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2017 IEEE 56th Annual Conference on Decision and Control (CDC)

Conference Location: Melbourne, Australia
Pagination info: 82–87

Ext Publisher Publication Date: December 2017

Defence Research and Development Canada

External Literature (P)
DRDC-RDDC-2018-P026
March 2018

IMPORTANT INFORMATIVE STATEMENTS

2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

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Abstract— **The problem of distributed connectivity optimization of an asymmetric sensor network represented by a weighted directed graph (digraph) is investigated in this paper. The notion of generalized algebraic connectivity is used to measure the connectivity of a time-varying weighted digraph. The generalized algebraic connectivity is regarded as a nonconcave and nondifferentiable continuous cost function, and a distributed approach, based on the subspace consensus algorithm, is developed to compute the supergradient vector of the network connectivity. By considering the above-mentioned network connectivity as a function of the transmission power vector of the network, a discrete-time update procedure is proposed to compute a stationary transmission power vector of the network which locally maximizes the network connectivity. The effectiveness of the developed algorithm is subsequently demonstrated by simulations.**

I. Introduction

Ad-hoc networks are composed of a collection of fixed or mobile sensors capable of exchanging data without the support of a pre-existing infrastructure [1]. The convergence rate of cooperative algorithms used for various objectives such as consensus, target localization and parameter estimation over ad-hoc networks highly depends on the connectivity degree of the network [2]. Normally, a network with a higher degree of connectivity is able to diffuse information more effectively throughout the network [3]. The algebraic connectivity has been used as a measure of connectivity for symmetric networks. It is defined as the second smallest eigenvalue of the Laplacian matrix of the undirected graph, representing the network [4]. A distributed procedure is developed in [5] to estimate and control the algebraic connectivity of symmetric networks. The notion of generalized algebraic connectivity (GAC) is introduced in [6], which captures the expected asymptotic convergence rate of the continuous-time consensus algorithms running on an asymmetric network.

The problem of distributed optimization has been investigated in different practical contexts including parallel computation [7] and statistical estimation [8]. The consensus-based approaches have been proven effective in addressing the distributed optimization problem in a sensor network [9], [10]. A subgradient-based distributed method is used in [9] to optimize the sum of some convex objective functions corresponding to multiple agents. A distributed subgradient method for solving a constrained multi-agent optimization problem is developed in [10] which uses consensus as a mechanism for distributing the computations throughout the network. A supergradient algorithm is used in [11] to reconfigure a symmetric network such that the algebraic connectivity of the undirected graph representing the network is maximized in a distributed manner.

Motivated by applications in underwater acoustic sensor networks with asymmetric communication links [12], [13], [14], the problem of connectivity optimization of asymmetric sensor networks is investigated in this paper. The notion of GAC is considered in this work as an effective measure of the connectivity of weighted digraphs. This notion represents the expected convergence rate of the distributed algorithms running on the network [6]. The elements of the weight matrix of the network are characterized as continuous functions of the transmission power of the nodes as well as some parameters of the corresponding communication channels [13], [15]. The transmission power vector of the network is then updated in a distributed manner such that the GAC of the network is maximized. Since the GAC is described as a nonconcave and nondifferentiable function of the transmission power vector of the network, a distributed supergradient-based optimization approach is proposed which is guaranteed to converge to an optimal transmission power vector corresponding to a local maximum of the GAC. An estimation of the GAC of the time-varying asymmetric network is also computed based on a modified subspace consensus approach [6]. The efficacy of the proposed algorithm is verified by simulations.

The remainder of the paper is organized as follows. The problem of connectivity maximization in asymmetric networks is formulated in Section II by computing the optimal transmission power vector of the network in a centralized fashion. Section III introduces an iterative distributed algorithm to solve the connectivity optimization problem in a discrete-time scenario. The simulation results are subsequently presented in Section IV, and finally the concluding remarks are given in Section V.

II. Connectivity Optimization Problem

A. Notations

Throughout this paper, the set of positive real numbers is denoted by $\mathbb{R}_{>0}$. Moreover, $\mathbb{N}_n$ is the finite set of natural numbers $\{1, 2, \ldots, n\}$. The transpose and conjugate transpose of a complex vector $\mathrm{v} \in \mathbb{C}^n$ are denoted by v^T and v^H, respectively. The inner product of two real vectors $\mathrm{v}, \mathrm{w} \in \mathbb{R}^n$ is represented by $\langle \mathrm{v}, \mathrm{w} \rangle$. The real part, imaginary part, and magnitude of a complex number c are respectively denoted by $\Re(c)$, $\Im(c)$, and $|c|$. The all-one column vector of length n and $n \times n$ identity matrix are represented by 1_n and $\mathbf{I}_n$, respectively. The matrix $\mathrm{Diag}(\mathrm{v}) \in \mathbb{R}^{n \times n}$ is defined as a diagonal matrix with the elements of the vector $\mathrm{v} \in \mathbb{R}^n$ on its main diagonal. Let $\mathbb{B}_\epsilon(\mathrm{w})$ denote a closed ball of radius

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This work has been supported by PWGSC contract no. W7707-145674/001/HAL funded by Defence Research and Development Canada.

$\epsilon > 0$ centered at $\mathrm{w} \in \mathbb{R}^n$, which is defined as $\mathbb{B}_\epsilon(\mathrm{w}) = \{\mathrm{v} \in \mathbb{R}^n \,|\, \|\mathrm{v} - \mathrm{w}\| \leq \epsilon\}$. Given a real vector $\mathrm{v} \in \mathbb{R}^n$, the function $\Omega(\cdot) : \mathbb{R}^n \to \mathbb{R}$ is defined as $\Omega(\mathrm{v}) = v_{i^*}$, where $\mathrm{v} = [v_1 \cdots v_n]^\mathsf{T}$ and $i^* = \mathrm{argmax}_{i \in \mathbb{N}_n} |v_i|$. Moreover, $\mathrm{e}_i \in \mathbb{R}^n$ denotes a column vector whose elements are all zero except for its i^{th} element which is one. The trace and determinant of a square matrix $\mathbf{A}$ are represented by $\mathrm{tr}(\mathbf{A})$ and $\det(\mathbf{A})$, respectively.

B. Problem Formulation

Consider a time-varying network composed of n stationary sensors whose information exchange topology is represented by a weighted directed graph (digraph) $\mathscr{G}(k) = (\mathscr{V}, \mathscr{E}(k), \mathbf{W}(k))$ in the time interval $[t_k, t_{k+1})$ for any $k \in \mathbb{N}$, with $t_1 = 0$. The digraph $\mathscr{G}(k)$ is assumed to be strongly connected for any $k \in \mathbb{N}$. Let $\mathscr{V} = \mathbb{N}_n$, and assume that $(i, j) \in \mathscr{E}(k)$ if node j receives information from node i in the time interval $[t_k, t_{k+1})$ for any pair of distinct nodes $i, j \in \mathbb{N}_n$ and any $k \in \mathbb{N}$. Note that $\mathbf{W}(k) = [w_{ij}(k)] \in \mathbb{R}^{n \times n}$ represents the weight matrix of the network in the time interval $[t_k, t_{k+1})$, where the positive scalar $w_{ji}(k)$ is the weight assigned to the link $(i, j) \in \mathscr{E}(k)$ for any $k \in \mathbb{N}$. Define $\mathcal{N}_i^{in}(k)$ and $\mathcal{N}_i^{out}(k)$ as the in-neighbor and out-neighbor sets associated with the i^{th} node of the network in the time interval $[t_k, t_{k+1})$, i.e.

$$\mathcal{N}_i^{in}(k) = \{j \in \mathscr{V} \backslash \{i\} \mid (j, i) \in \mathscr{E}(k)\}, \tag{1a}$$

$$\mathcal{N}_i^{out}(k) = \{j \in \mathscr{V} \backslash \{i\} \mid (i, j) \in \mathscr{E}(k)\}. \tag{1b}$$

The Laplacian of the weighted digraph $\mathscr{G}(k)$ is defined as a real matrix $\mathbf{L}(k) = [l_{ij}(k)] \in \mathbb{R}^{n \times n}$ such that $\mathbf{L}(k) = \mathrm{Diag}(\mathbf{W}(k)1_n) - \mathbf{W}(k)$ for any $k \in \mathbb{N}$. Define $\lambda_i(\mathbf{L}(k)) \in \mathbb{C}$, $\mathrm{v}_i(\mathbf{L}(k)) \in \mathbb{C}^n$, and $\mathrm{w}_i(\mathbf{L}(k)) \in \mathbb{C}^n$ as the i^{th} eigenvalue of matrix $\mathbf{L}(k)$, and the right and left eigenvectors associated with it, respectively, for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. The spectrum of the Laplacian matrix $\mathbf{L}(k)$ is then denoted by $\Lambda(\mathbf{L}(k)) = \{\lambda_i(\mathbf{L}(k)) \,|\, i \in \mathbb{N}_n\}$ for any $k \in \mathbb{N}$.

The transmission power vector of a network, represented by a time-varying weighted digraph $\mathscr{G}(k)$, is denoted by $\mathbf{P}(k) = [P_1(k) \;\cdots\; P_n(k)]^\mathsf{T} \in \mathbb{R}^n$ which is composed of the transmission powers of all nodes and $P_i^{min} \leq P_i(k) \leq P_i^{max}$ for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Moreover, $\mathbf{P}(k) \in \mathcal{K}$ for any $k \in \mathbb{N}$, where $\mathcal{K} = \Pi_{i=1}^n [P_i^{min}, P_i^{max}] \subset \mathbb{R}^n$ is a compact and convex set. A higher weight $w_{ji}(k)$, in general, represents a stronger and more reliable communication link (i, j) at the cost of a higher transmission power consumed by the i^{th} node in the time interval $[t_k, t_{k+1})$. Thus, one can describe the link weight $w_{ji}(k)$ as an increasing continuous function of the transmission power $P_i(k)$ in the following form [13]

$$w_{ji}(k) = h(P_i(k); \xi_{ji}), \tag{2}$$

for any $i \in \mathbb{N}_n$, $j \in \mathcal{N}_i^{out}(k)$, and $k \in \mathbb{N}$, where ξ_{ji} represents a set of constant parameters which characterize the communication channel (i, j). As a result, the i^{th} node directly impacts the weights of its outgoing links by adopting its transmission power within the interval $[P_i^{min}, P_i^{max}]$ for any $i \in \mathbb{N}_n$. For the simplicity of analysis, it is assumed in this paper that the weighted digraph $\mathscr{G}(k)$ remains *structurally static* as its transmission power vector $\mathbf{P}(k)$ vary within $\mathcal{K}$ for any $k \in \mathbb{N}$. In other words, neither an existing link is removed nor an additional link is created while the elements of the transmission power vector change; thus, the edge set $\mathscr{E}$ is static. This implies that if the digraph is strongly connected initially, it is guaranteed that it will remain so in any future time instant.

The *generalized algebraic connectivity* (GAC) of a weighted digraph $\mathscr{G}(k)$ with Laplacian matrix $\mathbf{L}(k)$, denoted by $\tilde{\lambda}(\mathbf{L}(k))$, is defined as the smallest real part of the nonzero eigenvalues of $\mathbf{L}(k)$, i.e.

$$\tilde{\lambda}(\mathbf{L}(k)) = \min_{\lambda_i(\mathbf{L}(k)) \neq 0,\ \lambda_i(\mathbf{L}(k)) \in \Lambda(\mathbf{L}(k))} \Re(\lambda_i(\mathbf{L}(k))), \tag{3}$$

for any $k \in \mathbb{N}$ [6]. A distributed optimization algorithm is proposed in this paper to choose the elements of the transmission power vector of the network such that the GAC measure of the underlying asymmetric network is maximized. In other words, by considering the GAC of the network as an implicit function of the transmission power vector $\mathbf{P}$, the objective is to find a local maximum $\mathbf{P}^* \in \mathrm{int}(\mathcal{K})$ such that there exists a positive constant ϵ where

$$\mathbf{P}^* = \underset{\mathbf{P} \in \mathbb{B}_\epsilon(\mathbf{P}^*) \cap \mathcal{K}}{\mathrm{argmax}} \; \tilde{\lambda}(\mathbf{P}). \tag{4}$$

It is worth noting that (4) is a nonconcave optimization problem, and the performance index $\tilde{\lambda}(\mathbf{P})$ is a C^0 function. The optimization complexity of $\tilde{\lambda}(\mathbf{P})$ stems from the fact that unlike the algebraic connectivity, increasing (or decreasing) the elements of the transmission power vector $\mathbf{P}$ (which results in the increase (or decrease) of the elements of the weight matrix $\mathbf{W}$) does not necessarily lead to a higher (or lower) GAC of the network [6]. In order to compute the desired transmission power vector $\mathbf{P}^*$ which (i) represents a stationary point of the function $\tilde{\lambda}(\mathbf{P})$, and (ii) locally maximizes the GAC of the network, the following continuous-time differential equation is to be solved

$$\dot{\mathbf{P}}(t) = \nabla_{\mathbf{P}} \tilde{\lambda}(\mathbf{P}(t)), \tag{5}$$

where $\mathbf{P}(t) \in \mathcal{K}$, and $\nabla_{\mathbf{P}} \tilde{\lambda}(\mathbf{P}(t)) \in \partial \tilde{\lambda}(\mathbf{P}(t))$ denotes the supergradient vector of $\tilde{\lambda}(\mathbf{P})$ w.r.t. $\mathbf{P}$ at time t. Moreover,

$$\nabla_{\mathbf{P}} \tilde{\lambda}(\mathbf{P}(t)) = [\nabla_{P_1} \tilde{\lambda}(\mathbf{P}(t)) \;\cdots\; \nabla_{P_n} \tilde{\lambda}(\mathbf{P}(t))]^\mathsf{T}, \tag{6}$$

and $\partial \tilde{\lambda}(\mathbf{P}(t))$ represents the superdifferential set of $\tilde{\lambda}(\mathbf{P})$ w.r.t. $\mathbf{P}$ at time t. By definition, there exists a positive constant ϵ for any $\hat{\mathbf{P}} \in \mathrm{int}(\mathcal{K})$ such that the supergradient vector of $\tilde{\lambda}(\mathbf{P})$ at $\hat{\mathbf{P}}$ meets the following inequality

$$(\nabla_{\mathbf{P}} \tilde{\lambda}(\hat{\mathbf{P}}))^\mathsf{T} (\mathbf{P} - \hat{\mathbf{P}}) \geq \tilde{\lambda}(\mathbf{P}) - \tilde{\lambda}(\hat{\mathbf{P}}), \tag{7}$$

for any $\mathbf{P} \in \mathbb{B}_\epsilon(\hat{\mathbf{P}}) \cap \mathcal{K}$. A discrete-time version of the differential equation (5) is derived in the sequel, which is used to develop a discrete update procedure for the transmission power vector. To this end, the following two assumptions are made throughout this work.

Assumption 1: The weighted digraph of the network is structurally static and strongly connected at all times.

Assumption 2: The step-size sequences $\{\alpha(k)\}_{k \in \mathbb{N}}$ and $\{\beta(k)\}_{k \in \mathbb{N}}$ (which are used in the discretization process) are chosen to satisfy the following set of conditions:

$$\lim_{k\to\infty} \alpha(k) = 0,\ \lim_{k\to\infty} \beta(k) = 0,\ \lim_{k\to\infty} \tfrac{\alpha(k)}{\beta(k)} = 0, \quad (8a)$$

$$\lim_{k\to\infty} \sum_{j=1}^{k} \alpha(j) = \infty,\ \lim_{k\to\infty} \sum_{j=1}^{k} \tfrac{\alpha(j)}{\beta(j)} = \infty, \quad (8b)$$

$$\lim_{k\to\infty} \sum_{j=1}^{k} \alpha(j)\beta(j) < \infty,\ \lim_{k\to\infty} \sum_{j=1}^{k} \tfrac{\alpha^2(j)}{\beta^2(j)} < \infty. \quad (8c)$$

Example 1: As an example of sequences which satisfies the conditions of Assumption 2, one can consider $\alpha(k) = \frac{\alpha_0}{k^\alpha}$ and $\beta(k) = \frac{\beta_0}{k^\beta}$ with positive real constants $\alpha \in [\alpha_{min}, \alpha_{max}]$, $\beta \in [\beta_{min}, \beta_{max}]$, $\alpha_0 \in \mathbb{R}_{>0}$, $\beta_0 \in \mathbb{R}_{>0}$, and $k \in \mathbb{N}$ such that

$$0.5 < \alpha_{min} - \beta_{max}, \quad (9a)$$

$$\alpha_{max} \leq 1, \quad (9b)$$

$$\alpha_{max} - \beta_{min} \leq 1, \quad (9c)$$

$$1 < \alpha_{min} + \beta_{min}. \quad (9d)$$

Note that the i^{th} element of the continuous-time derivative vector $\dot{\mathbf{P}}(t)$ w.r.t. time can be represented in discrete-time as

$$\dot{P}_i(t_k) = \lim_{\alpha(k)\to 0} \frac{1}{\alpha(k)}[P_i(t_k + \alpha(k)) - P_i(t_k)], \quad (10)$$

where $t_1 = 0$ and $t_{k+1} = \sum_{j=1}^{k} \alpha(j)$ for any $k \in \mathbb{N}$. By defining $P_i(k) = P_i(t_k)$ for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$, equation (10) can be rewritten as

$$\dot{P}_i(k) = \lim_{k\to\infty} \frac{1}{\alpha(k)}[P_i(k+1) - P_i(k)], \quad (11)$$

on noting that $\{\alpha(k)\}_{k\in\mathbb{N}}$ is a diminishing step-size sequence (the first condition of (8a)). Moreover, the i^{th} element of the supergradient vector (6) can be discretized using the *forward and backward finite difference method* at the k^{th} iteration of the algorithm such that

$$\nabla_{P_i}\tilde{\lambda}(\mathbf{P}(k)) = \lim_{k\to\infty} \frac{1}{\beta(k)}\Omega([\partial^+_{P_i}\tilde{\lambda}(\mathbf{P}(k)), \partial^-_{P_i}\tilde{\lambda}(\mathbf{P}(k))]), \quad (12)$$

where

$$\partial^+_{P_i}\tilde{\lambda}(\mathbf{P}(k)) = \tilde{\lambda}(\mathbf{P}(k)+\beta(k)\,\mathrm{e}_i) - \tilde{\lambda}(\mathbf{P}(k)), \quad (13a)$$

$$\partial^-_{P_i}\tilde{\lambda}(\mathbf{P}(k)) = \tilde{\lambda}(\mathbf{P}(k)) - \tilde{\lambda}(\mathbf{P}(k)-\beta(k)\,\mathrm{e}_i), \quad (13b)$$

for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$, with the diminishing step-size sequence $\{\beta(k)\}_{k\in\mathbb{N}}$ satisfying Assumption 2 (the second condition of (8a)). By employing equations (11) and (12) to approximate the i^{th} element of the left-hand and right-hand sides of the differential equation (5) at time t_k, a discrete-time update law for the transmission power of the i^{th} node at the k^{th} iteration of the optimization algorithm is obtained as follows

$$\hat{P}_i(k+1) = P_i(k) + \tfrac{\alpha(k)}{\beta(k)}\Omega([\partial^+_{P_i}\tilde{\lambda}(\mathbf{P}(k)), \partial^-_{P_i}\tilde{\lambda}(\mathbf{P}(k))]), \quad (14)$$

for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. The updated transmission power $P_i(k+1)$ is then obtained as $P_i(k+1) = \Pi(\hat{P}_i(k+1))$, where

$$\Pi(\hat{P}_i(k+1)) = \begin{cases} P_i^{min}, & \text{if } \hat{P}_i(k+1) < P_i^{min}, \\ \hat{P}_i(k+1), & \text{if } P_i^{min} \leq \hat{P}_i(k+1) \leq P_i^{max}, \\ P_i^{max}, & \text{if } P_i^{max} < \hat{P}_i(k+1). \end{cases}$$

Note that the projection map $\Pi(\cdot)$ ensures that $P_i(k+1)$ always remains inside the interval $[P_i^{min}, P_i^{max}]$ for any $i \in \mathbb{N}_n$. In order to implement the discrete-time update procedure (14) in a distributed manner based on the subspace consensus algorithm [6], a number of challenges should be dealt with, the most important of which are as follows:

1) By changing the transmission power of the nodes at each iteration the elements of the weight matrix $\mathbf{W}(k)$ change with time, resulting in a network with a time-varying weighted digraph $\mathscr{G}(k)$ for any $k \in \mathbb{N}$, even though the network topology is assumed to remain structurally static.
2) The exact value of the time-varying GAC of the network is not available to any node at each iteration. Instead, an estimate of the exact GAC, denoted by $\tilde{\lambda}_i^r(k)$ or $\tilde{\lambda}_i^c(k)$ corresponding to the cases where the GAC is associated with a real or complex eigenvalue of $\mathbf{L}(k)$ respectively, is available to every node $i \in \mathbb{N}_n$ at the k^{th} iteration based on the subspace consensus algorithm.
3) Since the weight matrix of the digraph is time-varying, the left eigenvector $\mathrm{w}_1(k)$, associated with the zero eigenvalue of the Laplacian matrix $\mathbf{L}(k)$ at the k^{th} iteration, is time-varying and has to be updated when the transmission power vector of the network is updated.

Note that the values of $\tilde{\lambda}(\mathbf{P}(k))$ and $\tilde{\lambda}(\mathbf{P}(k)\pm\beta(k)\mathrm{e}_i)$ in equation (13) are respectively estimated by $\tilde{\lambda}_i^r(k)$ (or $\tilde{\lambda}_i^c(k)$) and $\tilde{\lambda}_i^{r,\pm}(k)$ (or $\tilde{\lambda}_i^{c,\pm}(k)$) from the viewpoint of node i at the k^{th} iteration of the optimization algorithm after applying the subspace consensus procedure to matrices $\mathbf{L}(k)$ and $\mathbf{L}_i^{\pm}(k)$, respectively. The elements of the Laplacian matrix $\mathbf{L}(k)$ are determined based on the transmission power vector $\mathbf{P}(k)$ at the k^{th} iteration according to equation (2). Moreover, all elements of the Laplacian matrix $\mathbf{L}_i^+(k)$ (or $\mathbf{L}_i^-(k)$) are similar to those of $\mathbf{L}(k)$ except for its i^{th} column whose elements are determined based on the perturbed transmission power $P_i(k)+\beta(k)$ (or $P_i(k)-\beta(k)$) of the i^{th} node for any $i \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Hence, the difference between $\mathbf{L}(k)$ and $\mathbf{L}_i^{\pm}(k)$ stems from the discrepancy between the i^{th} element of their corresponding transmission power vectors.

III. A Distributed Connectivity Maximization Algorithm

A modified version of the subspace consensus algorithm [6] is introduced now to update the transmission power vector of the network in a distributed manner from the viewpoint of each node such that an optimal transmission power vector $\mathbf{P}^*$ (which satisfies equation (4)) is obtained asymptotically. This algorithm is elaborated in four steps:

(1) Let $\mathrm{x}_k^i \in \mathbb{R}^n$ denote the state vector of the i^{th} node at the k^{th} iteration of the algorithm. Analogously, let $\{\mathrm{x}_{k,j}^{\pm,i} \in \mathbb{R}^n \mid j \in \mathbb{N}_n\}$ be the set of perturbed state vectors from the viewpoint of the i^{th} node at the k^{th} iteration where the perturbation occurs in the j^{th} element of the transmission power vector of the network. Consider $y_k^i(0) = \langle \mathrm{x}_k^i, \mathrm{e_i}\rangle$ and $y_{k,j}^{\pm,i}(0) = \langle \mathrm{x}_{k,j}^{\pm,i}, \mathrm{e_i}\rangle$ as the i^{th} elements of the state vectors x_k^i and $\mathrm{x}_{k,j}^{\pm,i}$, respectively, which are obtained by the i^{th} node at the k^{th} iteration of the algorithm for every $j \in \mathbb{N}_n$. Define $\hat{\mathrm{y}}_k = \tilde{\mathbf{L}}(k)\mathrm{y}_k(0)$ and $\hat{\mathrm{y}}_{k,j}^{\pm} = \tilde{\mathbf{L}}_j^{\pm}(k)\mathrm{y}_{k,j}^{\pm}(0)$, where

$\mathrm{y}_k(0) = [y_k^1(0)\ y_k^2(0)\ \cdots\ y_k^n(0)]^\mathsf{T} \in \mathbb{R}^n$ and $\mathrm{y}_{k,j}^{\pm}(0) = [y_{k,j}^{\pm,1}(0)\ y_{k,j}^{\pm,2}(0)\ \cdots\ y_{k,j}^{\pm,n}(0)]^\mathsf{T} \in \mathbb{R}^n$ for any $j \in \mathbb{N}_n$ and $k \in \mathbb{N}$. Moreover, the *approximate modified Laplacian matrices* $\tilde{\mathbf{L}}(k)$ and $\tilde{\mathbf{L}}_j^{\pm}(k)$ are defined, respectively, as

$$\tilde{\mathbf{L}}(k) = \sum_{p=0}^{k} \frac{1}{p!}(\mathbf{I}_n - \delta\,\mathbf{L}(k))^p - \exp(1)\mathrm{w}_1(k)\mathrm{w}_1^\mathsf{T}(k),$$
$$\tilde{\mathbf{L}}_j^{\pm}(k) = \sum_{p=0}^{k} \frac{1}{p!}(\mathbf{I}_n - \delta\,\mathbf{L}_j^{\pm}(k))^p - \exp(1)\mathrm{w}_{1,j}^{\pm}(k)\mathrm{w}_{1,j}^{\pm,\mathsf{T}}(k),$$

for any $j \in \mathbb{N}_n$ and $k \in \mathbb{N}$ [6]. Note that the left eigenvectors associated with the zero eigenvalue of matrices $\mathbf{L}(k)$ and $\mathbf{L}_j^{\pm}(k)$ are denoted by $\mathrm{w}_1(k)$ and $\mathrm{w}_{1,j}^{\pm}(k)$, respectively, for any $j \in \mathbb{N}_n$, which can be obtained using the distributed procedure of [16] in finite-time at every iteration $k \in \mathbb{N}$ when the transmission power vector of the network is updated. In order to compute the vectors $\hat{\mathrm{y}}_k$ and $\hat{\mathrm{y}}_{k,j}^{\pm}$ in a distributed manner at the k^{th} iteration, define $y_k^i(l)$ and $y_{k,j}^{\pm,i}(l)$ as the i^{th} element of the vectors $\mathrm{y}_k(l) \in \mathbb{R}^n$ and $\mathrm{y}_{k,j}^{\pm}(l) \in \mathbb{R}^n$, respectively, where

$$\mathrm{y}_k(l) = (\mathbf{I}_n - \delta\,\mathbf{L}(k))^l\, \mathrm{y}_k(0), \tag{15a}$$
$$\mathrm{y}_{k,j}^{\pm}(l) = (\mathbf{I}_n - \delta\,\mathbf{L}_j^{\pm}(k))^l\, \mathrm{y}_{k,j}^{\pm}(0), \tag{15b}$$

for any $j \in \mathbb{N}_n$, $k \in \mathbb{N}$, and $l \in \mathbb{N}_k$. Then, the following update laws are used to compute $y_k^i(l)$ and $y_{k,j}^{\pm,i}(l)$ from the viewpoint of the i^{th} node

$$\begin{aligned} y_k^i(l) &= y_k^i(l-1) - \delta \sum_{p\in\mathcal{N}_i^{in}} w_{ip}(k)(y_k^i(l-1) - y_k^p(l-1)),\\ y_{k,j}^{\pm,i}(l) &= y_{k,j}^{\pm,i}(l-1) \\ &\quad - \delta \sum_{p\in\mathcal{N}_i^{in}\setminus\{j\}} w_{ip}(k)(y_{k,j}^{\pm,i}(l-1) - y_{k,j}^{\pm,p}(l-1)) \\ &\quad - \delta\, w_{ij}^{\pm,j}(k)(y_{k,j}^{\pm,i}(l-1) - y_{k,j}^{\pm,j}(l-1)), \end{aligned} \tag{16}$$

where $\mathbf{W}(k) = [w_{st}(k)]$ and $\mathbf{W}_j^{\pm}(k) = [w_{st}^{\pm,j}(k)]$ for any $s,t \in \mathbb{N}_n$, $i,j \in \mathbb{N}_n$, $k \in \mathbb{N}$, and $l \in \mathbb{N}_k$. After repeating the above procedure k times, the i^{th} elements of the vectors $\hat{\mathrm{y}}_k$ and $\hat{\mathrm{y}}_{k,j}^{\pm}$ from the viewpoint of node i are given by

$$\hat{y}_k^i = \sum_{l=0}^{k} \frac{1}{l!} y_k^i(l) - \exp(1)\langle \mathrm{w}_1(k), \mathrm{e}_i\rangle\langle \mathrm{w}_1(k), \mathrm{x}_k^i\rangle,$$
$$\hat{y}_{k,j}^{\pm,i} = \sum_{l=0}^{k} \frac{1}{l!} y_{k,j}^{\pm,i}(l) - \exp(1)\langle \mathrm{w}_{1,j}^{\pm}(k), \mathrm{e}_i\rangle\langle \mathrm{w}_{1,j}^{\pm}(k), \mathrm{x}_{k,j}^{\pm,i}\rangle,$$

for any $i,j \in \mathbb{N}_n$ and $k \in \mathbb{N}$.

(2) The concept of consensus observer is now used at the k^{th} iteration to propagate the elements of the vectors $\hat{\mathrm{y}}_k$ and $\hat{\mathrm{y}}_{k,j}^{\pm}$, $j \in \mathbb{N}_n$, throughout the network. Note that the left eigenvectors $\mathrm{w}_1(k)$ and $\mathrm{w}_{1,j}^{\pm}(k)$ for any $j \in \mathbb{N}_n$ are available to every node from the previous step of the algorithm. The consensus observer is updated in a distributed manner at the k^{th} iteration of the algorithm from the viewpoint of node i according to the following update laws

$$\begin{aligned} \mathrm{z}_k^i(m) &= \mathrm{z}_k^i(m-1) - \delta \sum_{p\in\mathcal{N}_i^{in}} w_{ip}(k)(\mathrm{z}_k^i(m-1) - \mathrm{z}_k^p(m-1)),\\ \mathrm{z}_{k,j}^{\pm,i}(m) &= \mathrm{z}_{k,j}^{\pm,i}(m-1) \\ &\quad - \delta \sum_{p\in\mathcal{N}_i^{in}\setminus\{j\}} w_{ip}(k)(\mathrm{z}_{k,j}^{\pm,i}(m-1) - \mathrm{z}_{k,j}^{\pm,p}(m-1)) \\ &\quad - \delta\, w_{ij}^{\pm,j}(k)(\mathrm{z}_{k,j}^{\pm,i}(m-1) - \mathrm{z}_{k,j}^{\pm,j}(m-1)), \end{aligned} \tag{17}$$

where $\mathbf{W}(k) = [w_{st}(k)]$ and $\mathbf{W}_j^{\pm}(k) = [w_{st}^{\pm,j}(k)]$ for any $s,t \in \mathbb{N}_n$, $i,j \in \mathbb{N}_n$, $k \in \mathbb{N}$, and $m \in \mathbb{N}_k$. It is to be noted that the state vectors of the consensus observer are initialized by setting $\mathrm{z}_k^i(0) = \hat{y}_k^i \mathrm{e}_i$ and $\mathrm{z}_{k,j}^{\pm,i}(0) = \hat{y}_{k,j}^{\pm,i}\mathrm{e}_i$ for any $i,j \in \mathbb{N}_n$ and $k \in \mathbb{N}$. After repeating the update procedure (17) k times, the updated state vectors x_{k+1}^i and $\mathrm{x}_{k+1,j}^{\pm,i}$ of the main optimization procedure, computed by the i^{th} node at the k^{th} iteration, are given by

$$\mathrm{x}_{k+1}^i = \frac{(\Xi(k))^{-1}\mathrm{z}_k^i(k)}{\|(\Xi(k))^{-1}\mathrm{z}_k^i(k)\|}, \tag{18a}$$
$$\mathrm{x}_{k+1,j}^{\pm,i} = \frac{(\Xi_j^{\pm}(k))^{-1}\mathrm{z}_{k,j}^{\pm,i}(k)}{\|(\Xi_j^{\pm}(k))^{-1}\mathrm{z}_{k,j}^{\pm,i}(k)\|}, \tag{18b}$$

where

$$\Xi(k) = \tfrac{1}{\langle \mathrm{w}_1(k), 1_n\rangle}\mathrm{Diag}(\mathrm{w}_1(k)), \tag{19a}$$
$$\Xi_j^{\pm}(k) = \tfrac{1}{\langle \mathrm{w}_{1,j}^{\pm}(k), 1_n\rangle}\mathrm{Diag}(\mathrm{w}_{1,j}^{\pm}(k)), \tag{19b}$$

for any $i,j \in \mathbb{N}_n$ and $k \in \mathbb{N}$.

(3) Define $\mathcal{W}_{k,i} = \mathrm{span}\{\mathrm{x}_k^i\}$ and $\mathcal{V}_{k,i} = \mathrm{span}\{\mathrm{x}_{k-1}^i, \mathrm{x}_k^i\}$ as the one-dimensional and two-dimensional subspaces formed to estimate, respectively, the real and complex dominant eigenvalues of the matrix $\tilde{\mathbf{L}}(k)$ at the k^{th} iteration of the distributed optimization algorithm by the i^{th} node. The one-dimensional and two-dimensional subspaces $\mathcal{W}_{k,i}^{\pm} = \mathrm{span}\{\mathrm{x}_{k,i}^{\pm,i}\}$ and $\mathcal{V}_{k,i}^{\pm} = \mathrm{span}\{\mathrm{x}_{k-1,i}^{\pm,i}, \mathrm{x}_{k,i}^{\pm,i}\}$ are defined in a similar manner to compute, respectively, the real and complex dominant eigenvalues of $\tilde{\mathbf{L}}_i^{\pm}(k)$ at the k^{th} iteration by the i^{th} node. After identifying the subspaces $\mathcal{W}_{k,i}$ and $\mathcal{V}_{k,i}$, the projection of $\tilde{\mathbf{L}}(k)$ onto subspaces $\mathcal{W}_{k,i}$ and $\mathcal{V}_{k,i}$, denoted respectively by $\mathbf{R}_{k,i}^r$ and $\mathbf{R}_{k,i}^c$, are computed as follows

$$\mathbf{R}_{k,i}^r = (\mathbf{Q}_{k,i}^{r,\mathsf{T}}\mathbf{Q}_{k,i}^r)^{-1}\mathbf{Q}_{k,i}^{r,\mathsf{T}}\,\tilde{\mathbf{L}}(k)\,\mathbf{Q}_{k,i}^r, \tag{20a}$$
$$\mathbf{R}_{k,i}^c = (\mathbf{Q}_{k,i}^{c,\mathsf{T}}\mathbf{Q}_{k,i}^c)^{-1}\mathbf{Q}_{k,i}^{c,\mathsf{T}}\,\tilde{\mathbf{L}}(k)\,\mathbf{Q}_{k,i}^c, \tag{20b}$$

where $\mathbf{Q}_{k,i}^r = \mathrm{x}_k^i$ and $\mathbf{Q}_{k,i}^c = [\mathrm{x}_{k-1}^i\ \mathrm{x}_k^i]$ for any node $i \in \mathbb{N}_n$ and any $k \in \mathbb{N}$. Moreover, the matrices $\mathbf{R}_{k,i}^{r,\pm}$ and $\mathbf{R}_{k,i}^{c,\pm}$ are defined as the projection of the Laplacian matrix $\tilde{\mathbf{L}}_i^{\pm}(k)$ onto subspaces $\mathcal{W}_{k,i}^{\pm}$ and $\mathcal{V}_{k,i}^{\pm}$, respectively, such that

$$\mathbf{R}_{k,i}^{r,\pm} = (\mathbf{Q}_{k,i}^{r,\pm,\mathsf{T}}\mathbf{Q}_{k,i}^{r,\pm})^{-1}\mathbf{Q}_{k,i}^{r,\pm,\mathsf{T}}\,\tilde{\mathbf{L}}_i^{\pm}(k)\,\mathbf{Q}_{k,i}^{r,\pm}, \tag{21a}$$
$$\mathbf{R}_{k,i}^{c,\pm} = (\mathbf{Q}_{k,i}^{c,\pm,\mathsf{T}}\mathbf{Q}_{k,i}^{c,\pm})^{-1}\mathbf{Q}_{k,i}^{c,\pm,\mathsf{T}}\,\tilde{\mathbf{L}}_i^{\pm}(k)\,\mathbf{Q}_{k,i}^{c,\pm}, \tag{21b}$$

where $\mathbf{Q}_{k,i}^{r,\pm} = \mathrm{x}_{k,i}^{\pm,i}$ and $\mathbf{Q}_{k,i}^{c,\pm} = [\mathrm{x}_{k-1,i}^{\pm,i}\ \mathrm{x}_{k,i}^{\pm,i}]$ for any node $i \in \mathbb{N}_n$ and any $k \in \mathbb{N}$. Then, the estimated GAC of the networks represented by Laplacian matrix $\mathbf{L}(k)$ and perturbed Laplacian matrix $\mathbf{L}_i^{\pm}(k)$ from the viewpoint of the i^{th} node, are given by

$$\tilde{\lambda}_i^r(k)=\Gamma\Big(\mathbf{R}_{k,i}^r\Big), \tag{22a}$$

$$\tilde{\lambda}_i^c(k)=\Gamma\Big(\tfrac{1}{2}\text{tr}(\mathbf{R}_{k,i}^c)+\text{j}\sqrt{\det(\mathbf{R}_{k,i}^c)-(\tfrac{1}{2}\text{tr}(\mathbf{R}_{k,i}^c))^2}\Big), \tag{22b}$$

$$\tilde{\lambda}_i^{r,\pm}(k)=\Gamma\Big(\mathbf{R}_{k,i}^{r,\pm}\Big), \tag{22c}$$

$$\tilde{\lambda}_i^{c,\pm}(k)=\Gamma\Big(\tfrac{1}{2}\text{tr}(\mathbf{R}_{k,i}^{c,\pm})+\text{j}\sqrt{\det(\mathbf{R}_{k,i}^{c,\pm})-(\tfrac{1}{2}\text{tr}(\mathbf{R}_{k,i}^{c,\pm}))^2}\Big), \tag{22d}$$

where function $\Gamma:\mathbb{C}\rightarrow\mathbb{R}$ is defined as $\Gamma(\cdot)=\frac{1}{\delta}\left[1-\log(|\cdot|)\right]$ for any $i\in\mathbb{N}_n$ and $k\in\mathbb{N}$. Note that (22a) and (22c) denote the estimates of the GAC of their corresponding networks assuming that the exact GACs are associated with real eigenvalues while (22b) and (22d) represent the estimated GACs of the networks corresponding to the case where the exact GACs are specified by complex eigenvalues. In order to determine whether the estimated GACs are associated with the real or complex eigenvalues of their corresponding Laplacian matrices, the global index $\hat{k}\in\{0,1\}$ at the k^{th} iteration of the optimization algorithm is defined as

$$\hat{k}=\begin{cases}0, & \text{if } \text{mod}(k,k^*+\text{Diam}(\mathscr{G}))\neq 0 \text{ and } k\geq 2,\\ 1, & \text{if } \text{mod}(k,k^*+\text{Diam}(\mathscr{G}))=0 \text{ and } k\geq 2,\end{cases} \tag{23}$$

where $\text{Diam}(\mathscr{G})$ denotes the diameter of the digraph $\mathscr{G}$ and k^* is equal to $\lceil|\frac{\log\bar{\epsilon}}{\log\bar{\kappa}}|\rceil$ for a sufficiently small positive constant $\bar{\epsilon}$. Moreover,

$$\bar{\kappa}=\max_{\mathbf{P}(k)\in\mathcal{K}}\left|\frac{\text{e}^{1-\delta\tilde{\tilde{\lambda}}(k)}}{\text{e}^{1-\delta\tilde{\lambda}(k)}}\right|, \tag{24}$$

such that

$$\tilde{\lambda}(k)=\min_{\lambda_i(\mathbf{L}(k))\neq 0,\,\lambda_i(\mathbf{L}(k))\in\Lambda(\mathbf{L}(k))}\Re(\lambda_i(\mathbf{L}(k))), \tag{25a}$$

$$\tilde{\tilde{\lambda}}(k)=\min_{\lambda_i(\mathbf{L}(k))\neq 0,\,\Re(\lambda_i(\mathbf{L}(k)))\neq\tilde{\lambda}(k),\,\lambda_i(\mathbf{L}(k))\in\Lambda(\mathbf{L}(k))}\Re(\lambda_i(\mathbf{L}(k))), \tag{25b}$$

for any $k\in\mathbb{N}$. Then, the distributed version of the update procedure (14) for the transmission power of the i^{th} node at the k^{th} iteration of the optimization algorithm is performed as

$$P_i(k+1)=\Pi\Big(P_i(k)+\hat{k}\tfrac{\alpha(k)}{\beta(k)}\Omega([\tilde{\lambda}_i^+(k)-\tilde{\lambda}_i(k),\tilde{\lambda}_i(k)-\tilde{\lambda}_i^-(k)])\Big),$$

where

$$\tilde{\lambda}_i(k)=\begin{cases}\tilde{\lambda}_i^r(k), & \text{if } |1-\mu_k^i|<\bar{\epsilon},\\ \tilde{\lambda}_i^c(k), & \text{if } |1-\mu_k^i|\geq\bar{\epsilon},\end{cases} \tag{26a}$$

$$\tilde{\lambda}_i^{\pm}(k)=\begin{cases}\tilde{\lambda}_i^{r,\pm}(k), & \text{if } |1-\mu_k^{\pm,i}|<\bar{\epsilon},\\ \tilde{\lambda}_i^{c,\pm}(k), & \text{if } |1-\mu_k^{\pm,i}|\geq\bar{\epsilon},\end{cases} \tag{26b}$$

for any $i\in\mathbb{N}_n$ and $k\in\mathbb{N}$. Note that the values of μ_k^i and $\mu_k^{\pm,i}$ for the case when $\hat{k}=0$ are obtained as

$$\mu_k^i=\min_{j\in\mathcal{N}_i^{in}\cup\{i\}}\left\{\mu_{k-1}^j,\frac{|\langle\text{x}_{k+1}^i,\text{x}_k^i\rangle|}{\|\text{x}_{k+1}^i\|\|\text{x}_k^i\|}\right\}, \tag{27a}$$

$$\mu_k^{\pm,i}=\min_{j\in\mathcal{N}_i^{in}\cup\{i\}}\left\{\mu_{k-1}^{\pm,j},\frac{|\langle\text{x}_{k+1,i}^{\pm,i},\text{x}_{k,i}^{\pm,i}\rangle|}{\|\text{x}_{k+1,i}^{\pm,i}\|\|\text{x}_{k,i}^{\pm,i}\|}\right\}, \tag{27b}$$

for any $i\in\mathbb{N}_n$ and $k\in\mathbb{N}$. However, if $\hat{k}=1$, the values of μ_k^i and $\mu_k^{\pm,i}$ are computed as

$$\mu_k^i=\max\left\{\mu_{k-1}^i,\frac{|\langle\text{x}_{k+1}^i,\text{x}_k^i\rangle|}{\|\text{x}_{k+1}^i\|\|\text{x}_k^i\|}\right\}, \tag{28a}$$

$$\mu_k^{\pm,i}=\max\left\{\mu_{k-1}^{\pm,i},\frac{|\langle\text{x}_{k+1,i}^{\pm,i},\text{x}_{k,i}^{\pm,i}\rangle|}{\|\text{x}_{k+1,i}^{\pm,i}\|\|\text{x}_{k,i}^{\pm,i}\|}\right\}, \tag{28b}$$

for any $i\in\mathbb{N}_n$ and $k\in\mathbb{N}$. Moreover, the values of μ_k^i and $\mu_k^{\pm,i}$ are initialized by setting $\mu_0^i=\mu_0^{\pm,i}=1$ for any $i\in\mathbb{N}_n$.

(4) To examine the convergence of the algorithm, the values of $\mathcal{D}_{k+1}^{r,i}$ and $\mathcal{D}_{k+1}^{c,i}$ for the case when $\hat{k}=0$ are updated as follows

$$\mathcal{D}_{k+1}^{r,i}=\max_{j\in\mathcal{N}_i^{in}\cup\{i\}}\big\{\mathcal{D}_k^{r,j},|\tilde{\lambda}_i^r(k)-\tilde{\lambda}_i^r(k-1)|\big\}, \tag{29a}$$

$$\mathcal{D}_{k+1}^{c,i}=\max_{j\in\mathcal{N}_i^{in}\cup\{i\}}\big\{\mathcal{D}_k^{c,j},|\tilde{\lambda}_i^c(k)-\tilde{\lambda}_i^c(k-1)|\big\}, \tag{29b}$$

for any $i\in\mathbb{N}_n$ and $k\geq 2$. Note that $|\tilde{\lambda}_i^r(k)-\tilde{\lambda}_i^r(k-1)|$ and $|\tilde{\lambda}_i^c(k)-\tilde{\lambda}_i^c(k-1)|$ measure the distance between the last two consecutive estimated GAC of the network corresponding to the cases where the GAC is associated with a real and complex eigenvalue of $\mathbf{L}(k)$, respectively, from the viewpoint of node i for any $k\geq 2$. As long as $\hat{k}$ is equal to 0, the termination condition of the algorithm will not be satisfied for any node, and the values of $\mathcal{D}_{k+1}^{r,i}$ and $\mathcal{D}_{k+1}^{c,i}$ of every node $i\in\mathbb{N}_n$ will be diffused throughout the network during the iteration cycles of length $k^*+\text{Diam}(\mathscr{G})$ according to (29). For the case when $\hat{k}=1$, the values of $\mathcal{D}_{k+1}^{r,i}$ and $\mathcal{D}_{k+1}^{c,i}$ are initialized for the next iteration cycle using the following update procedure

$$\mathcal{D}_{k+1}^{r,i}=\min\{\mathcal{D}_k^{r,i},|\tilde{\lambda}_i^r(k)-\tilde{\lambda}_i^r(k-1)|\}, \tag{30a}$$

$$\mathcal{D}_{k+1}^{c,i}=\min\{\mathcal{D}_k^{c,i},|\tilde{\lambda}_i^c(k)-\tilde{\lambda}_i^c(k-1)|\}, \tag{30b}$$

for any $i\in\mathbb{N}_n$ and $k\geq 2$. By setting $k=k+1$, the algorithm is said to have converged at the k^{th} iteration if $\hat{k}=1$ and $\min\{\mathcal{D}_k^{r,i},\mathcal{D}_k^{c,i}\}$ becomes sufficiently small (as defined by a prescribed threshold $\hat{\epsilon}$); otherwise, the iteration continues.

IV. Simulation Results

Consider an asymmetric network composed of four nodes represented by a strongly connected weighted digraph $\mathscr{G}=(\mathscr{V},\mathscr{E},\mathbf{W})$ whose initial weight matrix is given by

$$\mathbf{W}=\begin{bmatrix}0 & 0.431 & 0 & 0.900\\ 0.473 & 0 & 0 & 0\\ 0 & 0.400 & 0 & 0.467\\ 0 & 0 & 0.638 & 0\end{bmatrix}. \tag{31}$$

Consider $\mathbf{P}=[3.2\ 3.5\ 3.25\ 3]^{\mathsf{T}}$ as the initial transmission power vector of the network, which is constrained to the compact set $\mathcal{K}=[1\ 4]^4$ in this example. Let Assumption 1 hold, and choose $\alpha(k)=\frac{\alpha_0}{k^{\alpha}}$ and $\beta(k)=\frac{\beta_0}{k^{\beta}}$ for any $k\in\mathbb{N}$, with positive constants $\alpha=\alpha_0=0.85$ and $\beta=\beta_0=0.3$ such that the conditions of the inequality (9) (in relation to Assumption 2) hold. For this example, $\tilde{\lambda}(\mathbf{L})=0.6694$ is the initial value of the GAC of the network, which corresponds to a real eigenvalue of the Laplacian matrix of the digraph $\mathscr{G}$. The performance of the proposed optimization algorithm in a distributed implementation is evaluated by choosing

$\bar{\epsilon} = 10^{-4}$ and $\hat{\epsilon} = 10^{-3}$ in the termination condition and considering $\delta = \frac{1}{4}$. The value of the GAC of the network as the iteration index k increases is shown in Fig. 1. This figure demonstrates that the GAC of the network converges to the same local maximum from the viewpoint of all nodes in an asymptotical manner. Note that the optimal value of the GAC is given by $\tilde{\lambda}^* = 0.8182$, which corresponds to the pair of complex conjugate eigenvalues $0.8182 \pm \mathrm{j}0.0298$ of the Laplacian matrix associated with the following weight matrix

$$\mathbf{W}^* = \begin{bmatrix} 0 & 0.426 & 0 & 0.900 \\ 0.489 & 0 & 0 & 0 \\ 0 & 0.400 & 0 & 0.456 \\ 0 & 0 & 0.639 & 0 \end{bmatrix}. \tag{32}$$

The transmission power of every node versus the iteration index of the optimization algorithm is depicted in Fig. 2. It can be verified that $\mathbf{P}^* = [3.6696\ 2.8848\ 3.3168\ 2.6790]^\mathsf{T}$ is a vector in the set $\mathcal{K}$ which corresponds to a local maximum of the GAC of the network. It is worth mentioning that the transmission powers of nodes 1 and 3 increase in order to reach the optimal value, but unlike similar scenarios in symmetric networks, the transmission powers of nodes 2 and 4 decrease while converging to the optimal configuration.

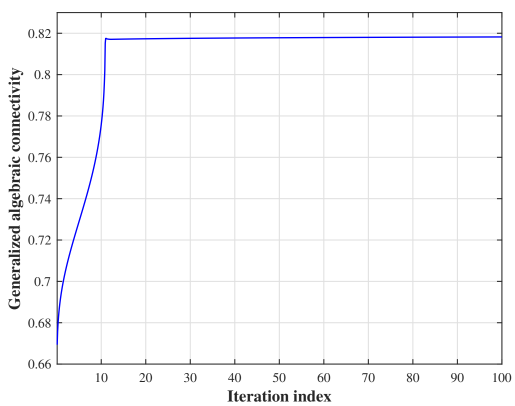

Fig. 1: Evolution of the GAC of the network.

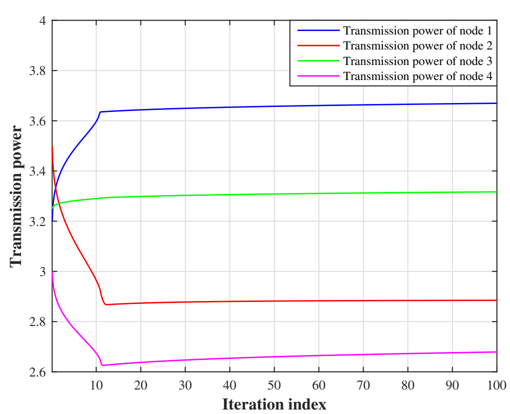

Fig. 2: Evolution of the transmission power of every node of the network.

V. Conclusions

A distributed optimization algorithm for maximizing the generalized algebraic connectivity of a weighted asymmetric network is introduced in this work. The generalized algebraic connectivity is formulated as an implicit function of the transmission power of the network. Since the generalized algebraic connectivity is a nonconcave and continuous (but not necessarily differentiable) function of the transmission power of the nodes, a distributed algorithm based on the subspace consensus approach is developed to compute the supergradient vector of the network connectivity w.r.t. the elements of the transmission power vector from the viewpoint of each node. It is guaranteed that the transmission power of every node converges to a local maximizer of the network connectivity in an asymptotic manner. The efficacy of the proposed algorithm is verified by simulations.

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3. TITLE (The document title and sub-title as indicated on the title page.)

Distributed Connectivity Optimization in Asymmetric Networks

4. AUTHORS (last name, followed by initials – ranks, titles, etc., not to be used)

Asadi, M.M.; Blouin, S.; Aghdam, A.G.

5. DATE OF PUBLICATION (Month and year of publication of document.)	6a. NO. OF PAGES (Total pages, including Annexes, excluding DCD, covering and verso pages.)	6b. NO. OF REFS (Total references cited.)
March 2018	6	16

7. DOCUMENT CATEGORY (e.g., Scientific Report, Contract Report, Scientific Letter.)

External Literature (P)

8. SPONSORING CENTRE (The name and address of the department project office or laboratory sponsoring the research and development.)

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9a. PROJECT OR GRANT NO. (If appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.)	9b. CONTRACT NO. (If appropriate, the applicable number under which the document was written.)

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DRDC-RDDC-2018-P026	

11a. FUTURE DISTRIBUTION WITHIN CANADA (Approval for further dissemination of the document. Security classification must also be considered.)

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12. KEYWORDS, DESCRIPTORS or IDENTIFIERS (Use semi-colon as a delimiter.)

network; connectivity; optimization

13. ABSTRACT/RESUME (When available in the document, the French version of the abstract must be included here.)

The problem of distributed connectivity optimization of an asymmetric sensor network represented by a weighted directed graph (digraph) is investigated in this paper. The notion of generalized algebraic connectivity is used to measure the connectivity of a time-varying weighted digraph. The generalized algebraic connectivity is regarded as a nonconcave and nondifferentiable continuous cost function, and a distributed approach, based on the subspace consensus algorithm, is developed to compute the supergradient vector of the network connectivity. By considering the above-mentioned network connectivity as a function of the transmission power vector of the network, a discrete-time update procedure is proposed to compute a stationary transmission power vector of the network which locally maximizes the network connectivity. The effectiveness of the developed algorithm is subsequently demonstrated by simulations.