# On the Difference Between Methods of Calculating the Neumann-Kelvin Wave Resistance

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## ABSTRACT

In the Neumann-Kelvin formulation of flow around a ship hull with a free surface, one can calculate the wave resistance using two different methods: Havelock's formula, which is an estimate of the resistance obtained from the radiated waves in the far field; and integration of the pressure over the hull. For a typical vessel, there is usually a discrepancy of the order of 20% in the two predictions.

An analysis of the momentum flux through a control volume enclosing the hull shows that the linearized free surface boundary condition allows the leakage of momentum through the free surface. Some of the momentum imparted to the fluid by the hull is lost through the free surface, so the resistance calculated using the Havelock formula is lower, for Froude numbers less than about 0.5, than the resistance calculated by integrating the pressure over the hull. It is not possible to say which method yields the more accurate predictions.

## 1 INTRODUCTION

In the Neumann-Kelvin formulation of flow around a ship hull with a free surface, one can calculate the wave resistance using two different methods: Havelock's formula, which is an estimate of the resistance obtained from the radiated waves in the far field; and integration of the pressure over the hull. For a typical vessel, there is usually a discrepancy of the order of 20% in the two predictions. This paper discusses the origin of this discrepancy.

We use a Cartesian coordinate system in which $x$ increases from bow to stern and $z$ is zero at the undisturbed free surface and increases upwards: see Fig. 1.

Incompressible potential flow is assumed:

$$\mathbf{V} = U\hat{x} + \mathbf{v}; \qquad \mathbf{v} = \nabla\phi; \qquad \nabla^2\phi = 0 \tag{1}$$

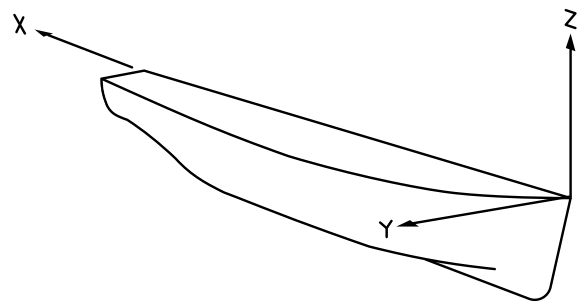


Figure 1: The coordinate system.

The flow is uniform at infinity: $|\mathbf{v}| \longrightarrow 0$ as $|\mathbf{x}| \longrightarrow \infty$. The pressure is obtained from Bernoulli's equation:

$$P + \tfrac{1}{2}\rho V^2 + \rho g z = \tfrac{1}{2}\rho U^2 \tag{2}$$

At the free surface the pressure is zero. There is no flux of fluid through the hull or the free surface.

The wave resistance is obtained by integrating the pressure over the wetted hull surface, $H$:

$$R = \int_H P n_x \, da \tag{3}$$

where $n_x$ is the $x$ component of a unit normal to the hull pointing into the hull. Havelock[5] showed that the resistance could also be evaluated from the energy in the waves in the far field. Let $S_d$ be any $yz$ plane downstream of the hull and let its intersection with the free surface be denoted by $D$. Then

$$R = \frac{\rho}{2}\int_{S_d} (v_y^2 + v_z^2 - v_x^2)\, da + \frac{\rho g}{2}\int_D h^2 \, dy \tag{4}$$

where $h(x,y)$ is the height of the free surface at $(x,y)$. Given the assumptions above, this equation is exact. However, when the free surface condition is linearized about the free stream, the Neumann-Kelvin approximation, then one finds that these two expressions for the resistance no longer match.

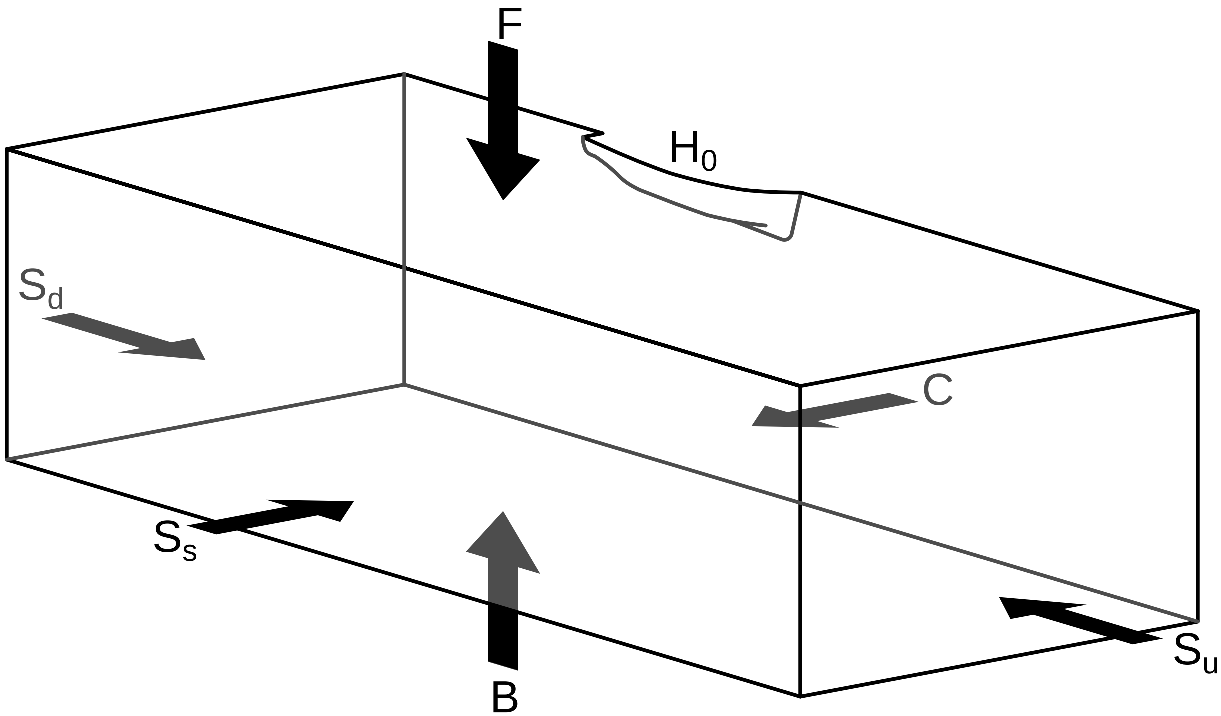


Figure 2: The control volume around the hull.

## 2 The Neumann-Kelvin Formulation

Let $\varepsilon$ be some small dimensionless quantity such that

$$\frac{|\mathbf{v}|}{U} \sim \varepsilon; \qquad |\nabla h| \sim \varepsilon \tag{5}$$

The first approximation states that the flow does not deviate much from the free stream, the second that the waves generated are not steep.

At the free surface the pressure is zero. Therefore, from Eq. (2):

$$k_0 h = -\frac{v_x}{U} + o(\varepsilon^2) \tag{6}$$

where the wavenumber $k_0$ is defined by $k_0 = g/U^2$. It follows that $k_0 h \sim \varepsilon$.

Since the free surface is a no-flux boundary:

$$\mathbf{V} \cdot \hat{n} = 0 \tag{7}$$

where $\hat{n}$ is a unit normal to the free surface. Since

$$\hat{n} = \frac{-\frac{\partial h}{\partial x}\hat{x} - \frac{\partial h}{\partial y}\hat{y} + \hat{z}}{\sqrt{\left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2 + 1}} = -\frac{\partial h}{\partial x}\hat{x} - \frac{\partial h}{\partial y}\hat{y} + \hat{z} + o(\varepsilon^2) \tag{8}$$

the no-flux condition becomes

$$\frac{\partial h}{\partial x} = \frac{v_z}{U} + o(\varepsilon^2) \tag{9}$$

In the Neumann-Kelvin approximation the flow is solved as if Eqs. (6) and (9) were exact:

$$k_0 h = -\frac{v_x}{U} \tag{10}$$

$$\frac{\partial h}{\partial x} = \frac{v_z}{U} \tag{11}$$

and that these boundary conditions are applied at the undisturbed free surface, $z = 0$. Eq. (10) becomes the *definition* of the free surface height.

Eliminating $h$ between Eqs. (10) and (11) one gets the linearized free surface boundary condition:

$$\frac{\partial^2 \phi}{\partial x^2} + k_0 \frac{\partial \phi}{\partial z} = 0 \tag{12}$$

## 3 Balance of Forces

We now derive the Havelock equation for resistance under the Neumann-Kelvin approximation. Imagine the hull enclosed by a large rectangular box with surfaces $F$ (the undisturbed free surface), $S_u$ (upstream face), $S_d$ (downstream face), $S_s$ (the side face), $C$

(the face in the centreplane), and $B$ (the bottom face). The remaining boundary is the hull surface below the undisturbed free surface, $H_0$. The surfaces are shown in Fig. 2.

On $F$ the linearized free surface condition holds. It is important to note that in this approximation one does not have zero flux through $F$ nor is $P = 0$; from Eqs. (6) and (9), $P$ and the flux through $F$ are both of order $\varepsilon^2$.

From the equation of continuity

$$\int_S \mathbf{V}\cdot\hat{n}\,da = 0 \tag{13}$$

where the integral over $S$ denotes the integral over all surfaces and $\hat{n}$ is a unit normal pointing out of the fluid.

The fluid must also obey Euler's equation for incompressible steady flow:

$$\rho\nabla\cdot(\mathbf{V}\mathbf{V}) = -\nabla P \tag{14}$$

Integrating over the fluid volume one obtains the equation for conservation of momentum:

$$\rho\int_S (\mathbf{V}\cdot\hat{n})\mathbf{V}\,da = -\int_S P\hat{n}\,da \tag{15}$$

where $\hat{n}$ is an outward pointing normal on the boundaries. Using Eq. (13) one has:

$$\begin{aligned} 0 &= \int_S \left[\rho(\mathbf{V}\cdot\hat{n})\mathbf{V} + P\hat{n}\right] da \\ &= \int_S \left[\rho(\mathbf{V}\cdot\hat{n})(U\hat{x}+\mathbf{v}) + P\hat{n}\right] da \\ &= \rho U\hat{x}\int_S (\mathbf{V}\cdot\hat{n})\,da + \int_S \left[\rho(\mathbf{V}\cdot\hat{n})\mathbf{v} + P\hat{n}\right] da \\ &= \int_S \left[\rho(\mathbf{V}\cdot\hat{n})\mathbf{v} + P\hat{n}\right] da \end{aligned} \tag{16}$$

Now consider the $x$-component of Eq. (16) as the box is extended to infinity. The pressure contribution on $F$, $S_s$, $C$, and $B$ are zero since the normals to these surfaces have no $x$-component. The momentum flux term on $S_u$, $S_s$, $C$, and $B$ are also zero since $v_x = 0$ on these surfaces as the box extends to infinity. On the hull, $\mathbf{V}\cdot\hat{n} = 0$, so that the momentum term also vanishes on $H_0$. Therefore, the conservation of momentum can be written

$$\begin{aligned} \tfrac{1}{2}R_P &\equiv \int_{H_0} P n_x\,da \\ &= \int_{S_u} P\,da - \rho\int_F V_z v_x\,da - \int_{S_d}\left[\rho V_x v_x + P\right] da \end{aligned} \tag{17}$$

where $R_P$ is the resistance obtained by integrating the pressure over the portion of the hull below the undisturbed free surface; the factor of one half accounts for the fact that $H_0$ includes only the portion of the hull on one side of the centreplane. Using Bernoulli's equation to substitute for $P$ one gets

$$\begin{aligned} \tfrac{1}{2}R_P = &-\rho g\int_{S_u} z\,da - \rho\int_F V_z v_x\,da \\ &- 2\rho\int_{S_d}\left[V_x v_x + \tfrac{1}{2}\left(U^2 - V^2\right) - gz\right] da \end{aligned} \tag{18}$$

Because the box is rectangular, the integrals of $z$ over $S_u$ and $S_d$ cancel:

$$\tfrac{1}{2}R_P = -\rho\int_{S_d}\left[V_x v_x + \tfrac{1}{2}\left(U^2 - V^2\right)\right] da - \rho\int_F V_z v_x\,da \tag{19}$$

Substituting $\mathbf{V} = U\hat{x} + \mathbf{v}$ one gets

$$\tfrac{1}{2}R_P = \tfrac{1}{2}\rho\int_{S_d}(v_y^2 + v_z^2 - v_x^2)\,da - \rho\int_F v_z v_x\,da \tag{20}$$

Using Eqs. (10) and (11) to replace $v_x$ and $v_z$ in the last term by $h$ we get:

$$\begin{aligned} \tfrac{1}{2}R_P &= \frac{\rho}{2}\int_{S_d}(v_y^2 + v_z^2 - v_x^2)\,da + \rho g\int_F h\frac{\partial h}{\partial x}\,da \\ &= \frac{\rho}{2}\int_{S_d}(v_y^2 + v_z^2 - v_x^2)\,da + \frac{\rho g}{2}\int_F \frac{\partial h^2}{\partial x}\,da \end{aligned} \tag{21}$$

Using Stokes' Theorem the integral over $F$ can be expressed as integrals along the boundaries of $F$:

$$\int_F \frac{\partial h^2}{\partial x}\,da = \int_F \hat{n}\cdot\nabla\times(h^2\hat{y})\,da = \oint_{\partial F} h^2\,dy \tag{22}$$

where $\partial F$ denotes the boundary of $F$. This curve must be traversed counterclockwise relative to the normal to $F$ ($\hat{z}$). The portions of $\partial F$ lying in the centreplane and the side boundary vanish since $dy$ will vanish. The portion of $\partial F$ in $S_u$ will also vanish as the box is extended to infinity. Therefore the only contribution from the line integral is the portion along the (undisturbed) waterline of the hull (denoted by $W$) and the portion in $S_d$ (denoted by $D$). Moreover, on $W$, we have $dy = n_x d\ell$, where $d\ell$ is an increment of arclength along the waterline. Therefore

$$\int_F \frac{\partial h^2}{\partial x}\,da = \int_D h^2\,dy + \int_W h^2 n_x\,d\ell \tag{23}$$

Substituting Eq. (23) into Eq. (21) one gets:

$$\tfrac{1}{2}R_P = \frac{\rho}{2}\int_{S_d}(v_y^2+v_z^2-v_x^2)\,da + \frac{\rho g}{2}\int_D h^2\,dy + \frac{\rho g}{2}\int_W h^2 n_x\,d\ell \tag{24}$$

The first two terms on the right hand side of Eq. (24) are Havelock's formula for the wave resistance, Eq. (4), except that the integral over $S_d$ is now bounded by the undisturbed free surface and the centreplane. If we define

$$E_P \equiv \rho g \int_W h^2 n_x\,d\ell \tag{25}$$

then

$$R_P = R_H + E_P \tag{26}$$

In all practical cases, at Froude numbers less than about 0.5, $E_P$ is positive. This is because its major contribution is from the bow wave where $n_x$ is positive; the stern wave does not contribution in the same way because most of it occurs astern of the hull. Therefore, in the Neumann-Kelvin approximation, the resistance predicted by Havelock's formula is consistently less than the resistance predicted by integration of the pressure over the hull.

## 4 Evaluating $R_H$ Using the Kochin Function

When the potential flow around a ship is calculated using Kelvin-Havelock sources, the velocity potential, $\phi$, can be expressed by:

$$\phi(x,y,z) = \int_{H_0}\sigma(x',y',z')G(x,y,z;x',y',z')\,da + \frac{1}{k_0}\oint_C \sigma(x',y',z')G(x,y,z;x',y',z')n_x\,dy \tag{27}$$

where $\sigma$ is a distribution of sources on the hull and $G$ is the Green function which satisfies the linearized free surface condition (see Hally[3], for example). The source strengths are normalized so that the flux through any surface enclosing a source of unit strength is $4\pi$. Substitution into the Havelock resistance formula, Eq. (4), yields the following well-known formula for the wave resistance:

$$R_H = 16\pi\rho k_0^2 \int_0^{\pi/2} \sec^3\theta\,|H(\theta)|^2\,d\theta \tag{28}$$

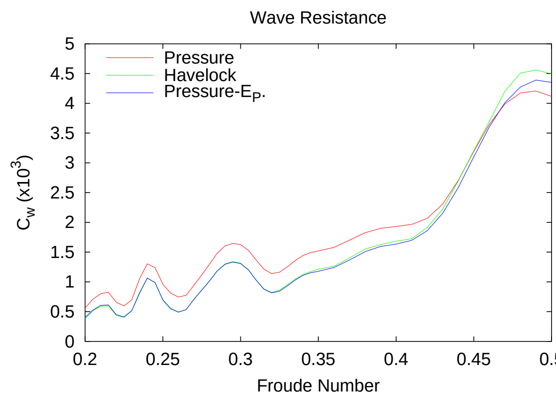


Figure 3: Comparison of the resistance of a Wigley hull calculated using integration of the pressure on the hull and Havelock's formula.

where $H(\theta)$ is the Kochin function:

$$H(\theta) = \int_{H_0}\sigma e^{-iu}e^{k_0 z\sec^2\theta}\,da + \frac{1}{k_0}\int_W \sigma n_x e^{-iu}e^{k_0 z\sec^2\theta}\,dy \tag{29}$$

## 5 Numerical Confirmation

To test the validity of Eq. (26), the flow around a Wigley hull was calculated using the potential flow program POTFLO[2], developed at DRDC Atlantic. The half hull was covered with 800 panels of roughly equal size: 40 from bow to stern and 20 from keel to waterline. Eq. (28) was used to evaluate $R_H$ and compared to the integration of the pressure over the panels on the hull. The comparison is shown in Fig. 3 for a range of Froude numbers. It can be seen that $R_P - E_P$ matches $R_H$ almost exactly at all Froude numbers less than about 0.45.

## 6 Physical Interpretation of $E_P$

We can give the term $E_P$ a physical meaning by examining the error incurred in the resistance calculation by integrating the pressure over the undisturbed free surface. If the hull is nearly wall sided, we can approximate this error as follows:

$$\begin{aligned}\tfrac{1}{2}\Delta R_P &\equiv \left(\int_H - \int_{H_0}\right) P n_x\,da \\ &\approx \int_W\int_0^h P n_x\,dz\,d\ell \\ &= \rho\int_W\int_0^h \left[\tfrac{1}{2}(U^2-V^2) - gz\right] n_x\,dz\,d\ell\end{aligned} \tag{30}$$

The integration over $z$ can be done if we assume that the velocity is nearly constant for $z$ from 0 to $h$:

$$\tfrac{1}{2}\Delta R_P \approx \frac{\rho}{2}\int_W \left[(U^2 - V^2)n_x h - g n_x h^2\right] d\ell \tag{31}$$

Since

$$U^2 - V^2 = -2Uv_x + o(U^2\varepsilon^2) = 2gh + o(U^2\varepsilon^2) \tag{32}$$

we have

$$\tfrac{1}{2}\Delta R_P \approx \frac{\rho}{2}\int_W g n_x h^2\, d\ell = \tfrac{1}{2}E_P \tag{33}$$

Therefore $E_P$ is an approximation of the error in the resistance incurred by integrating only over the hull below the undisturbed free surface. Notice that $E_P$ is of order $\rho L U^4 \varepsilon^3/g$.

## 7 Which is More Accurate?

The analysis of the previous section suggests that an accurate estimate of the wave resistance would be

$$R = R_P + E_P = R_H + 2E_P \tag{34}$$

but this is not necessarily the case. While $E_P$ is a good estimate of the error incurred by integrating only over the hull below the undisturbed free surface, $R_P$ contains errors which are of the same order as $E_P$ due to the use of the Neumann-Kelvin approximation.

The mismatch between $R_P$ and $R_H$ occurs because the Neumann-Kelvin approximation, by allowing a small flux of order $\varepsilon/U$ through the free surface, permits momentum to leak through the free surface. The total momentum lost is equal to $E_P$. If the free surface condition were exact, then the pressure distribution on the hull and the velocity distribution on $S_d$ would both be altered, such that the momentum flux on $H_0$ and $S_d$ would again balance. But it is not possible to say, without extensive further analysis, in what proportion the momentum fluxes over these two surface changes. Therefore, it cannot be said that one or the other is more accurate. Moreover, suggesting that $R_P + E_P$ is more accurate that $R_P$ is inconsistent, since it cannot be determined whether $R_P$ may contain offsetting errors that cancel $E_P$.

Fig. 4 shows the predicted resistance curves for the Wigley hull along with a shaded envelope of experimental data obtained from Rigby et al.[6]. In addition, the resistance calculated by a different program,

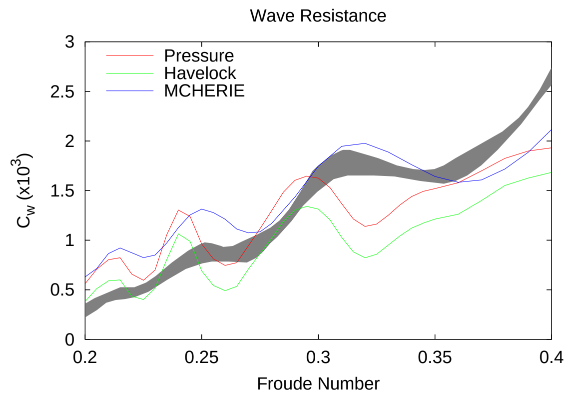


Figure 4: Comparison of different methods of calculating the resistance of a Wigley hull with experimental values (shaded envelope).

MCHERIE, is also shown. MCHERIE is a modification of the French program CHERIE[7] which implements Dawson's panel method[1]. In Dawson's method the free surface condition is linearized with respect to the potential flow at zero Froude number. The errors in the Dawson method are still proportional to $\varepsilon^3$ but, due to the different linearization used, these errors will differ from the errors in $R_P$ and $R_H$. Of course, there are also errors due to the panelization of the hull (and, in the case of MCHERIE, the free surface) but these are smaller than the error caused by the linearization of the free surface (for a discussion of the errors caused by the panelling of the free surface in MCHERIE, see Hally[4]).

Fig. 4 shows that, though MCHERIE is superior in the critical range $F_n = 0.25$–$0.35$, none of the three methods of calculating the resistance can claim to be accurate at all Froude numbers. Nevertheless, if we take the distance between the $R_P$ and $R_H$ curves to be indicative of an error proportional to $\varepsilon^3$, then we see that the distance of each curve from the experimental data is consistent with an error proportional to $\varepsilon^3$. We cannot expect more from a linear theory.

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