Equal Area Triangular Pixels on a Sphere

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1 Introduction

In a scenario where one wants to represent directional data, for example, from a detector which provides varying data as a function of angular origin, it is often useful to divide the angular space into finite sectors, in order to measure a set of quantities with associated statistical uncertainties. A natural approach could consist in the subdivision of the angular space into sectors having equal increments in azimuthal and polar, (ϕ, θ) coordinates, respectively. This however leads to sectors having different areas. Using increments in $(\phi, \cos\theta)$ prevents this issue, but the generated sectors still have non-uniform shapes, being more elongated near the poles.

A better approach consists in the tessellation of the space using more uniform sectors. This process can be initiated by defining a convex regular icosahedron whose vertices lie on the surface of a sphere, since it constitutes the platonic solid with the most faces. Each equilateral triangle composing the icosahedron can then be subdivided into six identical triangles. Additional subdivision of the sectors lead to non-identical faces, but the regularity of their shape can be optimise through the division of the longest edges. Section 2 of this report explains how this can be achieved using spherical triangles, which result from the projection of the triangular sectors onto the surface of the circumscribed sphere. The remaining sections focus on the calculation of the intersections between the triangular sectors of the tessellated sphere and a particular type of data constituted of conical surfaces whose apex is located at the centre of the sphere.

2 Division of a Spherical Triangle into Two Equal Area Triangles

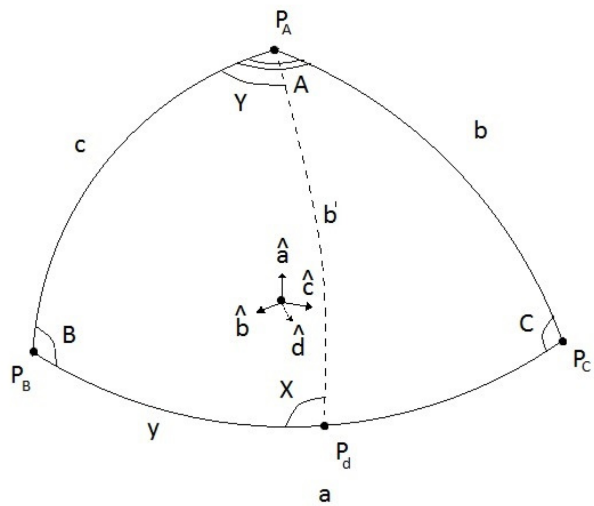


Figure 2.1: Spherical Triangle ABC divided into two equal area triangles ADB and ADC.

Consider the spherical triangle illustrated in Figure 2.1 made up of the points P_A, P_B, and P_C. The triangle is on the surface of a unit sphere centered on the origin. The three points comprising the triangle are located in the direction the vectors $\hat{a}$, $\hat{b}$, and $\hat{c}$. The interior angles are labelled A, B, and C, with corresponding arclengths a, b, and c.

The problem addressed here is the determination of the location of point P_D, located along the arc between P_B and P_C, which divides $\triangle$ABC into two triangles of equal area. In Figure 2.1 the interior angles of one of these triangles, $\triangle$ADB, are labelled Y, X, and B, with corresponding arclengths y, c, and b'. The point P_D can be constructed by rotating vector $\hat{b}$ toward $\hat{c}$, in the plane spanned by $[\hat{b}, \hat{c}]$, by an amount y. The value of y can be related to the two unknown interior angles of $\triangle$ADB (X and Y) by using the spherical triangle sine-law,

$$\frac{\sin(Y)}{y} = \frac{\sin(X)}{c}$$

$$y = \arcsin\left(\frac{\sin(Y)\sin(c)}{\sin(X)}\right) \tag{2.1}$$

The unknown intertior angles Y and X can be found by finding to equations which relate them. The first equation can be found by relating the area of the triangles. The area of △ABC is,

$$\triangle ABC = A + B + C - \pi \tag{2.2}$$

The area of △ADB is half the area of △ABC leading to the following equation,

$$\begin{aligned} \triangle ADB &= \frac{\triangle ABC}{2} \\ X + Y + B - \pi &= \frac{A + B + C - \pi}{2} \\ X &= \frac{A + C - B + \pi}{2} - Y \end{aligned} \tag{2.3}$$

A second equation relating angles Y and X can be found using the cosine-law for spherical triangles,

$$\cos(X) = -\cos(B)\cos(Y) + \sin(B)\sin(Y)\cos(c) \tag{2.4}$$

Taking th cosine of both sides of Equation 2.3 and making the substitution $\tau = \frac{A+C-B+\pi}{2}$ produces the following equation,

$$\cos(X) = \cos(\tau - Y) \tag{2.5}$$

Applying the sum-difference trig identity to Equation 2.5 produces,

$$\cos(X) = \cos(\tau)\cos(Y) - \sin(\tau)\sin(Y) \tag{2.6}$$

Setting Equation 2.4 equal to Equation 2.6 results in,

$$\cos(\tau)\cos(Y) - \sin(\tau)\sin(Y) = -\cos(B)\cos(Y) + \sin(B)\sin(Y)\cos(c) \tag{2.7}$$

Dividing both sides by $\cos(Y)$ and solving for $\tan(Y)$ gives the following,

$$\tan(Y) = \frac{\cos(B) + \cos(\tau)}{\sin(B)\cos(c) - \sin(\tau)} \tag{2.8}$$

Therefore, Y can be calculated by taking the arctan of both sides of Equation 2.8, and X can be found by substituting Y into Equation 2.3.

3 Intersection of a Cone with a Line Segment

A point, $\vec{x}$, lies on the surface of a cone, defined by axis $\hat{d}$ and opening angle Θ, if it satisfies the following equation,

$$\hat{d} \cdot \frac{\vec{x}}{|\vec{x}|} = \cos(\Theta) \tag{3.1}$$

Which can be rewritten to remove the square root as,

$$(\hat{d} \cdot \vec{x})^2 = \cos^2(\Theta)(\vec{x} \cdot \vec{x}) \tag{3.2}$$

To determine if a line segment, defined by two end points $\vec{e_0}$ and $\vec{e_1}$, intersects the cone one must determine if there is a point $\vec{x}$ along the line segment which satisfies Equation 3.2. To determine this the line segment can be written in parametric form with parameter $t \in [0, 1]$,

$$\vec{x}(t) = (1 - t)\vec{e_0} + t\vec{e_1} \tag{3.3}$$

By substituting Equation 3.3 into Equation 3.2, and using $u = 1 - t$ and $v = t$ for clarity,

$$(u\vec{d} \cdot \vec{e_0} + v\vec{d} \cdot \vec{e_1})^2 = \cos^2(\Theta)(u\vec{e_0} + v\vec{e_1})(u\vec{e_0} + v\vec{e_1}) \tag{3.4}$$

Expanding the brackets and collecting like terms produces,

$$\begin{aligned} u^2[(\vec{d} \cdot \vec{e_0})^2 - \cos^2(\Theta)]+ \\ 2uv[(\vec{d} \cdot \vec{e_0})(\vec{d} \cdot \vec{e_1})] - \cos^2(\Theta)(\vec{e_0} \cdot \vec{e_1}) \\ +v^2[(\vec{d} \cdot \vec{e_1})^2 - \cos^2(\Theta)] = 0 \end{aligned} \tag{3.5}$$

Making the following substitutions,

$$\alpha = (\vec{d} \cdot \vec{e_0})^2 - \cos^2(\Theta) \tag{3.6}$$

$$\beta = (\vec{d} \cdot \vec{e_0})(\vec{d} \cdot \vec{e_1}) - \cos^2(\Theta)(\vec{e_0} \cdot \vec{e_1}) \tag{3.7}$$

$$\gamma = (\vec{d} \cdot \vec{e_1})^2 - \cos^2(\Theta) \tag{3.8}$$

and replacing $u = 1 - t$ and $v = t$ gives,

$$(1 - t)^2\alpha + 2(1 - t)t\beta + t^2\gamma = 0 \tag{3.9}$$

Expanding the brackets and collecting like terms produces the following quadratic formula,

$$(\alpha - 2\beta + \gamma)t^2 + (2\beta - 2\alpha)t + \alpha = 0 \tag{3.10}$$

Applying the quadratic formula,

$$t = \frac{\alpha - \beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha - 2\beta + \gamma} \tag{3.11}$$

Two checks must be performed to determine if the line segment intersects the cone; one on the values of the parameter t, and the other on the resulting point $\vec{x}(t)$.

1. For the intersection point to lie between $\vec{e_0}$ and $\vec{e_1}$ at least one of the solutions to Equation 3.11 must be in the range $[0, 1]$. Otherwise the intersection point is before $\vec{e_0}$ $(t < 0)$, or after $\vec{e_1}$ $(t > 1)$

2. To determine if the point $\vec{x}(t)$, for a valid value of t, intersects the cone the condition $\vec{d} \cdot \vec{x}(t) > 0$ must be met. This check is required due to the squaring of both sides of Equation 3.1 which introduces intersections of the line segment with the cone defined by axis $-\vec{d}$ and opening angle Θ into the solution space.

4 Calculation of Arclength

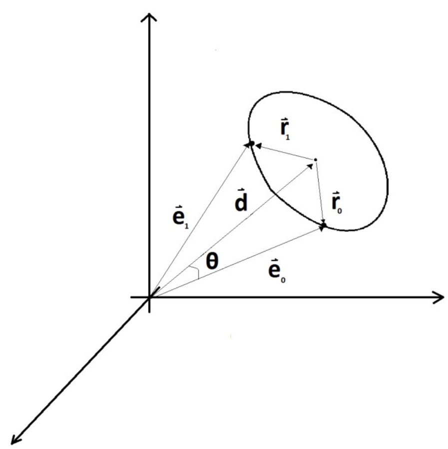


Figure 4.1: Caclulate of the arclength contained between two points of intersection.

Figure 4.1 illustrates the vectors used in the calculation of the arclength contained in a triangle. Vectors $\vec{e_o}$ and $\vec{e_1}$ point to the two intersections of a line segment with the a cone defined by axis $\hat{d}$ and opening angle Θ. The arclength, a, between $\vec{e_o}$ and $\vec{e_1}$ is equal to the angle between the vectors $\vec{r_o}$ and $\vec{r_1}$. Where $\vec{r_o}$ and $\vec{r_1}$ are defined as,

$$\vec{r_o} = \vec{d'} - \vec{e_o}$$

$$\vec{r_1} = \vec{d'} - \vec{e_1}$$

where,

$$\vec{d'} = \cos(\Theta)\vec{d}$$

The arclength can then be calculated as,

$$a = \arccos\left(\frac{\vec{r_o} \cdot \vec{r_1}}{|\vec{r_o}||\vec{r_1}|}\right)$$

5 Intersection Cases

The possible intersections of a circle with a triangle have been labelled 0 through 5 in Figure 5.1. There can be a maximum of two intersections per edge as seen in Section 4. The intersections of edge AB are 0 and 1, of edge BC 2 and 3, and of edge CA 4 and 5.

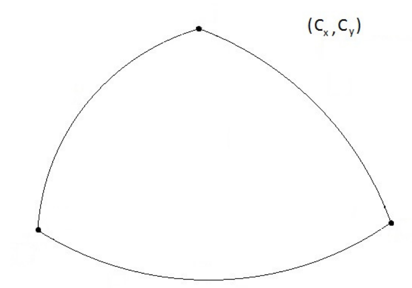


Figure 5.1: The scheme used to label the points of intersection on each of the three edges of a triangle.

Case 1: No Intersections but Circle Fully Contained in Triangle

- Use the Moller-Trumbore algorithm to determine if the circle centre is contained in triangle.
- Check that angle between triangle vertices and circle centre are greater than cone opening angle

If the above to statements are true then the circle is fully contained in a single triangle and the arclength value is 2π. This case is illustrated in Figure 5.2.

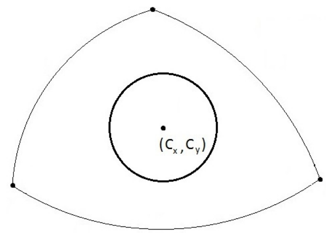


Figure 5.2: Case where a circle is fully contained with in a single triangle.

Case 2: All Sides Have 2 Intersections

If there are two intersections on each edge then the arclength contained in the triangle is the sum of the arc lengths between intersection point pairs (1,2), (3,4), and (5,0). This case is illustrated in Figure 5.3.

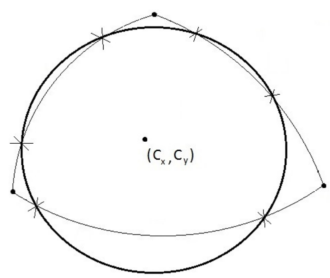


Figure 5.3: The case where a circle intersects all 3 edges of a triangle twice.

Case 3: 2 Intersections on the Same Edge

There are two cases depending on the location of the centre of the circle.

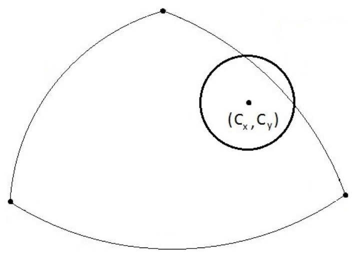
1. If the centre of the circle is inside the triangle (Figure 5.4a),

$$a = 2\pi - a$$

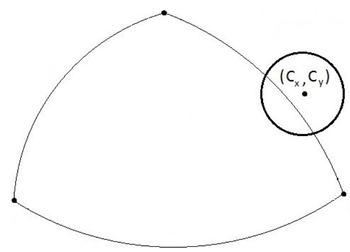
2. If the centre of the circle is outside the triangle (Figure 5.4b) then the arclength is equal to the value return by the calculation detailed in Section 4.

Case 4: 4 Intersections (2 on one edge, 1 on each of the other two)

The arclength contained in the triangle is the sum of two components. Each component is the arclength between the intersection point on a single edge with the nearest intersection point of the edge intersected twice. This case is illustrated in Figure 5.5.



(a) Case with circle centre inside triangle.



(b) Case with circle centre outside triangle.

Figure 5.4

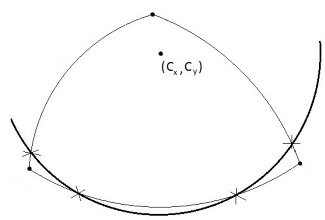


Figure 5.5: Case where the circle intersects one edge twice and the other two edges once.

Case 5: 4 Intersections (2 on one edge, and 2 on another edge)

There are two cases depending on the sum of the smaller 3 arcs.

1. If the sum of the smaller three arcs is less than π (Figure 5.6a) then

$$a = 2\pi - a$$

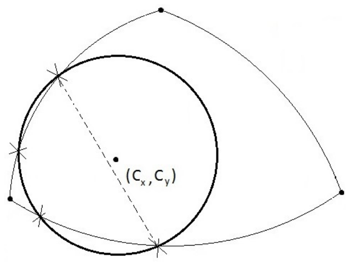
2. Else the arclength is equal to the value return by the calculation detailed in Section 4.This case is illustrated in Figure 5.6b.

Case 6: 2 Intersections on Different Edges

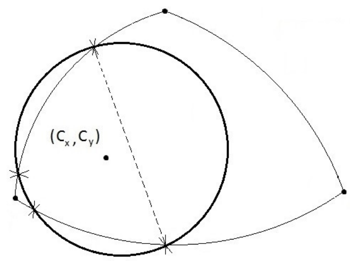
There are two cases depending on the location of the centre of the circle.

1. If the centre of the circle is outside the triangle formed by the two intersection points and the common vertice of the intersecting line segments (Figure 5.7b), and the caclulated arclength is greater than $\pi/2$ then,

$$a = 2\pi - a$$

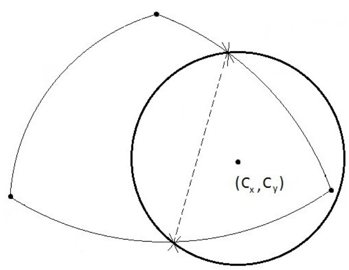


(a) The sum of the 3 smaller arcs is great than π

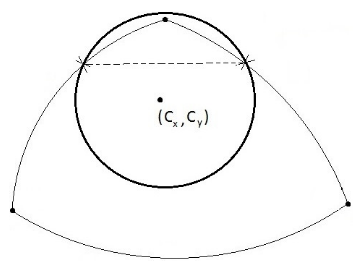


(b) The sum of the 3 larger arcs is great than π

2. If the centre of the circle is outside the triangle formed by the two intersection points and the common vertice of the intersecting line segments (Figure 5.7a), then the arclength is equal to the value return by the calculation detailed in Section 4.



(a) The circle centre is inside the triangle formed using the intersection points.



(b) The circle centre is outside the triangle formed using the intersection points.

6 Conclusion

In this report, a method to divide spherical triangles into equal-area triangles, such as desired to achieve the described angular space tessellation, was derived. Arc length calculation for the intersection of these triangles with a conical surface was also achieved. These results allow to more properly represent angular data for modelling and/or simulation purpose.