \\
& -\left[\mathbf{B}\mathbf{B}^H-(\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)\right]\times \\
& \quad \left[\left(\mathbf{B}^H\right)^{-2}\mathbf{A}\left(\mathbf{B}^H\right)^2-\lambda\mathbf{I}\right]\left(\mathbf{B}^H\right)^{-2}
\end{aligned}
$$

Combining (37), (39) and (40) and making simplifications, we obtain

$$
P_3(\lambda) = \tag{41}
$$

$$\det\left[-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B}\mathbf{B}^H\mathbf{B}^H+\mathbf{Q}_2\left[\left(\mathbf{B}^H\right)^{-2}\mathbf{A}\left(\mathbf{B}^H\right)^2-\lambda\mathbf{I}\right]\right]$$
$$=\det\left[\mathbf{Q}_2\left[\left(\mathbf{B}^H\right)^{-2}\mathbf{A}\left(\mathbf{B}^H\right)^2-\lambda\mathbf{I}\right]-(\mathbf{A}-\lambda\mathbf{I})\left(\mathbf{B}^H\right)^{-1}\mathbf{B}\mathbf{B}^H\mathbf{B}^H\right]$$

where

$$\mathbf{Q}_2=(\mathbf{A}-\lambda\mathbf{I})\left(\left(\mathbf{B}^H\right)^{-1}\mathbf{A}\mathbf{B}^H-\lambda\mathbf{I}\right)-\mathbf{B}\mathbf{B}^H \tag{42}$$

with

$$P_2(\lambda)=\det\left(\mathbf{Q}_2\right)$$

Since the entries of $\mathbf{B}$ are relatively small, we have $\mathbf{B}\mathbf{B}^H \approx \mathbf{0}$ and $P_3(\lambda) \approx \det(\mathbf{A}-\lambda\mathbf{I})^3$. Hence we can expect the eigenvalues of $\mathbf{H}$ to be small perturbations of the roots of $\det(\mathbf{A}-\lambda\mathbf{I})^3=0$. In other words, the eigenvalues of $\mathbf{H}$ should cluster into N groups with each group consisting of $L=3$ closely spaced eigenvalues.

In general, the characteristic polynomial $P_L(\lambda)$ for $L \geq 4$ can also be computed in terms of $\mathbf{A}$ and $\mathbf{B}$ in the same manner and if the entries of $\mathbf{B}$ are relatively small, we have $\mathbf{B}\mathbf{B}^H \approx \mathbf{0}$, $P_L(\lambda) \approx \det(\mathbf{A}-\lambda\mathbf{I})^L$ and we can expect the eigenvalues of $\mathbf{H}$ to be small perturbations of the roots of $\det(\mathbf{A}-\lambda\mathbf{I})^L=0$. In other words, eigenvalues of $\mathbf{H}$ should roughly cluster into N groups with each group consisting of L closely spaced eigenvalues.

To illustrate these observations, for $N=5$, $L=4$, $n=1$, $\gamma=\frac{1}{2}$, we group the 20 eigenvalues of $\mathbf{H}=E\left(\mathbf{Z}_n\mathbf{Z}_n^H\right)$ for the normalized Blackman window as follows:

Group 1	2.8652	2.7812	2.6807	2.6015
Group 2	1.7164	1.6139	1.4916	1.3960
Group 3	0.8358	0.6843	0.5041	0.3600
Group 4	0.2884	0.1802	0.1727	0.1258
Group 5	0.0638	0.0531	0.0517	0.0149

Clearly, the eigenvalues of $\mathbf{H}$ can be roughly divided into $N=5$ groups of $L=4$ eigenvalues each and each group with $L=4$ eigenvalues approximately form an arithmetic progression.

4 Probability of Detection ($0<\gamma\leq 1/2, N\geq 1, L\geq 1$)

In this section, we compute the probability of detection P_d for the FFT summation detector under the assumption that the overlapping ratio $0<\gamma\leq 1/2$ and the received signal is a complex pure tone embedded in additive white gaussian noise. Specifically, the received signal samples are given by

$$r_k = A\exp(2\pi jfk) + u_k = s_k + u_k \ , \quad k = 0, 1, 2, \cdots \ ,$$

where A and f are respectively the amplitude and normalized frequency of the pure tone, u_k is the additive white gaussian noise sample and $s_k = A\exp(j2\pi fk)$. The received signal sample vector $\mathbf{R}_m$ can now be written as the sum of the signal component $\mathbf{S}_m$ and the noise component $\mathbf{N}_m$

$$\mathbf{R}_m \ = \ \mathbf{S}_m + \mathbf{N}_m \ , \ 0 \leq m \leq L-1 \ ,$$

where $\mathbf{S}_m$ and $\mathbf{N}_m$ are defined by:

$$\left\{ \begin{array}{lcl} \mathbf{S}_m & = & \left[s_{m(1-\gamma)MK+MK-1}, s_{m(1-\gamma)MK+MK-2}, \cdots, s_{m(1-\gamma)MK}\right]^t \\ \mathbf{N}_m & = & \left[u_{m(1-\gamma)MK+MK-1}, u_{m(1-\gamma)MK+MK-2}, \cdots, u_{m(1-\gamma)MK}\right]^t \end{array} \right. \tag{43}$$

The additive white Gaussian noise samples u_k are independent and identically distributed with $E(u_k u_l^*) = \sigma^2 \delta_{k,l}$, where $\delta_{k,l} = 1$ if $k = l$ and $\delta_{k,l} = 0$ if $k \neq l$. It can be verified that for $0 \leq m, \ l \leq L-1, \ \ E\left(\mathbf{N}_m \mathbf{N}_l^t\right) = \mathbf{0}$. In fact, let $0 \leq n \leq M-1$ and define

$$\mathbf{V}_m \ = \ \mathbf{Z}_{n,m} - E\left(\mathbf{Z}_{n,m}\right) = \mathbf{F}_n \mathbf{W} \mathbf{R}_m - \mathbf{F}_n \mathbf{W} \mathbf{S}_m = \mathbf{F}_n \mathbf{W} \mathbf{N}_m \tag{44}$$

We have

$$\begin{array}{l} E\left(\begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \\ \cdots \\ \mathbf{V}_m \\ \cdots \\ \mathbf{V}_{L-1} \end{bmatrix} \begin{bmatrix} \mathbf{V}_0 \\ \mathbf{V}_1 \\ \cdots \\ \mathbf{V}_m \\ \cdots \\ \mathbf{V}_{L-1} \end{bmatrix}^t \right) \\ = \begin{bmatrix} E\left(\mathbf{V}_0 \mathbf{V}_0^t\right) & E\left(\mathbf{V}_0 \mathbf{V}_1^t\right) & \cdots & \cdots & E\left(\mathbf{V}_0 \mathbf{V}_{L-1}^t\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ E\left(\mathbf{V}_m \mathbf{V}_0^t\right) & E\left(\mathbf{V}_m \mathbf{V}_1^t\right) & \cdots & \cdots & E\left(\mathbf{V}_m \mathbf{V}_{L-1}^t\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ E\left(\mathbf{V}_{L-1} \mathbf{V}_0^t\right) & E\left(\mathbf{V}_{L-1} \mathbf{V}_1^t\right) & \cdots & \cdots & E\left(\mathbf{V}_{L-1} \mathbf{V}_{L-1}^t\right) \end{bmatrix} \\ = \mathbf{0} \end{array} \tag{45}$$

since for $0 \leq m, \ l \leq L-1$,

$$\begin{array}{rcl} E\left(\mathbf{V}_m \mathbf{V}_l^t\right) & = & E\left(\mathbf{F}_n \mathbf{W} \mathbf{N}_m \left(\mathbf{F}_n \mathbf{W} \mathbf{N}_l\right)^t\right) = E\left(\mathbf{F}_n \mathbf{W} \mathbf{N}_m \mathbf{N}_l^t \mathbf{W} \mathbf{F}_n^t\right) \\ & = & \mathbf{F}_n \mathbf{W} E\left(\mathbf{N}_m \mathbf{N}_l^t\right) \mathbf{W} \mathbf{F}_n^t = \mathbf{0} \end{array} \tag{46}$$

(46) implies that the components of the Gaussian random vector

$$\mathbf{Z}_n = \begin{bmatrix} \mathbf{Z}_{n,0} \\ \mathbf{Z}_{n,1} \\ \cdots \\ \mathbf{Z}_{n,m} \\ \cdots \\ \mathbf{Z}_{n,L-1} \end{bmatrix}$$

satisfy the two constraints (1) and (2) of [16]. Note that here the Gaussian random vector $\mathbf{Z}_n$ is no longer zero mean owing to the presence of the complex sinusoidal signal. It follows from [16] that the characteristic function $\phi(t)$ of the decision statistic z_n, which is the quadratic form defined by (12), can be computed using the formula (4a) in [16]:

$$\begin{aligned} \phi(t) &= (\det(\mathbf{I} - jt\mathbf{H}))^{-1} \exp\left(-\bar{\mathbf{Z}}_n^H \mathbf{H}^{-1} \left(\mathbf{I} - (\mathbf{I} - jt\mathbf{H})^{-1}\right) \bar{\mathbf{Z}}_n\right) \\ &= (\det(\mathbf{I} - jt\mathbf{H}))^{-1} \exp\left(\bar{\mathbf{Z}}_n^H \mathbf{H}^{-1} jt\mathbf{H} (\mathbf{I} - jt\mathbf{H})^{-1} \bar{\mathbf{Z}}_n\right) \\ &= (\det(\mathbf{I} - jt\mathbf{H}))^{-1} \exp\left(jt\bar{\mathbf{Z}}_n^H (\mathbf{I} - jt\mathbf{H})^{-1} \bar{\mathbf{Z}}_n\right) \end{aligned} \tag{47}$$

where $\mathbf{H}$ is the positive definite Hermitian matrix defined by (17) and $\bar{\mathbf{Z}}_n$ is the mean value or signal component of $\mathbf{Z}_n$ given by

$$\begin{aligned} \bar{\mathbf{Z}}_n &= E(\mathbf{Z}_n) \\ &= \begin{bmatrix} E(\mathbf{Z}_{n,0}) \\ E(\mathbf{Z}_{n,1}) \\ \cdots \\ E(\mathbf{Z}_{n,m}) \\ \cdots \\ E(\mathbf{Z}_{n,L-1}) \end{bmatrix} = \begin{bmatrix} \mathbf{F}_n\mathbf{W}\mathbf{S}_0 \\ \mathbf{F}_n\mathbf{W}\mathbf{S}_1 \\ \\ \cdots \\ \mathbf{F}_n\mathbf{W}\mathbf{S}_m \\ \cdots \\ \mathbf{F}_n\mathbf{W}\mathbf{S}_{L-1} \end{bmatrix} \end{aligned} \tag{48}$$

Let $\mathbf{U}$ be a complex unitary matrix of dimensions $LN \times LN$ such that

$$\mathbf{U}\mathbf{H}\mathbf{U}^{-1} = \mathbf{U}\mathbf{H}\mathbf{U}^H = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda_m & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \cdots & \lambda_{LN} \end{bmatrix} \tag{49}$$

where $\lambda_1,\ \lambda_2,\ \cdots, \lambda_{LN}$ are the LN distinct eigenvalues of $\mathbf{H}$. Then

$$(\det(\mathbf{I} - jt\mathbf{H}))^{-1} = \left(\prod_{m=1}^{LN} (1 - jt\lambda_m)\right)^{-1} \tag{50}$$

and

$$\bar{\mathbf{Z}}_n^H \left(\mathbf{I} - jt\mathbf{H}\right)^{-1} \bar{\mathbf{Z}}_n = \tag{51}$$

$$\begin{aligned} & \left(\mathbf{U}\bar{\mathbf{Z}}_n\right)^H \times \\ & \begin{bmatrix} (1-jt\lambda_1)^{-1} & 0 & \cdots & 0 \\ 0 & (1-jt\lambda_2)^{-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & (1-jt\lambda_m)^{-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & (1-jt\lambda_{LN})^{-1} \end{bmatrix} \mathbf{U}\bar{\mathbf{Z}}_n \\ = & \sum_{m=1}^{LN} \frac{|\alpha_m|^2}{1-jt\lambda_m} \end{aligned}$$

where

$$\mathbf{U}\bar{\mathbf{Z}}_n = \left(\alpha_1, \alpha_2, \cdots, \alpha_{LN}\right)^t \tag{52}$$

Combining (50), (51), (52), we obtain the following expression for the characteristic function of the random variable z_n:

$$\begin{aligned} \phi(t) &= \left(\det\left(\mathbf{I} - jt\mathbf{H}\right)\right)^{-1} \exp\left(jt\bar{\mathbf{Z}}_n^H \left(\mathbf{I} - jt\mathbf{H}\right)^{-1} \bar{\mathbf{Z}}_n\right) \\ &= \left(\prod_{m=1}^{LN} (1-jt\lambda_m)\right)^{-1} \exp \sum_{m=1}^{LN} \frac{jt|\alpha_m|^2}{1-jt\lambda_m} \end{aligned} \tag{53}$$

It can be verified that

$$\left(\prod_{m=1}^{LN} (1-jt\lambda_m)\right)^{-1} = \sum_{m=1}^{LN} \frac{A_m}{1-jt\lambda_m} \tag{54}$$

where $A_m = \dfrac{\lambda_m^{LN-1}}{\prod_{l \neq m} (\lambda_m - \lambda_l)}$ and

$$\begin{aligned} \sum_{m=1}^{LN} \frac{jt|\alpha_m|^2}{1-jt\lambda_m} &= \sum_{m=1}^{LN} \frac{(jt\lambda_m - 1 + 1)\frac{|\alpha_m|^2}{\lambda_m}}{1-jt\lambda_m} \\ &= -\sum_{m=1}^{LN} \frac{|\alpha_m|^2}{\lambda_m} + \sum_{m=1}^{LN} \frac{\frac{|\alpha_m|^2}{\lambda_m}}{1-jt\lambda_m} \end{aligned} \tag{55}$$

Hence the characteristic function $\phi(t)$ can be rewritten as

$$\phi(t) = \left(\sum_{m=1}^{LN} \frac{A_m}{1-jt\lambda_m}\right) \exp\left(-\sum_{m=1}^{LN} \frac{|\alpha_m|^2}{\lambda_m} + \sum_{m=1}^{LN} \frac{\frac{|\alpha_m|^2}{\lambda_m}}{1-jt\lambda_m}\right) \tag{56}$$

$$
\begin{aligned}
&= \left(\exp\left(-\sum_{m=1}^{LN}\frac{|\alpha_m|^2}{\lambda_m}\right)\right)\left(\sum_{m=1}^{LN}\frac{A_m}{1-jt\lambda_m}\right)\exp\sum_{m=1}^{LN}\frac{\frac{|\alpha_m|^2}{\lambda_m}}{1-jt\lambda_m} \\
&= \gamma_0\sum_{m=1}^{LN}\frac{A_m}{1-jt\lambda_m}\exp\sum_{l=1}^{LN}\frac{\beta_l}{1-jt\lambda_l}
\end{aligned}
$$

where $\beta_l = \frac{|\alpha_l|^2}{\lambda_l}$, $1 \le l \le LN$, and $\gamma_0 = \exp\left(-\sum_{m=1}^{LN}\frac{|\alpha_m|^2}{\lambda_m}\right) = \exp(-\sum_{m=1}^{LN}\beta_m)$.

Now we are ready to compute the probability of detection P_d for a given threshold $T > 0$. The derivations in the sequel of this report will not depend on the channel index n explicitly. To avoid any potential confusion, we now call the readers' attention to the fact that we shall from now on use the letter n freely to denote one of the indices in the rest of this document.

Let $\epsilon > 0$ be a small positive real number ($0 < \epsilon < \frac{1}{\max_{1\le l\le LN}\lambda_l}$) and let $\mathcal{L} = \{z = t - j\epsilon : -\infty < t < \infty\}$. $\mathcal{L}$ is a straight line that is in parallel with and below the real axis and the distance between these two lines is ϵ. Let $p(x)$ be the probability density function of the random variable z_n. We have

$$
p(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\phi(t)e^{-jxt}dt
$$

and it can be verified, using elaborate contour integral arguments from the theory of functions of one complex variable and also Fubini's Theorem from measure theory (details omitted), that

$$
\begin{aligned}
P_d &= \int_T^{\infty} p(x)\,dx = \frac{1}{2\pi}\int_T^{\infty}\int_{-\infty}^{\infty}\phi(t)e^{-jxt}dt\,dx \qquad (57)\\
&= \frac{1}{2\pi}\int_T^{\infty}\int_{\mathcal{L}}\phi(z)e^{-jxz}dz\,dx = \frac{1}{2\pi}\int_{\mathcal{L}}\int_T^{\infty}\phi(z)e^{-jxz}dx\,dz \\
&= \frac{1}{2\pi j}\int_{\mathcal{L}}\frac{\phi(z)e^{-jzT}}{z}\,dz \\
&= \frac{1}{2\pi j}\int_{\mathcal{L}}\frac{\gamma_0\sum_{m=1}^{LN}\left(\frac{A_m}{1-jz\lambda_m}\exp\sum_{l=1}^{LN}\frac{\beta_l}{1-jz\lambda_l}\right)e^{-jzT}}{z}\,dz \\
&= \frac{\gamma_0}{2\pi j}\sum_{m=1}^{LN}A_m\int_{\mathcal{L}}\left(\frac{1}{1-jz\lambda_m}\exp\sum_{l=1}^{LN}\frac{\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\,dz
\end{aligned}
$$

Let $\mathcal{C}_l$, $1 \le l \le LN$, be mutually disjoint small circles centered respectively at $\frac{1}{j\lambda_l}$ and oriented clockwise. Using the Residue Theorem from the theory of functions of one complex variable, one obtains, for any m, $1 \le m \le LN$,

$$
\int_{\mathcal{L}}\left(\frac{1}{1-jz\lambda_m}\exp\sum_{l=1}^{LN}\frac{\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\,dz \qquad (58)
$$

$$= \sum_{n=1}^{LN} \int_{\mathcal{C}_n} \left(\frac{1}{1-jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1-jz\lambda_l} \right) \frac{e^{-jzT}}{z}\ dz$$

$$= \sum_{n \neq m} \int_{\mathcal{C}_n} \left(\frac{1}{1-jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1-jz\lambda_l} \right) \frac{e^{-jzT}}{z}\ dz + \int_{\mathcal{C}_m} \left(\frac{1}{1-jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1-jz\lambda_l} \right) \frac{e^{-jzT}}{z}\ dz$$

Let

$$\psi_n(z) = \exp \sum_{l \neq n} \frac{\beta_l}{1-jz\lambda_l} \tag{59}$$

If $n \neq m$, we have

$$\begin{aligned} & \int_{\mathcal{C}_n} \left(\frac{1}{1-jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1-jz\lambda_l} \right) \frac{e^{-jzT}}{z}\ dz \\ = & \int_{\mathcal{C}_n} \frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z} \psi_n(z) \exp \frac{\beta_n}{1-jz\lambda_n} dz \\ = & \int_{\mathcal{C}_n} \frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z} \psi_n(z) \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\beta_n}{1-jz\lambda_n} \right)^p dz \\ = & \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\beta_n}{-j\lambda_n} \right)^p \int_{\mathcal{C}_n} \frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z} \psi_n(z) \frac{1}{\left(z - \frac{1}{j\lambda_n}\right)^p} dz \\ = & \sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{\beta_n}{-j\lambda_n} \right)^p \int_{\mathcal{C}_n} \frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z} \psi_n(z) \frac{1}{\left(z - \frac{1}{j\lambda_n}\right)^p} dz \end{aligned} \tag{60}$$

To simplify (60), we first develop the power series expansions of $\frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z}$ and $\psi_n(z)$ around the point $\frac{1}{j\lambda_n}$. We have

$$\begin{aligned} \frac{1}{1-jz\lambda_m} &= \frac{1}{1-j(z-\frac{1}{j\lambda_n}+\frac{1}{j\lambda_n})\lambda_m} = \frac{\frac{\lambda_n}{\lambda_n-\lambda_m}}{1-j(z-\frac{1}{j\lambda_n})\frac{\lambda_m\lambda_n}{\lambda_n-\lambda_m}} \\ &= \frac{\lambda_n}{\lambda_n-\lambda_m} \sum_{r=0}^{\infty} \left(j\left(z-\frac{1}{j\lambda_n}\right) \frac{\lambda_m\lambda_n}{\lambda_n-\lambda_m} \right)^r \\ &= \frac{\lambda_n}{\lambda_n-\lambda_m} \sum_{r=0}^{\infty} \left(j\frac{\lambda_m\lambda_n}{\lambda_n-\lambda_m} \right)^r \left(z-\frac{1}{j\lambda_n} \right)^r \end{aligned} \tag{61}$$

$$\frac{1}{z} = \frac{1}{z-\frac{1}{j\lambda_n}+\frac{1}{j\lambda_n}} = j\lambda_n \frac{1}{1+(z-\frac{1}{j\lambda_n})j\lambda_n} \tag{62}$$

$$= \quad j\lambda_n \sum_{s=0}^{\infty} (-j\lambda_n)^s \left(z - \frac{1}{j\lambda_n} \right)^s$$

$$\begin{aligned} e^{-jzT} &= \exp\left(-jT\left(z - \frac{1}{j\lambda_n} \right) - \frac{T}{\lambda_n} \right) \\ &= \exp\left(-\frac{T}{\lambda_n} \right) \exp\left(-jT\left(z - \frac{1}{j\lambda_n} \right) \right) \\ &= \exp\left(-\frac{T}{\lambda_n} \right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_n} \right)^t \end{aligned} \tag{63}$$

and for any $l \neq n$,

$$\begin{aligned} \exp\left(\frac{\beta_l}{1 - jz\lambda_l} \right) &= \exp\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l} \frac{1}{1 - j(z - \frac{1}{j\lambda_n})\frac{\lambda_l\lambda_n}{\lambda_n - \lambda_l}} \right) \\ &= 1 + \sum_{q=1}^{\infty} \frac{1}{q!} \left(\frac{\beta_l\lambda_n}{\lambda_n - \lambda_l} \right)^q \left(\frac{1}{1 - j(z - \frac{1}{j\lambda_n})\frac{\lambda_l\lambda_n}{\lambda_n - \lambda_l}} \right)^q \\ &= 1 + \sum_{q=1}^{\infty} \frac{1}{q!} \left(\frac{\beta_l\lambda_n}{\lambda_n - \lambda_l} \right)^q \sum_{u=0}^{\infty} \binom{u+q-1}{q-1} \left(\frac{j\lambda_l\lambda_n}{\lambda_n - \lambda_l} \right)^u \left(z - \frac{1}{j\lambda_n} \right)^u \\ &= 1 + \sum_{u=0}^{\infty} \left[\sum_{q=1}^{\infty} \frac{1}{q!} \left(\frac{\beta_l\lambda_n}{\lambda_n - \lambda_l} \right)^q \binom{u+q-1}{q-1} \right] \left(\frac{j\lambda_l\lambda_n}{\lambda_n - \lambda_l} \right)^u \left(z - \frac{1}{j\lambda_n} \right)^u \\ &= \sum_{u=0}^{\infty} \left[\sum_{q=0}^{\infty} \frac{1}{q!} \binom{u+q-1}{q-1} \left(\frac{\beta_l\lambda_n}{\lambda_n - \lambda_l} \right)^q \right] \left(\frac{j\lambda_l\lambda_n}{\lambda_n - \lambda_l} \right)^u \left(z - \frac{1}{j\lambda_n} \right)^u \\ &= \sum_{u=0}^{\infty} B_u \left(\frac{\beta_l\lambda_n}{\lambda_n - \lambda_l} \right) \left(\frac{j\lambda_l\lambda_n}{\lambda_n - \lambda_l} \right)^u \left(z - \frac{1}{j\lambda_n} \right)^u \end{aligned} \tag{64}$$

where $\binom{q}{p}$ is the binomial coefficient satisfying the following identities:

$$\begin{cases} \binom{q}{p} = \frac{q(q-1)\cdots(q-p+1)}{p!} = \frac{q!}{p!(q-p)!}\ ,\ q > p > 0\ , \\ \binom{p}{p} = 1\ ,\ p \text{ is any integer}\ , \\ \binom{p}{0} = 1\ ,\ p > 0\ , \\ \binom{q}{p} = 0\ ,\ p < 0\ ,\ \ q > p\ , \end{cases} \tag{65}$$

and

$$B_u(x) = \sum_{q=0}^{\infty} \frac{1}{q!} \left(\begin{array}{c} u+q-1 \\ q-1 \end{array} \right) x^q \tag{66}$$

Combining (59), (61) -(64), we obtain

$$\begin{aligned}
& \frac{1}{1-jz\lambda_m} \frac{e^{-jzT}}{z} \psi_n(z) \qquad (67) \\
= & \left(\frac{\lambda_n}{\lambda_n - \lambda_m} \sum_{r=0}^{\infty} \left(j \frac{\lambda_m \lambda_n}{\lambda_n - \lambda_m} \right)^r \left(z - \frac{1}{j\lambda_n} \right)^r \right) \times \\
& \left(j\lambda_n \sum_{s=0}^{\infty} (-j\lambda_n)^s \left(z - \frac{1}{j\lambda_n} \right)^s \right) \times \\
& \left(\exp\left(-\frac{T}{\lambda_n} \right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_n} \right)^t \right) \times \\
& \prod_{l \neq n} \left(\sum_{u_l=0}^{\infty} B_{u_l} \left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l} \right) \left(\frac{j\lambda_l \lambda_n}{\lambda_n - \lambda_l} \right)^{u_l} \left(z - \frac{1}{j\lambda_n} \right)^{u_l} \right) \\
= & \left(\frac{-j\lambda_n^2}{\lambda_m - \lambda_n} \exp\left(-\frac{T}{\lambda_n} \right) \right) \sum_{v=0}^{\infty} d_v(m,n) \left(z - \frac{1}{j\lambda_n} \right)^v
\end{aligned}$$

where

$$\begin{aligned}
& d_v(m,n) \qquad (68) \\
= & \sum_{r+s+t+\sum_{l\neq n} u_l = v} \left(j \frac{\lambda_m \lambda_n}{\lambda_n - \lambda_m} \right)^r (-j\lambda_n)^s \frac{(-jT)^t}{t!} \times \\
& \prod B_{u_l} \left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l} \right) \left(\frac{j\lambda_l \lambda_n}{\lambda_n - \lambda_l} \right)^{u_l} \\
= & (-j)^v \sum_{r+s+t+\sum_{l\neq n} u_l = v} \left(\frac{\lambda_m \lambda_n}{\lambda_m - \lambda_n} \right)^r \lambda_n^s \frac{T^t}{t!} \prod B_{u_l} \left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l} \right) \left(\frac{\lambda_l \lambda_n}{\lambda_l - \lambda_n} \right)^{u_l} \\
= & (-j)^v \lambda_n^v f_v(m,n)
\end{aligned}$$

with

$$\begin{aligned}
& f_v(m,n) \qquad (69) \\
= & \sum_{r+s+t+\sum_{l\neq n} u_l = v} \left(\frac{\lambda_m}{\lambda_m - \lambda_n} \right)^r \frac{\left(\frac{T}{\lambda_n} \right)^t}{t!} \prod B_{u_l} \left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l} \right) \left(\frac{\lambda_l}{\lambda_l - \lambda_n} \right)^{u_l}
\end{aligned}$$

$$= \sum_{s=0}^{v} \sum_{r+t+\sum_{l\neq n} u_l = v-s} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \prod B_{u_l}\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l}$$

$$= \sum_{s=0}^{v} \sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \sum_{r+\sum_{l\neq n} u_l = v-s-t} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \prod B_{u_l}\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l}$$

Substituting (67) into (60), we obtain, for $n \neq m$,

$$\int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1 - jz\lambda_l}\right) \frac{e^{-jzT}}{z}\, dz \tag{70}$$

$$= \sum_{p=1}^{\infty} \frac{1}{p!}\left(\frac{\beta_n}{-j\lambda_n}\right)^p \left(\frac{\lambda_n}{\lambda_m - \lambda_n} \exp\left(-\frac{T}{\lambda_n}\right)\right) \times$$

$$\sum_{v=0}^{\infty} (-j\lambda_n)^{1+v}\ f_v(m,n) \int_{\mathcal{C}_n} \left(z - \frac{1}{j\lambda_n}\right)^v \frac{1}{\left(z - \frac{1}{j\lambda_n}\right)^p} dz$$

$$= -2\pi j \left(\frac{\lambda_n}{\lambda_m - \lambda_n} \exp\left(-\frac{T}{\lambda_n}\right)\right) \sum_{p=1}^{\infty} \frac{1}{p!} \beta_n^p f_{p-1}(m,n)$$

Next let us compute

$$\int_{\mathcal{C}_m} \left(\frac{1}{1 - jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1 - jz\lambda_l}\right) \frac{e^{-jzT}}{z}\, dz \tag{71}$$

We have

$$\frac{1}{z} = \frac{1}{z - \frac{1}{j\lambda_m} + \frac{1}{j\lambda_m}} = j\lambda_m \frac{1}{1 + (z - \frac{1}{j\lambda_m}) j\lambda_m} \tag{72}$$

$$= j\lambda_m \sum_{s=0}^{\infty} (-j\lambda_m)^s \left(z - \frac{1}{j\lambda_m}\right)^s$$

and

$$e^{-jzT} = \exp\left(-jT\left(z - \frac{1}{j\lambda_m}\right) - \frac{T}{\lambda_m}\right) \tag{73}$$

$$= \exp\left(-\frac{T}{\lambda_m}\right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_m}\right)^t$$

Following (64), we have, for any $l \neq m$,

$$\exp\left(\frac{\beta_l}{1 - jz\lambda_l}\right) = \sum_{u=0}^{\infty} B_u\left(\frac{\beta_l \lambda_m}{\lambda_m - \lambda_l}\right)\left(\frac{j\lambda_l \lambda_m}{\lambda_m - \lambda_l}\right)^u \left(z - \frac{1}{j\lambda_m}\right)^u \tag{74}$$

It follows from (72)-(74) that

$$\left(\frac{1}{1-jz\lambda_m}\exp\sum_{l\neq m}\frac{\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z} \tag{75}$$

$$= \left(\frac{1}{-j\lambda_m}\right)\frac{1}{z-\frac{1}{j\lambda_m}}\times\left(j\lambda_m\sum_{s=0}^{\infty}(-j\lambda_m)^s\left(z-\frac{1}{j\lambda_m}\right)^s\right)\times$$

$$\left(\exp\left(-\frac{T}{\lambda_m}\right)\ \sum_{t=0}^{\infty}\frac{(-jT)^t}{t!}\left(z-\frac{1}{j\lambda_m}\right)^t\right)\times$$

$$\prod_{l\neq m}\sum_{u_l=0}^{\infty}B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{j\lambda_l\lambda_m}{\lambda_m-\lambda_l}\right)^{u_l}\left(z-\frac{1}{j\lambda_m}\right)^{u_l}$$

$$= \frac{-\exp\left(-\frac{T}{\lambda_m}\right)}{z-\frac{1}{j\lambda_m}}\sum_{v=0}^{\infty}d_v^*(m)\left(z-\frac{1}{j\lambda_m}\right)^v$$

where

$$d_v^*(m) = \sum_{s+t+\sum_{l\neq m}u_l=v}(-j\lambda_m)^s\frac{(-jT)^t}{t!}\prod B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{j\lambda_l\lambda_m}{\lambda_m-\lambda_l}\right)^{u_l}$$

$$= (-j)^v\sum_{s+t+\sum_{l\neq m}u_l=v}\lambda_m^s\frac{T^t}{t!}\prod B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l\lambda_m}{\lambda_l-\lambda_m}\right)^{u_l}$$

$$= (-j)^v\lambda_m^v g_v(m) \tag{76}$$

with

$$g_v(m) = \sum_{s+t+\sum_{l\neq m}u_l=v}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\prod B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \tag{77}$$

$$= \sum_{s=0}^{v}\sum_{t=0}^{v-s}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\sum_{\sum_{l\neq m}u_l=v-s-t}\prod B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l}$$

Utilizing (75)-(77), we finally obtain

$$\int_{\mathcal{C}_m}\left[\frac{1}{1-jz\lambda_m}\exp\sum_{l=1}^{LN}\frac{\beta_l}{1-jz\lambda_l}\right]\frac{e^{-jzT}}{z}\ dz \tag{78}$$

$$= -\exp\left(-\frac{T}{\lambda_m}\right)\times$$

$$
\begin{aligned}
& \int_{\mathcal{C}_m} \left[\frac{1}{z - \frac{1}{j\lambda_m}} \sum_{v=0}^{\infty} d_v^*(m) \left(z - \frac{1}{j\lambda_n} \right)^v \right] \sum_{p=0}^{\infty} \left(\frac{\beta_m}{-j\lambda_m} \right)^p \left(z - \frac{1}{j\lambda_m} \right)^{-p} dz \\
= \; & -\exp\left(-\frac{T}{\lambda_m}\right) \times \int_{\mathcal{C}_m} \left[\frac{1}{z - \frac{1}{j\lambda_m}} \sum_{v=0}^{\infty} \lambda_m^v g_v(m)(-j)^v \left(z - \frac{1}{j\lambda_n} \right)^v \right] \times \\
& \left[\sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{\beta_m}{-j\lambda_m} \right)^p \left(z - \frac{1}{j\lambda_m} \right)^{-p} \right] dz \\
= \; & 2\pi j \exp\left(-\frac{T}{\lambda_m}\right) \sum_{p=0}^{\infty} \frac{1}{p!} \beta_m^p g_p(m)
\end{aligned}
$$

Combining (57), (58), (70) and (78), we obtain the probability of detection P_d as follows:

$$
\begin{aligned}
P_d \; = \; & \frac{\gamma_0}{2\pi j} \sum_{m=1}^{LN} A_m \int_{\mathcal{L}} \left(\frac{1}{1 - jz\lambda_m} \exp \sum_{l=1}^{LN} \frac{\beta_l}{1 - jz\lambda_l} \right) \frac{e^{-jzT}}{z} \, dz \qquad (79) \\
= \; & \frac{\gamma_0}{2\pi j} \sum_{m=1}^{LN} A_m \sum_{n \neq m} -2\pi j \left(\frac{\lambda_n}{\lambda_m - \lambda_n} \exp\left(-\frac{T}{\lambda_n}\right) \right) \sum_{p=1}^{\infty} \frac{1}{p!} \beta_n^p f_{p-1}(m,n) \\
& + \frac{\gamma_0}{2\pi j} \sum_{m=1}^{LN} A_m \left[2\pi j \exp\left(-\frac{T}{\lambda_m}\right) \sum_{p=0}^{\infty} \frac{1}{p!} \beta_m^p g_p(m) \right] \\
= \; & \gamma_0 \sum_{m=1}^{LN} A_m \sum_{n \neq m} \left(\frac{\lambda_n \beta_n}{\lambda_n - \lambda_m} \exp\left(-\frac{T}{\lambda_n}\right) \right) \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \beta_n^p f_p(m,n) \\
& + \gamma_0 \sum_{m=1}^{LN} A_m \exp\left(-\frac{T}{\lambda_m}\right) \sum_{p=0}^{\infty} \frac{1}{p!} \beta_m^p g_p(m)
\end{aligned}
$$

The formula (79) can be further simplified. In fact, it can be shown (details omitted) that, if $u \geq 1$, $B_u(x)$ can be written as

$$
B_u(x) = \frac{x}{u!} \frac{d^u}{dx^u} \left(x^{u-1} \, e^x \right) = P_u(x) e^x \qquad (80)
$$

where $P_u(x)$ is a polynomial of degree u with non-negative coefficients and

$$
\begin{cases}
P_0(x) = 1 \\
P_1(x) = x \\
P_2(x) = x(1 + \frac{1}{2}x) \\
P_u(x) = \dfrac{x}{u} \displaystyle\sum_{k=0}^{u-1} \binom{u}{k+1} \frac{x^k}{k!} \quad , \quad u \geq 1 \, .
\end{cases} \qquad (81)
$$

As an illustration, we have

$$\begin{cases} B_0(x) &=& e^x \\ B_1(x) &=& xe^x \\ B_2(x) &=& (x+\frac{1}{2}x^2)e^x \end{cases} \tag{82}$$

From (77), (69) and (82), it follows that

$$\begin{cases} g_0(m) &=& \exp\sum_{l\neq m}\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l} \\ &=& \exp\sum_{l\neq m}\frac{\beta_l(\lambda_m-\lambda_l)+\beta_l\lambda_l}{\lambda_m-\lambda_l} = \left[\exp\sum_{l\neq m}\beta_l\right]\ \exp\sum_{l\neq m}\frac{\beta_l\lambda_l}{\lambda_m-\lambda_l} \\ &=& \frac{e^{-\beta_m}}{\gamma_0}\ \exp\sum_{l\neq m}\frac{\beta_l\lambda_l}{\lambda_m-\lambda_l} \\ f_0(m,n) &=& \exp\left(\sum_{l\neq n}\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}\right) = g_0(n) \\ &=& \frac{e^{-\beta_n}}{\gamma_0}\ \exp\sum_{l\neq n}\frac{\beta_l\lambda_l}{\lambda_n-\lambda_l} \quad , \quad n\neq m \end{cases} \tag{83}$$

Also we have, for $n \neq m$ and $v \geq 1$,

$$\begin{aligned} & f_v(m,n) \\ =\ & \sum_{r+s+t+\sum_{l\neq n}u_l=v}\left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r\frac{\left(\frac{T}{\lambda_n}\right)^t}{t!}\prod B_{u_l}\left(\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\ =\ & e^{\sum_{l\neq n}\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}}\times \\ & \sum_{r+s+t+\sum_{l\neq n}u_l=v}\left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r\frac{\left(\frac{T}{\lambda_n}\right)^t}{t!}\prod P_{u_l}\left(\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\ =\ & g_0(n)\sum_{r+s+t+\sum_{l\neq n}u_l=v}\left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r\frac{\left(\frac{T}{\lambda_n}\right)^t}{t!}\prod P_{u_l}\left(\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \end{aligned} \tag{84}$$

and for $1 \leq m \leq LN$ and $v \geq 1$,

$$\begin{aligned} & g_v(m) \\ =\ & \sum_{s+t+\sum_{l\neq m}u_l=v}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\prod B_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \\ =\ & g_0(m)\sum_{s+t+\sum_{l\neq m}u_l=v}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\prod P_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \end{aligned} \tag{85}$$

Substituting (83), (84) and (85) into (79), we obtain the following formula for P_d:

$$\begin{aligned}
&P_d = \\
&\gamma_0 \sum_{m=1}^{LN} A_m \sum_{n \neq m} \left(\frac{\lambda_n \beta_n}{\lambda_n - \lambda_m} \exp\left(-\frac{T}{\lambda_n}\right)\right) \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \beta_n^p g_0(n) \times \\
&\sum_{s=0}^{p} \sum_{r+t+\sum_{l\neq n} u_l = p-s} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \times \\
&\prod P_{u_l}\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l}\right) \left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l} \\
&+\gamma_0 \sum_{m=1}^{LN} A_m \exp\left(-\frac{T}{\lambda_m}\right) \sum_{p=0}^{\infty} \frac{1}{p!} \beta_m^p g_0(m) \times \\
&\sum_{s=0}^{p} \sum_{t+\sum_{l\neq m} u_l = p-s} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} \prod P_{u_l}\left(\frac{\beta_l \lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{\lambda_l}{\lambda_l - \lambda_m}\right)^{u_l} \\
= &\sum_{m=1}^{LN} A_m \sum_{n \neq m} \left(\frac{\lambda_n \beta_n}{\lambda_n - \lambda_m} \gamma_0 \; g_0(n) \; \exp\left(-\frac{T}{\lambda_n}\right)\right) \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \beta_n^p \times \\
&\sum_{s=0}^{p} \sum_{r+t+\sum_{l\neq n} u_l = p-s} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \times \\
&\prod P_{u_l}\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l}\right) \left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l} \\
&+\sum_{m=1}^{LN} A_m \; \gamma_0 \; g_0(m) \; \exp\left(-\frac{T}{\lambda_m}\right) \sum_{p=0}^{\infty} \frac{1}{p!} \beta_m^p \times \\
&\sum_{s=0}^{p} \sum_{t+\sum_{l\neq m} u_l = p-s} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} \prod P_{u_l}\left(\frac{\beta_l \lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{\lambda_l}{\lambda_l - \lambda_m}\right)^{u_l} \\
= &\sum_{m=1}^{LN} A_m \sum_{n \neq m} \left[\frac{\lambda_n \beta_n}{\lambda_n - \lambda_m} \; e^{-\beta_n} \; \left[\exp \sum_{l \neq n} \frac{\beta_l \lambda_l}{\lambda_n - \lambda_l}\right] \; \exp\left(-\frac{T}{\lambda_n}\right)\right] \\
&\times \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \beta_n^p \; \sum_{s=0}^{p} \sum_{r+t+\sum_{l\neq n} u_l = p-s} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \times \\
&\prod P_{u_l}\left(\frac{\beta_l \lambda_n}{\lambda_n - \lambda_l}\right) \left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{LN} A_m\ e^{-\beta_m}\ \left(\exp\sum_{l\neq m}\frac{\beta_l\lambda_l}{\lambda_m-\lambda_l}\right)\ \exp\left(-\frac{T}{\lambda_m}\right)\sum_{p=0}^{\infty}\frac{1}{p!}\beta_m^p \times \\
&\sum_{s=0}^{p}\quad\sum_{t+\sum_{l\neq m}u_l=p-s}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\prod P_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \\
= &\sum_{m=1}^{LN} A_m\sum_{n\neq m}\left[\frac{\lambda_n\beta_n}{\lambda_n-\lambda_m}\ \left(\exp\sum_{l\neq n}\frac{\beta_l\lambda_l}{\lambda_n-\lambda_l}\right)\right]\sum_{p=0}^{\infty}\frac{1}{(p+1)!}\beta_n^p e^{-\beta_n} \\
&\times\sum_{s=0}^{p}\quad\sum_{r+t+\sum_{l\neq n}u_l=p-s}\left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r\frac{\left(\frac{T}{\lambda_n}\right)^t}{t!}e^{-\frac{T}{\lambda_n}}\times \\
&\prod P_{u_l}\left(\frac{\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\
&+\sum_{m=1}^{LN} A_m\ \left(\exp\sum_{l\neq m}\frac{\beta_l\lambda_l}{\lambda_m-\lambda_l}\right)\ \sum_{p=0}^{\infty}\frac{1}{p!}\beta_m^p\ e^{-\beta_m}\times \\
&\sum_{s=0}^{p}\quad\sum_{t+\sum_{l\neq m}u_l=p-s}\frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\ e^{-\frac{T}{\lambda_m}} \\
&\times\prod P_{u_l}\left(\frac{\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \qquad (86)
\end{aligned}
$$

5 Probability of Detection ($\gamma = 0, N > 1, L > 1$)

In this section, we compute the probability of detection P_d for the FFT summation detector under the assumption that the received signal is a complex pure tone embedded in additive white Gaussian noise, the overlapping ratio γ is zero and $N > 1$, $L > 1$. Since $\gamma = 0$, the data blocks are not overlapped and it follows from (18), (33), (47) and (53) that the characteristic function $\phi(t)$ is given by

$$
\phi(t) \;=\; \gamma_0^L\ \left(\frac{1}{\prod_{m=1}^{N}(1-jt\lambda_m)}\right)^L\ e^{\sum_{l=1}^{N}\frac{L\beta_l}{1-jt\lambda_l}} \qquad (87)
$$

Here λ_m, $1 \le m \le N$, are the N distinct eigenvalues of the Hermitian matrix $\mathbf{A}$ defined by (21) and (22), $\beta_m = \frac{\alpha_m^2}{\lambda_m}$, with α_m defined by (52), where $L = 1$, $\mathbf{H} = \mathbf{A}$, $\gamma_0 = e^{-\sum_{m=1}^{N}\beta_m}$. The rational function $\left(\frac{1}{\prod_{m=1}^{N}(1-jt\lambda_m)}\right)^L$ admits the following fractional decomposition:

$$\left(\frac{1}{(1-jt\lambda_1)(1-jt\lambda_2)\cdots(1-jt\lambda_N)}\right)^L = \sum_{m=1}^{N}\sum_{k=1}^{L} A_{m,k}\,\frac{1}{(1-jt\lambda_m)^k} \tag{88}$$

where it can be shown that

$$\left\{\begin{array}{rcl} A_{m,L} & = & \left(\dfrac{\lambda_m^{N-1}}{\prod_{l\neq m}(\lambda_m-\lambda_l)}\right)^L ,\ 1\leq m\leq N\ , \\ A_{m,k} & = & A_{m,L}\displaystyle\sum_{\sum_{l\neq m}k_l=L-k}\prod\left(\begin{array}{c} L+k_l-1 \\ k_l\end{array}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{k_l} , \\ & & 1\leq k\leq L-1\ , 1\leq m\leq N\ . \end{array}\right. \tag{89}$$

As in the case of overlapping data blocks, we have

$$\begin{aligned} P_d &= \int_T^{\infty} p(x)\ dx = \frac{1}{2\pi}\int_T^{\infty}\int_{-\infty}^{\infty}\phi(t)e^{-jxt}dt\ dx \\ &= \frac{1}{2\pi}\int_T^{\infty}\int_{\mathcal{L}}\phi(z)e^{-jxz}dz\ dx = \frac{1}{2\pi}\int_{\mathcal{L}}\int_T^{\infty}\phi(z)e^{-jxz}dx\ dz \\ &= \frac{1}{2\pi j}\int_{\mathcal{L}}\frac{\phi(z)e^{-jzT}}{z}\ dz \\ &= \frac{1}{2\pi j}\int_{\mathcal{L}}\left(\gamma_0^L\sum_{m=1}^{N}\sum_{k=1}^{L}\frac{A_{m,k}}{(1-jz\lambda_m)^k}\right)\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\frac{e^{-jzT}}{z}\ dz \\ &= \frac{\gamma_0^L}{2\pi j}\sum_{m=1}^{N}\ \sum_{k=1}^{L}A_{m,k}\int_{\mathcal{L}}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\ dz \end{aligned} \tag{90}$$

Let $\mathcal{C}_l$, $1\leq l\leq N$, be mutually disjoint small circles centered respectively at $\frac{1}{j\lambda_l}$ and oriented clockwise. Using the Residue Theorem from the theory of functions of one complex variable, we obtain, for any m, k, $1\leq m\leq N$, $1\leq k\leq L$,

$$\begin{aligned} &\int_{\mathcal{L}}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\ dz \\ &= \sum_{n=1}^{N}\int_{\mathcal{C}_n}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\ dz \\ &= \sum_{n\neq m}\int_{\mathcal{C}_n}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\ dz + \\ &\quad \int_{\mathcal{C}_m}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\ dz \end{aligned} \tag{91}$$

Let

$$\Psi_n(z) = \exp \sum_{l \neq n} \frac{L\beta_l}{1 - jz\lambda_l} \tag{92}$$

If $n \neq m$, we have

$$\begin{aligned}
&\int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \right)^k \left(\exp \sum_{l=1}^{N} \frac{L\beta_l}{1 - jz\lambda_l} \right) \frac{e^{-jzT}}{z} \, dz && (93) \\
= &\int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \right)^k \frac{e^{-jzT}}{z} \Psi_n(z) \exp \frac{L\beta_n}{1 - jz\lambda_n} dz \\
= &\int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \right)^k \frac{e^{-jzT}}{z} \Psi_n(z) \sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{L\beta_n}{1 - jz\lambda_n} \right)^p dz \\
= &\sum_{p=0}^{\infty} \frac{1}{p!} \left(\frac{L\beta_n}{-j\lambda_n} \right)^p \int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \right)^k \left(\frac{e^{-jzT}}{z} \right) \Psi_n(z) \frac{1}{\left(z - \frac{1}{j\lambda_n} \right)^p} dz \\
= &\sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{L\beta_n}{-j\lambda_n} \right)^p \int_{\mathcal{C}_n} \left(\frac{1}{1 - jz\lambda_m} \right)^k \left(\frac{e^{-jzT}}{z} \right) \Psi_n(z) \frac{1}{\left(z - \frac{1}{j\lambda_n} \right)^p} dz
\end{aligned}$$

To evaluate (93), we first develop the power series expansions of $\left(\frac{1}{1 - jz\lambda_m} \right)^k \frac{e^{-jzT}}{z}$ and $\psi_n(z)$ around the point $\frac{1}{j\lambda_n}$. We have

$$\begin{aligned}
&\left(\frac{1}{1 - jz\lambda_m} \right)^k = \left(\frac{1}{1 - j(z - \frac{1}{j\lambda_n} + \frac{1}{j\lambda_n})\lambda_m} \right)^k = \left(\frac{\frac{\lambda_n}{\lambda_n - \lambda_m}}{1 - j(z - \frac{1}{j\lambda_n}) \frac{\lambda_m \lambda_n}{\lambda_n - \lambda_m}} \right)^k \\
= &\left(\frac{\lambda_n}{\lambda_n - \lambda_m} \right)^k \sum_{r=0}^{\infty} \binom{r + k - 1}{k - 1} \left(j \left(z - \frac{1}{j\lambda_n} \right) \frac{\lambda_m \lambda_n}{\lambda_n - \lambda_m} \right)^r \\
= &\left(\frac{\lambda_n}{\lambda_n - \lambda_m} \right)^k \sum_{r=0}^{\infty} \binom{r + k - 1}{k - 1} \left(j \frac{\lambda_m \lambda_n}{\lambda_n - \lambda_m} \right)^r \left(z - \frac{1}{j\lambda_n} \right)^r && (94)
\end{aligned}$$

where $\binom{r + k - 1}{k - 1}$ is a binomial coefficient defined by (65),

$$\frac{1}{z} = j\lambda_n \sum_{s=0}^{\infty} (-j\lambda_n)^s \left(z - \frac{1}{j\lambda_n} \right)^s \tag{95}$$

and

$$e^{-jzT} = \exp\left(-\frac{T}{\lambda_n} \right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_n} \right)^t \tag{96}$$

and from (64) it follows that, for any $l \neq n$,

$$\exp\left(\frac{L\beta_l}{1-jz\lambda_l}\right) = \sum_{u=0}^{\infty} B_u\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{j\lambda_l\lambda_n}{\lambda_n-\lambda_l}\right)^u\left(z-\frac{1}{j\lambda_n}\right)^u \tag{97}$$

where

$$B_u(x) = \sum_{q=0}^{\infty}\frac{1}{q!}\binom{u+q-1}{q-1}x^q = P_u(x)e^x$$

with $P_u(x)$ defined by (80) and (81). Combining the preceding equations, we obtain

$$\begin{aligned}
&\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\frac{e^{-jzT}}{z}\right)\psi_n(z) && (98)\\
= \;&\left[\left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right)^k\sum_{r=0}^{\infty}\binom{r+k-1}{k-1}\left(j\frac{\lambda_m\lambda_n}{\lambda_n-\lambda_m}\right)^r\left(z-\frac{1}{j\lambda_n}\right)^r\right]\\
&\times\left[j\lambda_n\sum_{s=0}^{\infty}(-j\lambda_n)^s\left(z-\frac{1}{j\lambda_n}\right)^s\right]\times\\
&\left[\exp\left(-\frac{T}{\lambda_n}\right)\sum_{t=0}^{\infty}\frac{(-jT)^t}{t!}\left(z-\frac{1}{j\lambda_n}\right)^t\right]\times\\
&\prod_{l\neq n}\sum_{u_l=0}^{\infty}B_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{j\lambda_l\lambda_n}{\lambda_n-\lambda_l}\right)^{u_l}\left(z-\frac{1}{j\lambda_n}\right)^{u_l}\\
= \;& j\lambda_n\left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right)^k\exp\left(-\frac{T}{\lambda_n}\right)\sum_{v=0}^{\infty}D_v(k,m,n)\left(z-\frac{1}{j\lambda_n}\right)^v
\end{aligned}$$

where

$$\begin{aligned}
&D_v(k,m,n) && (99)\\
= \;&\sum_{r+s+t+\sum_{l\neq n}u_l=v}\binom{r+k-1}{k-1}\left(j\frac{\lambda_m\lambda_n}{\lambda_n-\lambda_m}\right)^r(-j\lambda_n)^s\frac{(-jT)^t}{t!}\\
&\times\prod B_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{j\lambda_l\lambda_n}{\lambda_n-\lambda_l}\right)^{u_l}\\
= \;&(-j)^v\lambda_n^v F_v(k,m,n)
\end{aligned}$$

and

$$F_v(k,m,n) \tag{100}$$

$$
\begin{aligned}
&= \sum_{s=0}^{v} \sum_{r+t+\sum_{l\neq n} u_l = v-s} \binom{r+k-1}{k-1} \left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \times \\
&\qquad \prod B_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\
&= \sum_{s=0}^{v} \sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \sum_{r+\sum_{l\neq n} u_l = v-s-t} \binom{r+k-1}{k-1} \left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r \times \\
&\qquad \prod B_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\
&= e^{\sum_{l\neq n} \frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}} \sum_{s=0}^{v} \sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \sum_{r+\sum_{l\neq n} u_l = v-s-t} \binom{r+k-1}{k-1} \left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r \\
&\qquad \times \prod P_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l} \\
&= e^{\sum_{l\neq n} L\beta_l}\, e^{\sum_{l\neq n} \frac{L\beta_l\lambda_l}{\lambda_n-\lambda_l}} \sum_{s=0}^{v} \sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} \sum_{r+\sum_{l\neq n} u_l = v-s-t} \binom{r+k-1}{k-1} \left(\frac{\lambda_m}{\lambda_m-\lambda_n}\right)^r \\
&\qquad \times \prod P_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_n}\right)^{u_l}
\end{aligned}
$$

Substituting (98) into (93), we finally obtain, for $n \neq m$,

$$
\begin{aligned}
&\int_{\mathcal{C}_n} \left(\frac{1}{1-jz\lambda_m}\right)^k \left(\exp \sum_{l=1}^{N} \frac{L\beta_l}{1-jz\lambda_l}\right) \frac{e^{-jzT}}{z}\, dz \qquad (101) \\
&= -\sum_{p=1}^{\infty} \frac{1}{p!} \left(\frac{L\beta_n}{-j\lambda_n}\right)^p \left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right)^k \left(\exp\left(-\frac{T}{\lambda_n}\right)\right) \times \\
&\qquad \sum_{v=0}^{\infty} (-j\lambda_n)^{1+v}\; F_v(k,m,n) \int_{\mathcal{C}_n} \left(z-\frac{1}{j\lambda_n}\right)^v \frac{1}{\left(z-\frac{1}{j\lambda_n}\right)^p} dz \\
&= 2\pi j \left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right)^k \left(\exp -\frac{T}{\lambda_n}\right) \sum_{p=1}^{\infty} \frac{1}{p!} (L\beta_n)^p F_{p-1}(k,m,n)
\end{aligned}
$$

It remains to compute

$$
\int_{\mathcal{C}_m} \left(\frac{1}{1-jz\lambda_m}\right)^k \left(\exp \sum_{l=1}^{N} \frac{L\beta_l}{1-jz\lambda_l}\right) \frac{e^{-jzT}}{z}\, dz \qquad (102)
$$

We have

$$\frac{1}{z} = j\lambda_m \sum_{s=0}^{\infty} (-j\lambda_m)^s \left(z - \frac{1}{j\lambda_m}\right)^s \tag{103}$$

and

$$e^{-jzT} = \exp\left(-\frac{T}{\lambda_m}\right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_m}\right)^t \tag{104}$$

From (64) it follows that, for any $l \neq m$,

$$\exp\left(\frac{L\beta_l}{1 - jz\lambda_l}\right) = \sum_{u=0}^{\infty} B_u \left(\frac{L\beta_l\lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{j\lambda_l\lambda_m}{\lambda_m - \lambda_l}\right)^u \left(z - \frac{1}{j\lambda_m}\right)^u \tag{105}$$

The preceding identities imply that

$$\begin{aligned}
& \left(\frac{1}{1 - jz\lambda_m}\right)^k \left(\exp \sum_{l \neq m} \frac{L\beta_l}{1 - jz\lambda_l}\right) \frac{e^{-jzT}}{z} \\
= \; & \left(\frac{1}{-j\lambda_m}\right)^k \left(\frac{1}{z - \frac{1}{j\lambda_m}}\right)^k \times \left(j\lambda_m \sum_{s=0}^{\infty} (-j\lambda_m)^s \left(z - \frac{1}{j\lambda_m}\right)^s\right) \times \\
& \left(\exp\left(-\frac{T}{\lambda_m}\right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_m}\right)^t\right) \times \\
& \prod_{l \neq m} \sum_{u_l=0}^{\infty} B_{u_l} \left(\frac{L\beta_l\lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{j\lambda_l\lambda_m}{\lambda_m - \lambda_l}\right)^{u_l} \left(z - \frac{1}{j\lambda_m}\right)^{u_l} \\
= \; & \frac{-\exp\left(-\frac{T}{\lambda_m}\right)}{(-j\lambda_m)^{k-1}} \sum_{v=0}^{\infty} D_v^*(m) \left(z - \frac{1}{j\lambda_m}\right)^{v-k}
\end{aligned} \tag{106}$$

where

$$\begin{aligned}
& D_v^*(m) \\
= \; & \sum_{s+t+\sum_{l \neq m} u_l = v} (-j\lambda_m)^s \frac{(-jT)^t}{t!} \prod B_{u_l} \left(\frac{L\beta_l\lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{j\lambda_l\lambda_m}{\lambda_m - \lambda_l}\right)^{u_l} \\
= \; & (-j)^v \sum_{s+t+\sum_{l \neq m} u_l = v} \lambda_m^s \frac{T^t}{t!} \prod B_{u_l} \left(\frac{L\beta_l\lambda_m}{\lambda_m - \lambda_l}\right) \left(\frac{\lambda_l\lambda_m}{\lambda_l - \lambda_m}\right)^{u_l} \\
= \; & (-j)^v \lambda_m^v G_v(m)
\end{aligned} \tag{107}$$

and

$$G_v(m) = \sum_{s+t+\sum_{l\neq m} u_l = v} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} \prod B_{u_l}\left(\frac{L\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l} \tag{108}$$

$$= \sum_{s=0}^{v}\sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} \sum_{\sum_{l\neq m} u_l = v-s-t} \prod B_{u_l}\left(\frac{L\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l}$$

$$= e^{\sum_{l\neq m} L\beta_l}\, e^{\sum_{l\neq m}\frac{L\beta_l\lambda_l}{\lambda_m-\lambda_l}} \times$$

$$\sum_{s=0}^{v}\sum_{t=0}^{v-s} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} \sum_{\sum_{l\neq m} u_l = v-s-t} \prod P_{u_l}\left(\frac{L\beta_l\lambda_m}{\lambda_m-\lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l-\lambda_m}\right)^{u_l}$$

Combining (106)-(108) yields

$$\int_{\mathcal{C}_m}\left(\frac{1}{1-jz\lambda_m}\right)^k\left(\exp\sum_{l=1}^{N}\frac{L\beta_l}{1-jz\lambda_l}\right)\frac{e^{-jzT}}{z}\,dz \tag{109}$$

$$= \frac{-\exp\left(-\frac{T}{\lambda_m}\right)}{(-j\lambda_m)^{k-1}} \times \int_{\mathcal{C}_m}\left(\sum_{v=0}^{\infty}(-j\lambda_m)^v G_v(m)\left(z-\frac{1}{j\lambda_m}\right)^{v-k}\right)\times$$

$$\left(\sum_{p=0}^{\infty}\left(\frac{L\beta_m}{-j\lambda_m}\right)^p\left(z-\frac{1}{j\lambda_m}\right)^{-p}\right)dz$$

$$= -\exp\left(-\frac{T}{\lambda_m}\right)\times\int_{\mathcal{C}_m}\left(\sum_{v=0}^{\infty}(-j\lambda_m)^{v-k+1}G_v(m)\left(z-\frac{1}{j\lambda_m}\right)^{v-k}\right)\times$$

$$\left(\sum_{p=0}^{\infty}\frac{1}{p!}\left(\frac{L\beta_m}{-j\lambda_m}\right)^p\left(z-\frac{1}{j\lambda_m}\right)^{-p}\right)dz$$

$$= 2\pi j\exp\left(-\frac{T}{\lambda_m}\right)\sum_{p=0}^{\infty}\frac{1}{p!}(L\beta_m)^p\; G_{p+k-1}(m)$$

Substituting (101) and (109) into (91) and (90), we obtain:

$$P_d = e^{-L\sum_{m=1}^{N}\beta_m} \times \tag{110}$$

$$\sum_{m=1}^{N}\sum_{k=1}^{L} A_{m,k}\; \exp\left(-\frac{T}{\lambda_m}\right)\sum_{p=0}^{\infty}\frac{1}{p!}(L\beta_m)^p\; G_{p+k-1}(m)$$

$$+e^{-L\sum_{m=1}^{N}\beta_m} \times$$

$$\sum_{m=1}^{N}\sum_{k=1}^{L} A_{m,k}\sum_{n\neq m}\left(\frac{\lambda_n}{\lambda_n-\lambda_m}\right)^k\left(\exp-\frac{T}{\lambda_n}\right)\sum_{p=1}^{\infty}\frac{1}{p!}(L\beta_n)^p F_{p-1}(k,m,n)$$

$$= \sum_{m=1}^{N} e^{\sum_{l\neq m} \frac{L\beta_l \lambda_l}{\lambda_m - \lambda_l}} \sum_{k=1}^{L} A_{m,k} \sum_{p=0}^{\infty} \frac{1}{p!}(L\beta_m)^p e^{-L\beta_m} \times$$

$$\sum_{s=0}^{p+k-1} \sum_{t=0}^{p+k-1-s} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!} e^{-\frac{T}{\lambda_m}} \sum_{\sum_{l\neq m} u_l = p+k-1-s-t} \prod P_{u_l}\left(\frac{L\beta_l\lambda_m}{\lambda_m - \lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l - \lambda_m}\right)^{u_l}$$

$$+ \sum_{m=1}^{N}\sum_{k=1}^{L} A_{m,k} \sum_{n\neq m} \left(\frac{\lambda_n}{\lambda_n - \lambda_m}\right)^k e^{\sum_{l\neq n} \frac{L\beta_l\lambda_l}{\lambda_n - \lambda_l}} \sum_{p=1}^{\infty} \frac{1}{p!}(L\beta_n)^p e^{-L\beta_n} \times$$

$$\sum_{s=0}^{p-1}\sum_{t=0}^{p-1-s} \frac{\left(\frac{T}{\lambda_n}\right)^t}{t!} e^{-\frac{T}{\lambda_n}} \sum_{r+\sum_{l\neq n} u_l = p-1-s-t} \binom{r+k-1}{k-1} \left(\frac{\lambda_m}{\lambda_m - \lambda_n}\right)^r \times$$

$$\prod P_{u_l}\left(\frac{L\beta_l\lambda_n}{\lambda_n - \lambda_l}\right)\left(\frac{\lambda_l}{\lambda_l - \lambda_n}\right)^{u_l}$$

6 Probability of Detection ($\gamma = 0, K = N = 1, L \geq 1$)

In this section, we compute the probability of detection P_d for the FFT summation detector under the assumption that the received signal is a complex pure tone embedded in additive white Gaussian noise and $\gamma = 0$, $K = N = 1$, $L \geq 1$. Since $\gamma = 0$, the data blocks are non-overlapped and it follows from (18), (33), (47) and (53) that the characteristic function $\phi(t)$ is given by

$$\phi(t) = \gamma_0^L \left(\frac{1}{1 - jt\lambda_1}\right)^L e^{\frac{L\beta_1}{1-jt\lambda_1}} \tag{111}$$

where $\lambda_1 = \sigma^2 \sum_{l=0}^{M-1} w_l^2$ is the single eigenvalue of the 1×1 Hermitian matrix $\mathbf{A}$ defined by (21) and (22), $\gamma_0 = e^{-\beta_1}$, $\beta_1 = \frac{|\alpha_1|^2}{\lambda_1}$ and $\alpha_1 = A \sum_{l=0}^{M-1} w_l \exp(-2\pi jl(f - f_n))$ with $f_n = \frac{n}{M}$. Here the integer n, $0 \leq n \leq M-1$, denotes the channel index (or bin index) since only one FFT bin is assigned to each channel ($K = N = 1$). As in the previous two sections, it can be shown that

$$\begin{aligned} P_d &= \int_T^\infty p(x)\,dx = \frac{1}{2\pi}\int_T^\infty \int_{-\infty}^{\infty} \phi(t) e^{-jxt} dt\,dx \quad (112) \\ &= \frac{1}{2\pi}\int_T^\infty \int_{\mathcal{L}} \phi(z) e^{-jxz} dz\,dx = \frac{1}{2\pi}\int_{\mathcal{L}}\int_T^\infty \phi(z)e^{-jxz}dx\,dz \\ &= \frac{1}{2\pi j}\int_{\mathcal{L}} \gamma_0^L \left(\frac{1}{1-jz\lambda_1}\right)^L e^{\frac{L\beta_1}{1-jz\lambda_1}} \frac{e^{-jzT}}{z} dz \end{aligned}$$

We have

$$\frac{1}{z} = j\lambda_1 \sum_{s=0}^{\infty} (-j\lambda_1)^s \left(z - \frac{1}{j\lambda_1}\right)^s \tag{113}$$

and

$$e^{-jzT} = \exp\left(-\frac{T}{\lambda_1}\right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_1}\right)^t \tag{114}$$

It follows that

$$\begin{aligned}
& \left(\frac{1}{1 - jz\lambda_1}\right)^L e^{\frac{L\beta_1}{1-jz\lambda_1}} \frac{e^{-jzT}}{z} && (115)\\
= & \left(\frac{1}{-j\lambda_1}\right)^L \left(z - \frac{1}{j\lambda_1}\right)^{-L} \left[\sum_{p=0}^{+\infty} \frac{1}{p!} \left(\frac{L\beta_1}{1 - jz\lambda_1}\right)^p\right] \times \\
& j\lambda_1 \sum_{s=0}^{\infty} (-j\lambda_1)^s \left(z - \frac{1}{j\lambda_1}\right)^s \times \exp\left(-\frac{T}{\lambda_1}\right) \sum_{t=0}^{\infty} \frac{(-jT)^t}{t!} \left(z - \frac{1}{j\lambda_1}\right)^t \\
= & -\exp\left(-\frac{T}{\lambda_1}\right) (-j\lambda_1)^{-L+1} \left[\sum_{p=0}^{+\infty} \frac{1}{p!} \left(\frac{L\beta_1}{-j\lambda_1}\right)^p \left(z - \frac{1}{j\lambda_1}\right)^{-p-L}\right] \times \\
& \sum_{v=0}^{+\infty} \left[\sum_{s+t=v} (-j\lambda_1)^s \frac{(-jT)^t}{t!}\right] \left(z - \frac{1}{j\lambda_1}\right)^v
\end{aligned}$$

and hence

$$\begin{aligned}
P_d &= \frac{\gamma_0^L}{2\pi j} \int_{\mathcal{L}} \left(\frac{1}{1 - jz\lambda_1}\right)^L e^{\frac{L\beta_1}{1-jz\lambda_1}} \frac{e^{-jzT}}{z} dz && (116)\\
&= \gamma_0^L \exp\left(-\frac{T}{\lambda_1}\right) \sum_{p=0}^{+\infty} \frac{(L\beta_1)^p}{p!} \sum_{s+t=p+L-1} \frac{\left(\frac{T}{\lambda_1}\right)^t}{t!} \\
&= \sum_{p=0}^{+\infty} \frac{(L\beta_1)^p}{p!} e^{-L\beta_1} \sum_{t=0}^{p+L-1} \frac{\left(\frac{T}{\lambda_1}\right)^t}{t!} e^{-\frac{T}{\lambda_1}}
\end{aligned}$$

where

$$\beta_1 = \frac{|\alpha_1|^2}{\lambda_1} = \frac{\left|A \sum_{l=0}^{M-1} w_l \exp\left(-2\pi jl(f - f_n)\right)\right|^2}{\sigma^2 \sum_{l=0}^{M-1} w_l^2} \tag{117}$$

We remark here that A is the amplitude of the complex pure tone defined in section 4 in the paragraph before (43) and $f_n = \frac{n}{M}$ is the normalized center frequency of the n-th channel, $0 \leq n \leq M - 1$.

7 Probability of False Alarm ($0 < \gamma \leq 1/2, N \geq 1, L \geq 1$)

For a given threshold $T > 0$, the corresponding probability of false alarm P_{fa} for the FFT summation detector with $0 < \gamma \leq 1/2$ can be immediately obtained from the formula (86). When the amplitude A of the complex pure tone $s(t) = A\exp(2\pi jfF_st)$ vanishes (F_s is the sampling frequency), the received signal samples r_k contain white Gaussian noise only. In this case, the formula (86) actually computes the probability of false alarm P_{fa} for a given threshold T and the α sequence defined by (52) and the β sequence defined by $\beta_l = \frac{|\alpha_l|^2}{\lambda_l}$, $1 \leq l \leq LN$, both vanish. It follows from the formula (86) that for a given threshold $T > 0$, the corresponding probability of false alarm P_{fa} for the FFT summation detector with $0 < \gamma \leq 1/2$ is given by

$$P_{fa} = \sum_{l=1}^{LN} A_l e^{-\frac{T}{\lambda_l}} \tag{118}$$

where λ_l, $1 \leq l \leq LN$, are the LN distinctive eigenvalues of the covariance matrix $\mathbf{H} = E\left(\mathbf{Z}_n\mathbf{Z}_n^H\right)$ given by (17) and $A_l = \frac{\lambda_l^{LN-1}}{\prod_{m\neq l}(\lambda_l-\lambda_m)}$, $1 \leq l \leq LN$. Note that the formula (118) was derived in [13] via a different method.

8 Probability of False Alarm ($\gamma = 0, N > 1, L > 1$)

Using basically the same argument as in the previous section, we can derive the formula for the probability of false alarm P_{fa} for the FFT summation detector with $\gamma = 0, N > 1, L > 1$. In fact, it follows from (110) that, for a given threshold $T > 0$, the probability of false alarm P_{fa} for the FFT summation detector with $\gamma = 0, N > 1, L > 1$ is given by

$$P_{fa} = \sum_{m=1}^{N}\sum_{k=1}^{L} A_{m,k} \sum_{t=0}^{k-1} \frac{\left(\frac{T}{\lambda_m}\right)^t}{t!}\; e^{-\frac{T}{\lambda_m}} \tag{119}$$

where the constants $A_{m,k}$, defined by (89), are reproduced here for easy reference

$$\begin{cases} A_{m,L} = \left(\dfrac{\lambda_m^{N-1}}{\prod\limits_{l \neq m} (\lambda_m - \lambda_l)} \right)^L , \ 1 \leq m \leq N , \\ A_{m,k} = A_{m,L} \sum\limits_{\sum_{l \neq m} k_l = L-k} \prod \left(\begin{array}{c} L + k_l - 1 \\ k_l \end{array} \right) \left(\dfrac{\lambda_l}{\lambda_l - \lambda_m} \right)^{k_l} , \\ \qquad 1 \leq k \leq L-1 \ , 1 \leq m \leq N \ . \end{cases} \tag{120}$$

Here the numbers λ_l, $1 \leq l \leq N$, are the N distinct eigenvalues of the Hermitian matrix $\mathbf{A}$ computed by (21).

9 Probability of False Alarm ($\gamma = 0, K = N = 1, L \geq 1$)

Finally, from (116) it follows that for a given threshold $T > 0$, the probability of false alarm P_{fa} for the FFT summation detector for $\gamma = 0, K = N = 1, L \geq 1$ is given by

$$P_{fa} = \sum_{t=0}^{L-1} \frac{\left(\frac{T}{\lambda_1}\right)^t}{t!} e^{-\frac{T}{\lambda_1}} \tag{121}$$

where

$$\lambda_1 = \sigma^2 \sum_{l=0}^{M-1} w_l^2$$

We remark here that the formula (121) was also derived in [3] and [13], using different techniques.

10 Summary

We have presented a full derivation of formulas for the probability of detection P_d and the probability of false alarm P_{fa} for the FFT filter bank-based summation detector when the received signal is a complex pure tone embedded in additive white Gaussian noise. Specifically, the formulas for the probability of detection P_d are given in the identities (86), (110), (116) and the formulas for the probability of false alarm P_{fa} are given in the identities (118), (119), (121). The formulas (118) and (121) were derived in [13] by different approaches but they are presented here as corollaries of formulas for P_d for the sake of completeness. The results presented in this report provide the basis for evaluating the performance of the FFT summation detector for different scenarios.

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