Localization of a target in clutter by multistatic active sonar

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Abstract

A method is presented for localizing a single target with multistatic active sonar, from a single ping, by means of a numerical search for the highest peak in a probability distribution function. That function is constructed using Bayesian principles. The method takes into account errors in the locations of the transmitter and the receivers, errors in the estimate of the local speed of sound, errors in time and bearing measurements, an imperfect probability of detection, and the appearance of false contacts due to noise. The efficacy of the method is demonstrated by means of simulated multistatic data. Because of the false contacts and the imperfect probability of detection, it is inevitable that the resulting localization will sometimes be very inaccurate. But there is a tendency for multiple receivers to corroborate each other, leading to frequent successful localization in spite of these difficulties. The distribution of the error in localization over many runs is similar in form to the corresponding distribution in the case of perfect detection and no false contacts, except for the appearance of a long tail in the distribution. However, the computational load is very heavy. A streamlined version of the probability distribution function, in which some of the uncertainties are neglected, speeds up the computation by several orders of magnitude with the localization performance being degraded only slightly. The single-ping localization method presented here appears to be a promising foundation for future approaches to the tracking of underwater targets by multistatic active sonar.

Significance for defence and security

The localization of underwater targets via multistatic active sonar is of interest in tactical picture compilation. This paper is concerned with the localization of a target in response to a single ping. Previous work established the superiority of a Bayesian method in an idealized scenario with no false alarms. The present work expands this Bayesian method to deal with more realistic data in which false alarms arising from noise are included.

Résumé

Le présent document porte sur la méthode de localisation d'une cible unique, à partir d'une seule impulsion et au moyen d'un sonar actif multistatique, par la recherche numérique de la crête la plus élevée selon une fonction de distribution des probabilités. Cette fonction est fondée sur les principes de Bayes. La méthode tient compte des erreurs de localisation de l'émetteur et des récepteurs, des erreurs d'estimation de la vitesse locale du son, des erreurs de mesure du temps et du gisement, d'une probabilité de détection imparfaite et de l'apparition de faux contacts causés par le bruit. On a démontré l'efficacité de la méthode au moyen de données multistatiques simulées. En raison des faux contacts et de la probabilité de détection imparfaite, il est inévitable que la localisation qui en découle soit parfois très imprécise. Les nombreux récepteurs tendent toutefois à se corroborer et la localisation est donc souvent réussie malgré ces difficultés. De plus, la distribution des erreurs de localisation sur plusieurs passages est de forme analogue à la distribution correspondante dans le cas d'une détection parfaite sans faux contact, sauf pour l'apparition d'une longue queue de distribution. La charge de calcul est toutefois très lourde. Une version simplifiée de la fonction de distribution des probabilités, dans laquelle on ne tient pas compte de certaines incertitudes, permet d'accélérer le calcul de plusieurs ordres de grandeur et ne réduit que légèrement les performances de localisation. Par ailleurs, la méthode de localisation à une seule impulsion présentée ici semble être une assise prometteuse pour les approches futures de la poursuite de cibles sous marines au moyen d'un sonar actif multistatique.

Importance pour la défense et la sécurité

La localisation de cibles sous-marines au moyen d'un sonar actif multistatique est intéressante pour la compilation de la situation tactique. Le présent document porte sur la localisation d'une cible en réponse à une seule impulsion. Les travaux antérieurs ont permis d'établir la supériorité d'une méthode de Bayes dans un scénario idéalisé sans fausses alarmes. Les travaux actuels étendent cette méthode de Bayes pour traiter des données plus réalistes comportant des fausses alarmes causées par le bruit.

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1 Introduction

A previous paper [1] presented a Bayesian approach to the single-ping localization of a target by multistatic active sonar. It was assumed there that each of N receivers measures both the time of arrival (TOA) and the direction of arrival (DOA) of each received echo. It was further assumed that the problem of association was already overcome—i.e., that we could take a single echo from each receiver and treat them as belonging to the same target. The problem was treated as two-dimensional, with the depths of the source, of the receivers, and of the target all being neglected. Spatial variation in the speed of sound was also neglected, but the local speed of sound in the region of interest was not assumed to be known with perfect precision.

The following quantities were assumed to be either measured or known *a priori*, within certain known error statistics, and all assumed to be Gaussian and mutually uncorrelated:

- the local speed of sound in water, c, assumed to be constant over the region of interest, with uncertainty σ_c
- the position of the transmitter, $\mathbf{x}_0 = (x_0 \quad y_0)^{\mathrm{T}}$, with uncertainty σ_x in each dimension (assuming no correlation between the two dimensions)
- the position of each receiver, $\mathbf{x}_k = (x_k \quad y_k)^{\mathrm{T}}$ for k from 1 to N, with uncertainty σ_x in each dimension (again assuming no correlation between the two dimensions)
- the ping time t (at the transmitter) with uncertainty σ_t
- the TOA of the echo at each receiver, T_k for k from 1 to N, with uncertainty σ_T
- the DOA of the echo at each receiver, θ_k for k from 1 to N, with uncertainty σ_θ (each direction defined as counterclockwise from east)

The method consisted of deriving a rather complicated probability distribution function (pdf) for the position of the target, and finding numerically the position of the peak of that function. The method is described here as "Bayesian" because the derivation of the pdf is based on Bayesian principles; however, a purist might argue that it departs from Bayesian thought insofar as a point estimate for the target position is generated as output.

The Bayesian method was shown, for both simulated [1] and real [2] data, to be superior to a more conventional method [3] that was treated as the baseline.

Acoustic sensing in real-world bodies of water is a noisy affair. Generally, there is a high rate of false alarms ("clutter")[1]. If we retain the assumption that we have a single target, but drop the assumption that we know which echo at each receiver corresponds to that target, the question naturally arises whether we can still localize the target by using the full set of echoes received from a single ping. In this Scientific Report it is argued that we can do so, because of the tendency of multiple receivers to corroborate each other.

[1] The word *clutter* as used in this report refers to false alarms arising from noise. Some authors use the word differently. In sonar applications, that word is sometimes used to refer to persistent, target-like returns from discrete sea-bottom features. Such things are outside the scope of this report.

Section 2 is concerned with the derivation of the full pdf with clutter included, and with methods for finding the position that maximizes the pdf. Section 3 presents the results of a test of the method using simulated multistatic data.

The computational load turns out to be very heavy. Section 4 presents a "streamlined" version of the method, which cuts down greatly on the computation time required, at the cost of neglecting some of the uncertainties. Section 5 presents the results of a test of the streamlined method.

Finally, Section 6 provides discussion and conclusions.

2 The localization method

Subsection 2.1 presents the probability distribution function for the target position. Subsection 2.2 discusses ways of finding the position of the nearest peak from an initial guess. The generation of initial guesses is addressed in Subsection 2.3.

2.1 The probability distribution function

Let n_k denote the number of contacts (echoes) from the k^{th} receiver, for all k from 1 to N. Let T_{kj} and θ_{kj} denote respectively the time of arrival (TOA) and the direction of arrival (DOA) of the j^{th} contact belonging to the k^{th} receiver. Let us assume that the uncertainties σ_T and σ_θ apply to all of the echoes. Let us assume further that we have a single target, with unknown position $\mathbf{X} = (X \quad Y)^{\mathrm{T}}$.

Assuming a uniform and unbounded prior for the target position $\mathbf{X}$, the probability density for the target position is proportional to the corresponding likelihood density:

$$P\left(\mathbf{X} \middle| t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c\right) \propto P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}\right). \tag{1}$$

That likelihood density can be expressed as an integral:

$$\begin{aligned} &P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}\right) \\ &= \int \dots \int P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}\right) P(\tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) d\tilde{t}\, d\tilde{\mathbf{x}}_0 d\tilde{\mathbf{x}}_1 \dots d\tilde{\mathbf{x}}_N d\tilde{c}, \end{aligned} \tag{2}$$

where $\tilde{t}$, $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{x}}_k$, and $\tilde{c}$ are assumed for the sake of argument as the true values corresponding respectively to the measurements t, $\mathbf{x}_0$, $\mathbf{x}_k$, and c. Taking a uniform and unbounded prior for each of these variables, as in [1], we get

$$\begin{aligned} &P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}\right) \\ &\propto \int \dots \int P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}\right) d\tilde{t}\, d\tilde{\mathbf{x}}_0 d\tilde{\mathbf{x}}_1 \dots d\tilde{\mathbf{x}}_N d\tilde{c}. \end{aligned} \tag{3}$$

The integrand in Equation (3) above can be decomposed into a product:

$$\begin{aligned} &P\left(t, \{n_k\}, \{T_{kj}\}, \{\theta_{kj}\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}\right) \\ &\qquad = P(t, \mathbf{x}_0, \{\mathbf{x}_k\}, c | \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) P\left(\{n_k\}, \{T_{kj}\}, \{\theta_{kj}\} \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}\right). \end{aligned} \tag{4}$$

The first factor is given by

$$P(t,\mathbf{x}_0,\{\mathbf{x}_k\},c|\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}) = \frac{1}{(2\pi)^{N+2}\sigma_t\sigma_c\sigma_x^{2N+2}}\exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2}+\frac{(c-\tilde{c})^2}{\sigma_c^2}+\frac{(x_0-\tilde{x}_0)^2}{\sigma_x^2} + \frac{(y_0-\tilde{y}_0)^2}{\sigma_x^2}+\sum_k\left(\frac{(x_k-\tilde{x}_k)^2}{\sigma_x^2}+\frac{(y_k-\tilde{y}_k)^2}{\sigma_x^2}\right)\right]\right). \tag{5}$$

The second factor in (4) can be expressed as a sum over hypotheses, where each hypothesis specifies which, if any, of the returns from each receiver corresponds to a successful detection of the target. Thus there are, all together, $\prod_k(1+n_k)$ possible hypotheses. For any hypothesis h, let $L(h)$ denote the number of returns (out of N) that correspond to successful detections according to that hypothesis. Also, let h_k denote the k^{th} element of the hypothesis h, defined as follows: If the k^{th} receiver detects the target according to the hypothesis h, then let h_k take the value of the index of the return from that receiver that corresponds to a successful detection according to h. If the k^{th} receiver does not detect the target according to the hypothesis h, then $h_k = 0$. Now

$$P\big(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\big|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\big) = \sum_h P\big(\{n_k\},\{T_{kj}\},\{\theta_{kj}\},h\big|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\big). \tag{6}$$

and

$$P\big(\{n_k\},\{T_{kj}\},\{\theta_{kj}\},h\big|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\big) = P(\{n_k\},h|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c})P\big(\{T_{kj}\},\{\theta_{kj}\}\big|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c},\{n_k\},h\big). \tag{7}$$

Let us assume a fixed probability of detection P_D and a Poisson distribution for the number of false alarms, with λ denoting the expected number of false alarms. Then the first factor in (7) is given by

$$P(\{n_k\},h|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}) = \prod_k P(n_k,h_k), \tag{8}$$

where

$$P(n_k,h_k) = \begin{cases} \dfrac{1}{n_k}P_D\dfrac{e^{-\lambda}\lambda^{n_k-1}}{(n_k-1)!}, h_k > 0 \\ (1-P_D)\dfrac{e^{-\lambda}\lambda^{n_k}}{n_k!}, h_k = 0. \end{cases} \tag{9}$$

The extra factor of $1/n_k$ in the case of $h_k > 0$ arises because there are n_k equal hypotheses whose summed probability is the probability that we have a detection and $n_k - 1$ false alarms. Meanwhile, the probability for the case of $h_k = 0$ is the probability that we have no detection and n_k false alarms. Therefore (8) becomes

$$P(\{n_k\}, h | \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) = \frac{e^{-N\lambda} \lambda^{\sum_k (n_k - 1)}}{\prod_k n_k!} P_D^{L(h)} \big((1 - P_D)\lambda\big)^{N - L(h)}. \tag{10}$$

Now let us turn to the second factor in (7). The notation $\{T_{kj}\}$ is a shorthand for the set of all TOA measurements for all k from 1 to N and for all j from 1 to n_k. Similarly for $\{\theta_{kj}\}$, the set of all DOA measurements. We need a notation to refer to the set of all TOA (or all DOA) measurements for a given (fixed) k. Let us use $\{T_{\cdot j}\}_k$ and $\{\theta_{\cdot j}\}_k$ for this purpose. So now

$$P\big(\{T_{kj}\}, \{\theta_{kj}\} \big| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}, \{n_k\}, h\big) = \prod_k P\left(\{T_{\cdot j}\}_k, \{\theta_{\cdot j}\}_k \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}, n_k, h_k\right). \tag{11}$$

We assume that the pdf for a false alarm is uniform over the space of measurements. Let Λ denote the total "volume" of that space. Then, for the case of $h_k = 0$, we have

$$P\left(\{T_{\cdot j}\}_k, \{\theta_{\cdot j}\}_k \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}, n_k, h_k\right) = \frac{n_k!}{\Lambda^{n_k}}, \tag{12}$$

where the factor of $n_k!$ is needed to account for the permutations of the false alarms, since the hypothesis does not specify an ordering thereof. For the case of $h_k > 0$, we have similarly

$$\begin{aligned} &P\left(\{T_{\cdot j}\}_k, \{\theta_{\cdot j}\}_k \middle| \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}, n_k, h_k\right) \\ &\qquad = \frac{(n_k - 1)!}{\Lambda^{n_k - 1}} \frac{1}{2\pi\sigma_\theta\sigma_T} \exp\left[-\tfrac{1}{2}\left(\frac{\big(\theta_{kh_k} - \tilde{\theta}_k\big)^2}{\sigma_\theta^2} + \frac{\big(T_{kh_k} - \tilde{T}_k\big)^2}{\sigma_T^2}\right)\right] \end{aligned} \tag{13}$$

where

$$\tilde{T}_k = \tilde{t} + \frac{\tilde{S}_0 + \tilde{S}_k}{\tilde{c}}, \tag{14}$$

$$\tilde{S}_0 = \sqrt{(\tilde{x}_0 - X)^2 + (\tilde{y}_0 - Y)^2}, \tag{15}$$

$$\tilde{S}_k = \sqrt{(\tilde{x}_k - X)^2 + (\tilde{y}_k - Y)^2}, \tag{16}$$

$$\sin\tilde{\theta}_k = \frac{Y - \tilde{y}_k}{\tilde{S}_k}, \tag{17}$$

$$\cos\tilde{\theta}_k = \frac{X - \tilde{x}_k}{\tilde{S}_k}, \tag{18}$$

and the DOA measurements are understood to be interpreted in such a way that $\left|\theta_{kj} - \tilde{\theta}_k\right| \leq \pi$ always. So we have

$$P\left(\{T_{kj}\},\{\theta_{kj}\}\middle|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c},\{n_k\},h\right)$$
$$= \left(\frac{\prod_k n_k!}{\Lambda^{\sum_k(n_k-1)}}\right)\frac{1}{\Lambda^{N-L(h)}}\left(\frac{1}{2\pi\sigma_\theta\sigma_T}\right)^{L(h)}$$
$$\times\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)\exp\left[-\tfrac{1}{2}\sum_{k:h_k>0}\left(\frac{\left(\theta_{kh_k}-\tilde{\theta}_k\right)^2}{\sigma_\theta^2}+\frac{\left(T_{kh_k}-\tilde{T}_k\right)^2}{\sigma_T^2}\right)\right]. \tag{19}$$

And (6) becomes

$$P\left(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\middle|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\right)$$
$$= e^{-N\lambda}\left(\frac{\lambda}{\Lambda}\right)^{\sum_k(n_k-1)}\sum_h\left((1-P_D)\frac{\lambda}{\Lambda}\right)^{N-L(h)}\left(\frac{P_D}{2\pi\sigma_\theta\sigma_T}\right)^{L(h)}$$
$$\times\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)\exp\left[-\tfrac{1}{2}\sum_{k:h_k>0}\left(\frac{\left(\theta_{kh_k}-\tilde{\theta}_k\right)^2}{\sigma_\theta^2}+\frac{\left(T_{kh_k}-\tilde{T}_k\right)^2}{\sigma_T^2}\right)\right]. \tag{20}$$

Now the fraction λ/Λ is the expected rate of false alarms in both time and bearing. Let φ denote the expected false alarm rate in time only, $\varphi = 2\pi\lambda/\Lambda$. (Let us assume further that both φ and P_D are known.) Then we have

$$P\left(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\middle|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\right)$$
$$= \frac{e^{-N\lambda}\varphi^{\sum_k(n_k-1)}}{(2\pi)^{\sum_k n_k}}\sum_h\left((1-P_D)\varphi\right)^{N-L(h)}\left(\frac{P_D}{\sigma_\theta\sigma_T}\right)^{L(h)}$$
$$\times\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)\exp\left[-\tfrac{1}{2}\sum_{k:h_k>0}\left(\frac{\left(\theta_{kh_k}-\tilde{\theta}_k\right)^2}{\sigma_\theta^2}+\frac{\left(T_{kh_k}-\tilde{T}_k\right)^2}{\sigma_T^2}\right)\right]. \tag{21}$$

Now, for any hypothesis h, let

$$J(h) = \exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2}+\frac{(c-\tilde{c})^2}{\sigma_c^2}+\frac{(x_0-\tilde{x}_0)^2}{\sigma_x^2}+\frac{(y_0-\tilde{y}_0)^2}{\sigma_x^2}\right.\right.$$
$$\left.\left.+\sum_{k:h_k>0}\left(\frac{(x_k-\tilde{x}_k)^2}{\sigma_x^2}+\frac{(y_k-\tilde{y}_k)^2}{\sigma_x^2}+\frac{\left(\theta_{kh_k}-\tilde{\theta}_k\right)^2}{\sigma_\theta^2}+\frac{\left(T_{kh_k}-\tilde{T}_k\right)^2}{\sigma_T^2}\right)\right]\right). \tag{22}$$

Now we can write our integrand in (3) as

$$P\big(t,\{n_k\},\{T_{kj}\},\{\theta_{kj}\},\mathbf{x}_0,\{\mathbf{x}_k\},c\big|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}\big)$$
$$\propto \sum_h \Bigg[\big((1-P_D)\varphi\big)^{N-L(h)}\left(\frac{P_D}{\sigma_\theta\sigma_T}\right)^{L(h)}\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)\frac{J(h)}{(2\pi)^{N+2}\sigma_t\sigma_c\sigma_x^{2N+2}}$$
$$\times\exp\left(-\tfrac{1}{2}\sum_{k:h_k=0}\left[\frac{(x_k-\tilde{x}_k)^2}{\sigma_x^2}+\frac{(y_k-\tilde{y}_k)^2}{\sigma_x^2}\right]\right)\Bigg]. \tag{23}$$

The integral is (2N+4)-fold, over the variables $\tilde{t}$, $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{x}}_k$, and $\tilde{c}$. If we express the integral of a sum as a sum of integrals, then for each hypothesis (i.e., each term of the sum) we can immediately get rid of 2(N – L(h)) of the variables of integration, because the integrals over the $\tilde{\mathbf{x}}_k$, for those values of k for which $h_k = 0$, are simple Gaussians. So we get

$$P\big(t,\{n_k\},\{T_{kj}\},\{\theta_{kj}\},\mathbf{x}_0,\{\mathbf{x}_k\},c\big|\mathbf{X}\big)$$
$$\propto \sum_h \Bigg[\frac{\big((1-P_D)\varphi\big)^{N-L(h)}P_D^{L(h)}}{(2\pi)^{L(h)+2}\sigma_t\sigma_c\sigma_x^{2L(h)+2}\sigma_\theta^{L(h)}\sigma_T^{L(h)}}\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)$$
$$\times\int\dots\int J(h)d\tilde{t}\,d\tilde{c}\,d\tilde{\mathbf{x}}_0\prod_{k:h_k>0}d\tilde{\mathbf{x}}_k\Bigg]. \tag{24}$$

Now, for each hypothesis, the remaining integral is (2L(h)+4)-fold, over the variables $\tilde{t}$, $\tilde{\mathbf{x}}_0$, and $\tilde{c}$ as well as the $\tilde{\mathbf{x}}_k$ for those values of k corresponding to a receiver that made a successful detection (according to the hypothesis in question). The integrand $J(h)$ is the same as the integrand considered in [1], except that the sum is now only over a subset of the full range of values of k. An approximate closed-form result for the integral was given in Equations (16)–(18) of [1]. That result, applied separately to each hypothesis, can be used here—but wherever N appears, we need to use $L(h)$ instead, and every sum over k now includes only those values of k for which $h_k > 0$. Also, T_k and θ_k are replaced respectively by T_{kh_k} and θ_{kh_k}. Let $I(h)$ denote the result of the integral as reported in [1], with the few modifications just described. A constant factor was neglected in that paper, because there was no combination of hypotheses and because the final pdf will not be normalized in any case. Now, however, we must restore that constant factor, so that the results from different hypotheses can be combined properly. With that factor included, the result of the integral is

$$\int\dots\int J(h)d\tilde{t}\,d\tilde{c}\,d\tilde{\mathbf{x}}_0\prod_{k:h_k>0}d\tilde{\mathbf{x}}_k=\frac{(2\pi)^{L(h)+2}\sigma_x^{2L(h)+2}\sigma_t I(h)}{\sqrt{2}\left(1+\frac{\sigma_x^2}{c^2\sigma_T^2}\right)^{L(h)/2}\sqrt{1+L(h)\left(\frac{\sigma_x^2+c^2\sigma_t^2}{\sigma_x^2+c^2\sigma_T^2}\right)}}, \tag{25}$$

whence we get

$$P\big(\mathbf{X}\big|t,\{n_k\},\{T_{kj}\},\{\theta_{kj}\},\mathbf{x}_0,\{\mathbf{x}_k\},c\big) \propto \sum_h \frac{\big((1-P_D)\varphi\big)^{N-L(h)}(P_D c)^{L(h)} I(h)\left(\prod_{k:h_k>0}\frac{1}{n_k}\right)}{\sigma_\theta^{L(h)}(\sigma_x^2+c^2\sigma_T^2)^{L(h)/2}\sqrt{1+L(h)\left(\frac{\sigma_x^2+c^2\sigma_t^2}{\sigma_x^2+c^2\sigma_T^2}\right)}} \tag{26}$$

as the final expression for the (non-normalized) probability distribution function. See Annex A for an explicit presentation of $I(h)$.

The value of $\mathbf{X}$ for which the pdf is maximized will be taken as the estimate of the target position.

2.2 Finding the nearest peak to a prior guess

Suppose we have a pdf, $P(\mathbf{X})$, and that this function has a local maximum at $\mathbf{X}=\mathbf{X}_0$. Suppose that the pdf is roughly Gaussian in shape, in the near vicinity of the peak, and that we have a rough guess $\mathbf{X}_{\text{guess}}$ for the position of the peak.

$$P(\mathbf{X}) \approx K\exp\left[-\tfrac{1}{2}(\mathbf{X}-\mathbf{X}_0)^{\mathrm{T}}\mathbf{P}_0^{-1}(\mathbf{X}-\mathbf{X}_0)\right]. \tag{27}$$

Let D be a length that is characteristic of the width of the peak, or smaller. In the present application, we might take $D=c\sigma_T$. Let $\mathbf{u}(\theta)=(\cos\theta \quad \sin\theta)^{\mathrm{T}}$ denote the unit vector in the direction given by its argument. Let

$$g(\theta)=\frac{2}{D^2}\ln\left(\frac{P\big(\mathbf{X}_{\text{guess}}\big)}{P\left(\mathbf{X}_{\text{guess}}+D\mathbf{u}(\theta)\right)}\right). \tag{28}$$

Then from a rearrangement of Equation (27), we get

$$\mathbf{P}_0 \approx \begin{pmatrix} \frac{1}{2}\big(g(0)+g(\pi)\big) & \frac{1}{4}\left(g\left(\frac{\pi}{4}\right)-g\left(-\frac{\pi}{4}\right)-g\left(\frac{3\pi}{4}\right)+g\left(-\frac{3\pi}{4}\right)\right) \\ \frac{1}{4}\left(g\left(\frac{\pi}{4}\right)-g\left(-\frac{\pi}{4}\right)-g\left(\frac{3\pi}{4}\right)+g\left(-\frac{3\pi}{4}\right)\right) & \frac{1}{2}\left(g\left(\frac{\pi}{2}\right)+g\left(-\frac{\pi}{2}\right)\right) \end{pmatrix}^{-1} \tag{29}$$

and

$$\mathbf{X}_0 \approx \mathbf{X}_{\text{guess}}-\frac{D}{4}\mathbf{P}_0\begin{pmatrix} g(0)-g(\pi) \\ g\left(\frac{\pi}{2}\right)-g\left(-\frac{\pi}{2}\right)\end{pmatrix}. \tag{30}$$

Equations (29) and (30) immediately suggest an iterative process for finding the position of the peak that is nearest to some initial guess, and also for finding a covariance matrix that describes the shape of the pdf in the near vicinity of that peak. Whenever such a method shows signs of failing to converge—for example, if there is growth from one iteration to the next in the length of the vector between two consecutive guesses—then we can fall back on a brute-force method to find the peak position, with Equation (29) then being used for the covariance.

For one possible brute-force method, let *D* be a variable that is set initially to $D = c\sigma_T$. Consider the greatest value of $P\left(\mathbf{X}_{\text{guess}} + D\mathbf{u}(\theta)\right)$ for θ being any integer multiple of $\pi/6$. If the latter is greater than $P(\mathbf{X}_{\text{guess}})$, then $\mathbf{X}_{\text{guess}}$ is changed to that new position; otherwise, *D* is decreased by a factor of two, unless *D* is already less than some pre-specified error tolerance, in which case the procedure is terminated.

2.3 Finding the top peaks

An initial guess for a peak position can be derived simply from each contact from each receiver, by intercepting the bearing measurement with the ellipse defined by the time measurement. For any given contact (from the k^{th} receiver), with TOA and DOA measurements denoted by T and θ, respectively, the initial guess for that contact is given by

$$\begin{pmatrix} X_{\text{contact}} \\ Y_{\text{contact}} \end{pmatrix} = \begin{pmatrix} x_k + r\cos\theta \\ y_k + r\sin\theta \end{pmatrix}, \tag{31}$$

where

$$r = \frac{c^2(T-t)^2 - L^2}{2(c(T-t) - L\cos\alpha)}, \tag{32}$$

$$L = \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}, \tag{33}$$

$$\cos(\alpha + \theta) = \frac{x_0 - x_k}{L}, \tag{34}$$

$$\sin(\alpha + \theta) = \frac{y_0 - y_k}{L}. \tag{35}$$

For each contact, the nearest peak to the initial guess can be found by the methods described in Subsection 2.2. To find the highest peak overall, these peaks can all be compared.

To save computing time, we can evaluate the pdf at the initial guess positions, and sort the contacts from highest to lowest pdf value, then find the nearest peak to each initial guess in turn, stopping when *n* distinct peaks (for some *n*) have been found. This was the approach taken for the results in the next section, using *n* = 4.

3 Test of the localization method

A simple simulation, described in Subsection 3.1, was used to demonstrate the efficacy of the localization method presented in Section 2. Subsection 3.2 presents the results.

3.1 Simulation

One thousand "runs" of a simple simulation were made. Each run consisted of the following steps:

1. A position is generated for a stationary target within a square region of sea, 10 km on each side, using a uniform distribution.
2. A position is generated similarly for a transmitter, and again for each of three receivers, all within the same square region, and using a uniform distribution in each case.
3. "Measured" positions of the transmitter and receivers are generated from the true values, by adding a random Gaussian error to each dimension of each position, with a standard deviation of $\sigma_x = 10$ m.
4. A "measured" value of the ping time is generated from the true value (which is set to 0 s) by adding a random Gaussian error with a standard deviation of $\sigma_t = 0.1$ s.
5. A "measured" value of the local speed of sound in water is generated from the true value (which is set to 1500 m/s) by adding a random Gaussian error with a standard deviation of $\sigma_c = 10 \text{ m/s}$.
6. Each receiver generates contacts as follows:
    a. With a probability of $P_D = 0.8$, the receiver generates a contact based on the true position of the target. If this contact is generated, its TOA deviates from the true TOA (which is computed from the true positions, the true ping time, and the true speed of sound) by adding a random Gaussian error with a standard deviation of $\sigma_T = 0.1$ s, and its DOA deviates from the true DOA (which is computed from the true positions) by adding a random Gaussian error with a standard deviation of $\sigma_\theta = 0.1$ rad.
    b. False contacts are generated. The number of false contacts is random, taken from a Poisson distribution with a mean of 7.5. The TOA of each false contact is random and uniformly distributed over a 15 second interval that starts with the "measured" ping time. (Thus a false alarm rate of $\varphi = 0.5 \text{ s}^{-1}$ is used.) The DOA of each false contact is random and uniformly distributed from zero to 2π.
7. The location of the highest peak in the pdf is found by the methods of Section 2. The values of σ_x, σ_c, σ_t, σ_T, σ_θ, P_D, and φ are treated as known. The resulting location is taken as the estimate of the target's position. A covariance matrix for that estimate is also generated as described in Subsection 2.2.
8. The absolute error and relative error of the estimate are computed and recorded.

For an estimated position $\mathbf{X}_{\text{est}}$ with covariance $\mathbf{P}_{\text{est}}$, the absolute and relative errors are defined as

$$E_{\text{abs}} = |\mathbf{X}_{\text{est}} - \mathbf{X}_{\text{true}}|, \tag{36}$$

$$E_{\text{rel}} = \sqrt{(\mathbf{X}_{\text{est}} - \mathbf{X}_{\text{true}})^{\text{T}} \mathbf{P}_{\text{est}}^{-1} (\mathbf{X}_{\text{est}} - \mathbf{X}_{\text{true}})}. \tag{37}$$

The relative error can be viewed as the distance between the estimated position and the true position, measured as a number of standard deviations.

In addition to the thousand runs done as described, another twenty runs were performed with five receivers instead of three.

All runs were performed using *Mathematica*®.

3.2 Results

We consider first the thousand runs performed with three receivers. The median absolute error was 210 m, and the median relative error was 1.4, over all of these runs.

In several runs of the simulation, the peak that was selected to represent the estimate was not related to the contacts (if any) that were generated from the true target position. Such "misses" are inevitable, given that we are dealing with single-ping localization, and it is easy for false contacts from several receivers to coincide by chance, thus creating a peak in the pdf that is higher than any peak that is nearer to the true target position. This means that some runs resulted in very high absolute and relative errors. As shown in Figure 1, the relative error roughly follows a chi-squared distribution, with a peak near unity. However, the relative error distribution also has a long tail due to the "misses". The figure cuts off at a relative error value of 5, so this tail is not visible there.

Because of the "misses", there is no use in presenting the mean or the RMS of the absolute or relative errors over all runs. We can, however, present the RMS of the errors over all runs without a "miss", provided that we can find a provisional definition of a "miss". Let Z denote a variable threshold value for the relative error. For any given value of Z, we note the number of runs for which the relative error is less than Z, as well as the RMS relative error and the RMS absolute error among those runs. These results are plotted in Figures 2–4, with Z running from 2 to 10. Overall, if we confine our attention to those runs without a "miss", the relative error has an RMS value of about 1.2 to 1.9 while the absolute error has an RMS value of about 250 m to 400 m. For the case of $Z = 5$ in particular, there were 803 runs in which the relative error was less than 5, and among those runs the RMS absolute error was 330 m and the RMS relative error was 1.5. These results are, of course, dependent on the many arbitrary parameters used in the simulation.

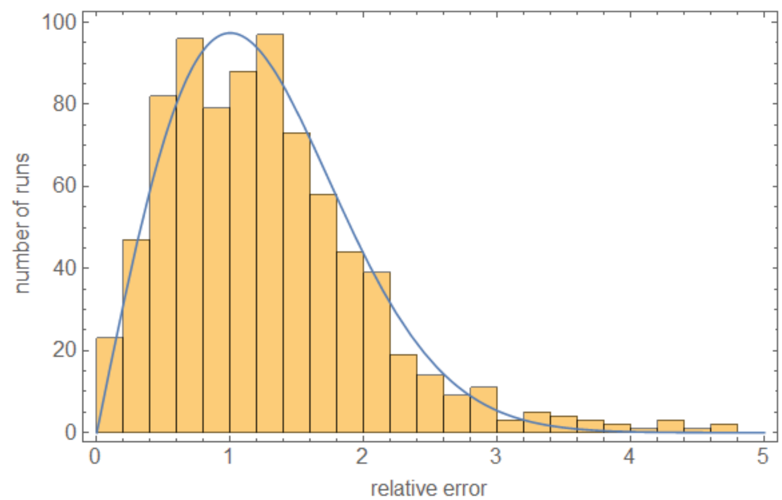

Figure 1: *A histogram of the runs according to the relative error. The superimposed curve shows a chi-squared distribution for comparison.*

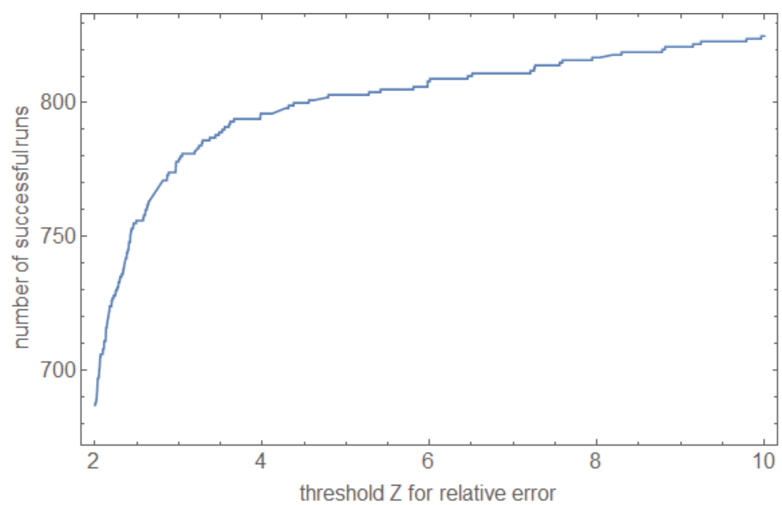

Figure 2: *The number of runs with relative error less than Z, shown as a function of Z.*

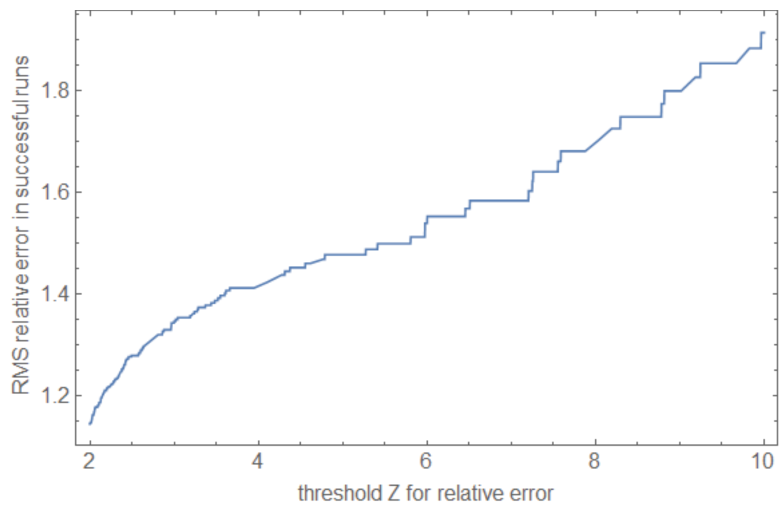

Figure 3: *The RMS relative error among runs in which the relative error is less than Z, shown as a function of Z.*

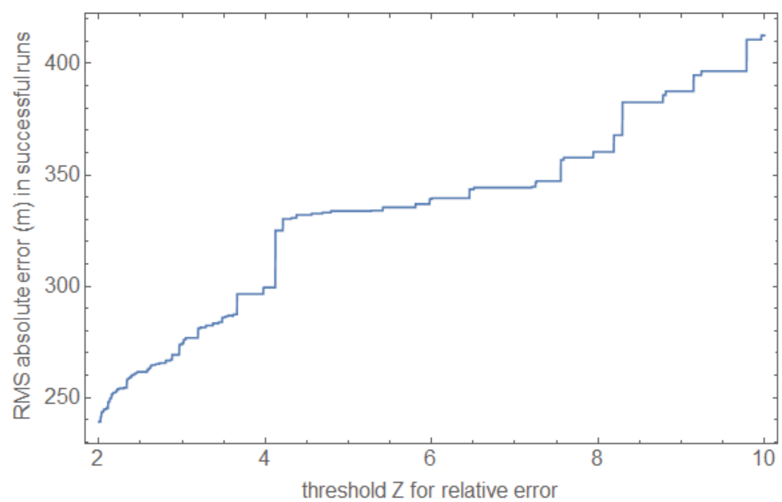

Figure 4: *The RMS absolute error (in metres) among runs in which the relative error is less than Z, shown as a function of Z.*

The results above are all concerned with the thousand runs that used three receivers.

Among the twenty runs with five receivers, the median absolute error was 130 m and the median relative error was 1.4. Eighteen of the twenty runs had a relative error less than 5. Among these eighteen runs, the RMS absolute error was 220 m and the RMS relative error was 1.4.

The mean elapsed time for each run was 2.0 minutes, for the runs with three receivers, and 5.8 hours, for the runs with five receivers. The very large amount of computing time in the latter case accounts for why only twenty runs were performed. Each evaluation of the pdf involves a sum over hypotheses (see Equation (26)), and the number of hypotheses proliferates rapidly with an increase in the number of receivers (see the discussion before Equation (6)). Hence the large difference in computing time between the cases of three receivers and of five receivers.

4 Streamlining the pdf

The pdf is evaluated many times in order to find the position estimate, and each evaluation involves a sum over hypotheses, as shown in Equation (26). Moreover, the summand is itself rather complicated. The upshot is that a very large processing time was required for each run. A tracking system based on these principles will need to perform even more computation for each ping, and therefore some way of streamlining the process will have to be found, in order to enable such a tracking system to process the data in real time.

There is a very simple way to cut down greatly on the computational load, and that is to neglect the uncertainties in transmitter and receiver positions, in the speed of sound, and in the ping time, while still taking into account the uncertainties in the measurements of TOA and DOA. It is worthwhile to investigate how the results presented in Section 3 will be impacted by such a modification to the method.

We can rewrite Equation (1) as

$$P\big(\mathbf{X}\big|\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\big) \propto P\big(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\big|\mathbf{X}\big), \tag{38}$$

with no integral required. Following the discussion up to Equation (21), we get

$$\begin{aligned} &P\big(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\big|\mathbf{X}\big) \\ &\quad= \frac{e^{-N\lambda}\varphi^{\sum_k(n_k-1)}}{(2\pi)^{\sum_k n_k}} \sum_h \big((1-P_D)\varphi\big)^{N-L(h)} \left(\frac{P_D}{\sigma_\theta\sigma_T}\right)^{L(h)} \\ &\quad\times \left(\prod_{k:h_k>0}\frac{1}{n_k}\right) \exp\left[-\tfrac{1}{2}\sum_{k:h_k>0}\left(\frac{\big(\theta_{kh_k}-\breve{\theta}_k\big)^2}{\sigma_\theta^2}+\frac{\big(T_{kh_k}-\breve{T}_k\big)^2}{\sigma_T^2}\right)\right]. \end{aligned} \tag{39}$$

where we have

$$\breve{T}_k = t + \frac{S_0+S_k}{c}, \tag{40}$$

$$S_0 = \sqrt{(x_0-X)^2+(y_0-Y)^2}, \tag{41}$$

$$S_k = \sqrt{(x_k-X)^2+(y_k-Y)^2}, \tag{42}$$

$$\sin\breve{\theta}_k = \frac{Y-y_k}{S_k}, \tag{43}$$

$$\cos\breve{\theta}_k = \frac{X-x_k}{S_k}, \tag{44}$$

in place of (14) through (18). But (39) can be expressed more simply as

$$P\left(\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\middle|\mathbf{X}\right)$$
$$= \frac{e^{-N\lambda}\varphi^{\sum_k(n_k-1)}}{(2\pi)^{\sum_k n_k}} \prod_{k:n_k=0} \left((1-P_D)\varphi\right) \prod_{k:n_k>0} \left((1-P_D)\varphi + \frac{P_D}{n_k\sigma_\theta\sigma_T} \sum_j \exp\left[-\tfrac{1}{2}\left(\frac{\left(\theta_{kj}-\breve{\theta}_k\right)^2}{\sigma_\theta^2} + \frac{\left(T_{kj}-\breve{T}_k\right)^2}{\sigma_T^2}\right)\right]\right). \tag{45}$$

Therefore we have

$$P\left(\mathbf{X}\middle|\{n_k\},\{T_{kj}\},\{\theta_{kj}\}\right)$$
$$\propto \prod_{k:n_k>0} \left((1-P_D)\varphi + \frac{P_D}{n_k\sigma_\theta\sigma_T} \sum_j \exp\left[-\tfrac{1}{2}\left(\frac{\left(\theta_{kj}-\breve{\theta}_k\right)^2}{\sigma_\theta^2} + \frac{\left(T_{kj}-\breve{T}_k\right)^2}{\sigma_T^2}\right)\right]\right). \tag{46}$$

Equation (46) replaces Equation (26).

5 Test of the streamlined method

The procedure of Section 3 was repeated, with a thousand runs using three receivers and another thousand runs using five receivers, but now with the simplified pdf of Section 4, in which σ_x, σ_c, and σ_t are neglected. The methods of Section 2 were otherwise unchanged.

Again, we consider first the thousand runs performed with three receivers. The median absolute error was 230 m, and the median relative error was 2.1, over all of these runs. The mean elapsed time was 0.44 seconds per run.

As shown in Figure 5, the relative error no longer appears to follow a chi-squared distribution.

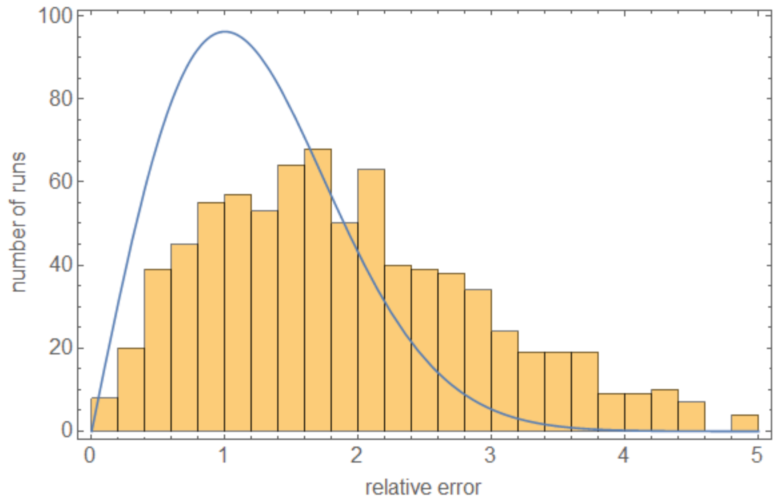

Figure 5: *A histogram of the runs according to the relative error. The superimposed curve shows a chi-squared distribution for comparison. (Streamlined case.)*

RMS errors are presented here the same way they were in Section 3. Again, let Z denote a variable threshold value for the relative error. For any given value of Z, we note the number of runs for which the relative error is less than Z, as well as the RMS relative error and the RMS absolute error among those runs. These results are plotted in Figures 6–8, with Z running from 2 to 10. For the case of $Z = 5$ in particular, there were 793 runs in which the relative error was less than 5, and among those runs the RMS absolute error was 330 m and the RMS relative error was 2.2.

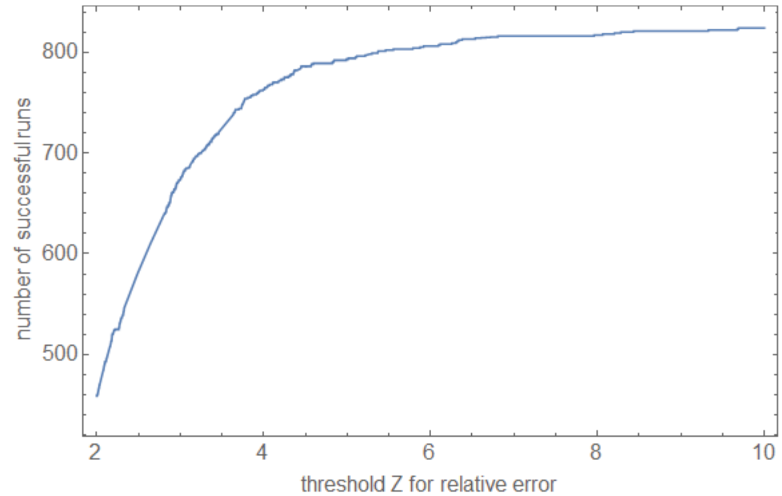

Figure 6: *The number of runs with relative error less than Z, shown as a function of Z. (Streamlined case.)*

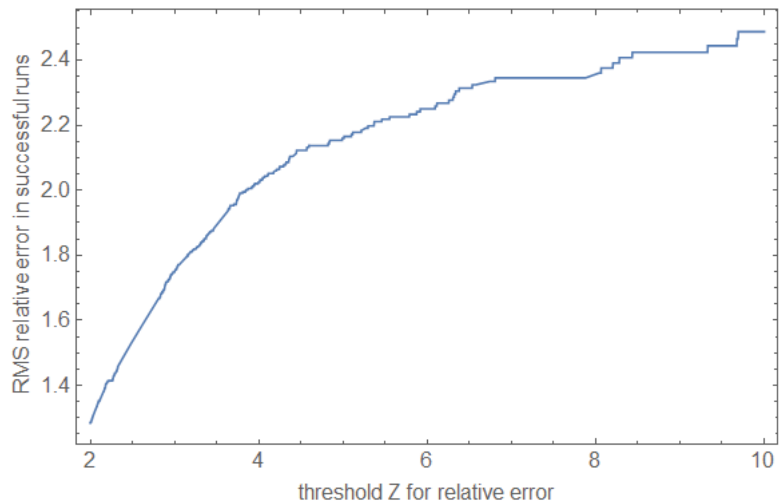

Figure 7: *The RMS relative error among runs in which the relative error is less than Z, shown as a function of Z. (Streamlined case.)*

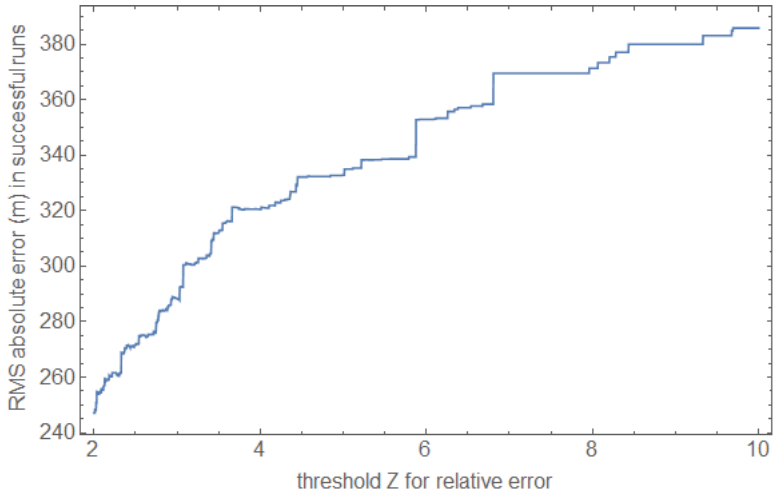

Figure 8: *The RMS absolute error (in metres) among runs in which the relative error is less than Z, shown as a function of Z. (Streamlined case.)*

The higher relative error arising from neglecting several of the uncertainties indicates that the amount of error in the final estimated position is being underestimated. This shortcoming can be addressed very roughly by inflating the covariance by an amount related to σ_x and σ_t. Let the "adjusted relative error" be defined as

$$E_{\text{adjrel}} = \sqrt{(\mathbf{X}_{\text{est}} - \mathbf{X}_{\text{true}})^{\text{T}} \left(\mathbf{P}_{\text{est}} + \begin{pmatrix} \sigma_x^2 + c^2\sigma_t^2 & 0 \\ 0 & \sigma_x^2 + c^2\sigma_t^2 \end{pmatrix} \right)^{-1} (\mathbf{X}_{\text{est}} - \mathbf{X}_{\text{true}})}. \qquad (47)$$

Then the median adjusted relative error was 1.1. There were 819 runs in which the adjusted relative error was less than 5, and among those runs the RMS absolute error was 360 m and the RMS adjusted relative error was 1.3.

There were also a thousand runs using five receivers. The median absolute error was 150 m, and the median relative error was 1.8, over all of these runs. The mean elapsed time was 0.98 seconds per run. There were 925 runs in which the relative error was less than 5, and among those runs the RMS absolute error was 220 m and the RMS relative error was 2.2.

If we again define the adjusted relative error as in Equation (47), then the median adjusted relative error was 0.8. There were 956 runs in which the adjusted relative error was less than 5, and among those runs the RMS absolute error was 240 m and the RMS adjusted relative error was 1.1.

Table 1 presents a summary of the results. For each of the four cases (either three or five receivers, and either the full pdf or the streamlined pdf), the table indicates the number of runs performed, the mean computation time per run, the number of runs for which the relative error was less than 5, the median relative error, the median absolute error, the RMS relative error for the cases in which that error was less than 5, and the RMS absolute error for the cases in which the relative error was less than 5. Numbers in parentheses indicate where the adjusted relative error was used in place of the relative error.

Table 1: *Summary of results. Numbers in parentheses indicate where the adjusted relative error is used in place of the relative error.*

Number of receivers	3	3	5	5
Type of pdf	full	streamlined	full	streamlined
Number of runs	1000	1000	20	1000
Mean computation time	2.0 min	0.44 s	5.8 hr	0.98 s
Number of runs with $E_{\mathrm{rel}} < 5$	803	793 (819)	18	925 (956)
Median E_{rel}	1.4	2.1 (1.1)	1.4	1.8 (0.8)
Median E_{abs}	210 m	230 m	130 m	150 m
RMS E_{rel} for runs with $E_{\mathrm{rel}} < 5$	1.5	2.2 (1.3)	1.4	2.2 (1.1)
RMS E_{abs} for runs with $E_{\mathrm{rel}} < 5$	330 m	330 m (360 m)	220 m	220 m (240 m)

6 Conclusion

This report presents a Bayesian method for the localization of a target by means of a single ping of a multistatic active sonar system. In its fullest form, the method accounts for transmitter and receiver location errors, errors in the local estimate of the speed of sound, and errors in the measurements of both time and bearing. The method consists mainly of a numerical search for the highest peak in a rather complicated pdf.

The method also deals with clutter and with imperfect detection probabilities. Hence, it sometimes happens that the resulting estimate of the target position is far off from the true target position. These occasional failures ("misses") are an unavoidable consequence of the problem under consideration, because it may happen that false contacts from several receivers coincide, and it may happen that the target is not detected at all.

Aside from the "misses", the distribution of absolute and relative error are similar to what one would expect from localization in the ideal case of perfect detection probability and no clutter.

In order to reduce or mitigate the "misses", a system needs to consider several pings in succession. Therefore, the natural next step in the development of this work will be to create a tracking algorithm that makes use of the principles that underlie the single-ping method considered here. Such a development has already appeared for the clutter-free case, in the form of the "Bayesian pseudo-Kalman filter" [2]. An analogous method for the cluttered case should surely be feasible.

The computational load is a concern. The mean elapsed time for each run, in the simulation of Section 3, was about 120 seconds in the case of three receivers, and over two orders of magnitude greater in the case of five receivers. A tracking system based on these principles will need to perform even more computation for each ping, and therefore some way of streamlining the process is required in order to enable the tracking system to process the data in real time.

Section 4 presented a streamlined version of the pdf in which the transmitter and receiver location errors, the error in the speed of sound, and the error in the ping generation time are all neglected. The localization results are not much worse, while the computational load is reduced greatly.

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[2] D. J. Peters, "Application of Bayesian multistatic localization to sea trial data," DRDC-RDDC-2016-R245, Defence Research and Development Canada, December 2016.

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Annex A Evaluation of the key integral

We evaluate (approximately) the integral in (25), with the integrand given explicitly in (22).

First, let S_0, S_k, and $\breve{\theta}_k$ be defined as in (41) through (44). Then we make the transformation

$$\begin{pmatrix} \tilde{x}'_k \\ \tilde{y}'_k \end{pmatrix} = \begin{pmatrix} -\cos\breve{\theta}_k & -\sin\breve{\theta}_k \\ \sin\breve{\theta}_k & -\cos\breve{\theta}_k \end{pmatrix} \begin{pmatrix} \tilde{x}_k - X \\ \tilde{y}_k - Y \end{pmatrix}, \tag{A.1}$$

$$\begin{pmatrix} x'_k \\ y'_k \end{pmatrix} = \begin{pmatrix} -\cos\breve{\theta}_k & -\sin\breve{\theta}_k \\ \sin\breve{\theta}_k & -\cos\breve{\theta}_k \end{pmatrix} \begin{pmatrix} x_k - X \\ y_k - Y \end{pmatrix}. \tag{A.2}$$

Thus $x'_k = S_k$ and $y'_k = 0$, and we can substitute an integral over $\tilde{x}'_k$ and $\tilde{y}'_k$ for the integral over $\tilde{x}_k$ and $\tilde{y}_k$. If we write

$$\tilde{\theta}_k = \breve{\theta}_k + \varepsilon, \tag{A.3}$$

then to first order in ε we have $\tilde{x}'_k \approx \tilde{S}_k$ and $\tilde{y}'_k \approx \varepsilon\tilde{S}_k \approx \varepsilon S_k$. So $J(h)$, in Equation (22), can be rewritten with the substitutions

$$(x_k - \tilde{x}_k)^2 + (y_k - \tilde{y}_k)^2 = (S_k - \tilde{x}'_k)^2 + \tilde{y}'^2_k, \tag{A.4}$$

$$\left(\theta_{kh_k} - \tilde{\theta}_k\right)^2 = \left(\theta_{kh_k} - \breve{\theta}_k - \frac{\tilde{y}'_k}{S_k}\right)^2, \tag{A.5}$$

$$\left(T_{kh_k} - \tilde{T}_k\right)^2 = \left(T_{kh_k} - \tilde{t} - \frac{\tilde{S}_0}{\tilde{c}} - \frac{\tilde{x}'_k}{\tilde{c}}\right)^2, \tag{A.6}$$

whence the $2L(h)$ integrals over the $\tilde{x}'_k$ and $\tilde{y}'_k$ become simply Gaussian, and they result in

$$\int \dots \int J(h) \prod_{k:h_k>0} d\tilde{\mathbf{x}}_k$$
$$= (2\pi)^{L(h)} \left(\frac{\tilde{c}^2 \sigma_T^2 \sigma_x^2}{\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \right)^{\frac{L(h)}{2}} \left(\prod_{k:h_k>0} \sqrt{\frac{S_k^2 \sigma_\theta^2 \sigma_x^2}{\sigma_x^2 + S_k^2 \sigma_\theta^2}} \right) \exp\left(-\frac{1}{2} \left[\frac{(t-\tilde{t})^2}{\sigma_t^2} + \frac{(c-\tilde{c})^2}{\sigma_c^2} \right.\right.$$
$$\left.\left. + \frac{(x_0 - \tilde{x}_0)^2}{\sigma_x^2} + \frac{(y_0 - \tilde{y}_0)^2}{\sigma_x^2} + \sum_{k:h_k>0} \left(\frac{S_k^2 \left(\theta_k - \breve{\theta}_k\right)^2}{\sigma_x^2 + S_k^2 \sigma_\theta^2} + \frac{\left(\tilde{c}(T_k - \tilde{t}) - \tilde{S}_0 - S_k\right)^2}{\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \right) \right] \right). \tag{A.7}$$

The integration over $\tilde{x}_0$ and $\tilde{y}_0$ is handled similarly: These variables are transformed by a rotation and a translation to $\tilde{x}'_0$ and $\tilde{y}'_0$ such that $\tilde{x}'_0 \approx \tilde{S}_0$ and $(x_0 - \tilde{x}_0)^2 + (y_0 - \tilde{y}_0)^2 = (S_0 - \tilde{x}'_0)^2 + \tilde{y}'^2_0$, and the integral results in

$$\int \dots \int J(h)\, d\tilde{\mathbf{x}}_0 \prod_{k:h_k>0} d\tilde{\mathbf{x}}_k$$
$$= (2\pi)^{L(h)+1} \sigma_x^2 \sqrt{\frac{\sigma_x^2 + \tilde{c}^2 \sigma_T^2}{(N+1)\sigma_x^2 + \tilde{c}^2 \sigma_T^2}} \left(\frac{\tilde{c}^2 \sigma_T^2 \sigma_x^2}{\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \right)^{\frac{L(h)}{2}} \left(\prod_{k:h_k>0} \sqrt{\frac{S_k^2 \sigma_\theta^2 \sigma_x^2}{\sigma_x^2 + S_k^2 \sigma_\theta^2}} \right) \exp\left(-\frac{1}{2} \left[\frac{(t-\tilde{t})^2}{\sigma_t^2} \right.\right.$$
$$+ \frac{(c-\tilde{c})^2}{\sigma_c^2} + \frac{S_0^2}{\sigma_x^2}$$
$$+ \sum_{k:h_k>0} \left(\frac{S_k^2 \left(\theta_k - \breve{\theta}_k\right)^2}{\sigma_x^2 + S_k^2 \sigma_\theta^2} + \frac{(\tilde{c}(T_k - \tilde{t}) - S_k)^2}{\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \right) \tag{A.8}$$
$$\left.\left. - \frac{\sigma_x^2 (\sigma_x^2 + \tilde{c}^2 \sigma_T^2)}{(N+1)\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \left(\frac{S_0}{\sigma_x^2} + \frac{\sum_{k:h_k>0} (\tilde{c}(T_k - \tilde{t}) - S_k)}{\sigma_x^2 + \tilde{c}^2 \sigma_T^2} \right)^2 \right] \right).$$

The integration over $\tilde{t}$ is straightforward, as the integrand (A.8) is already Gaussian in that variable. Thus we get

$$
\begin{aligned}
&\int \dots \int J(h) d\tilde{t}\, d\tilde{\mathbf{x}}_0 \prod_{k:h_k>0} d\tilde{\mathbf{x}}_k \\
&\quad = \frac{(2\pi)^{L(h)+3/2}\sigma_x^2\sigma_t}{\sqrt{1+L(h)\left(\frac{\sigma_x^2+\tilde{c}^2\sigma_t^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)}}\left(\frac{\tilde{c}^2\sigma_T^2\sigma_x^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^{\frac{L(h)}{2}}\left(\prod_{k:h_k>0}\sqrt{\frac{S_k^2\sigma_\theta^2\sigma_x^2}{\sigma_x^2+S_k^2\sigma_\theta^2}}\right)\exp\left(-\tfrac{1}{2}\left[\frac{t^2}{\sigma_t^2}\right.\right. \\
&\quad + \frac{(c-\tilde{c})^2}{\sigma_c^2} + \frac{S_0^2}{\sigma_x^2} \\
&\quad + \sum_{k:h_k>0}\left(\frac{S_k^2\left(\theta_k-\check{\theta}_k\right)^2}{\sigma_x^2+S_k^2\sigma_\theta^2} + \frac{(\tilde{c}T_k-S_k)^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right) \\
&\quad - \frac{\sigma_x^2(\sigma_x^2+\tilde{c}^2\sigma_T^2)}{(L(h)+1)\sigma_x^2+\tilde{c}^2\sigma_T^2}\left(\frac{S_0}{\sigma_x^2}+\frac{\sum_{k:h_k>0}(\tilde{c}T_k-S_k)}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^2 \\
&\quad \left.\left. - \left(\frac{\sigma_t^2((L(h)+1)\sigma_x^2+\tilde{c}^2\sigma_T^2)}{(L(h)+1)\sigma_x^2+\tilde{c}^2(\sigma_T^2+L(h)\sigma_t^2)}\right)\left(\frac{t}{\sigma_t^2}+\frac{\tilde{c}\left(\sum_{k:h_k>0}(\tilde{c}T_k-S_k)-L(h)S_0\right)}{(L(h)+1)\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^2\right]\right).
\end{aligned}
\tag{A.9}
$$

The final integration is over $\tilde{c}$. Given the term $(c-\tilde{c})^2/\sigma_c^2$ in the argument of the exponential, most of the contribution to this integral will be in the near vicinity of c. This constraint suggests that a low-order expansion in powers of $(\tilde{c}-c)$ should suffice. The expression in (A.9) expands to

$$
\begin{aligned}
&\int \dots \int J(h) d\tilde{t}\, d\tilde{\mathbf{x}}_0 \prod_{k:h_k>0} d\tilde{\mathbf{x}}_k \\
&\quad = \frac{(2\pi)^{L(h)+3/2}\sigma_x^{2L(h)+2}\sigma_t \exp(-C-B(\tilde{c}-c)-A(\tilde{c}-c)^2+O(\tilde{c}-c)^3)}{\left(1+\frac{\sigma_x^2}{c^2\sigma_T^2}\right)^{\frac{L(h)}{2}}\sqrt{1+L(h)\left(\frac{\sigma_x^2+c^2\sigma_t^2}{\sigma_x^2+c^2\sigma_T^2}\right)}},
\end{aligned}
\tag{A.10}
$$

where

$$
\begin{aligned}
A = \frac{1}{2\sigma_c^2} - \frac{\sigma_T^2\beta\mu}{2\alpha^3} - \frac{2c\sigma_T^2\chi}{\alpha^2} + \frac{\upsilon}{2\alpha} + \frac{L(h)\sigma_x^2(3\alpha-\beta)}{4c^2\alpha^2} \\
- \frac{L(h)\sigma_x^2(\sigma_T^2-\sigma_t^2)}{2\alpha^2\gamma'^2}\Big[(L(h)+1)\sigma_x^4 - \sigma_x^2c^2\big((L(h)+2)\sigma_T^2 + L(h)\sigma_t^2\big) \\
- 3c^4\sigma_T^2(\sigma_T^2 + L(h)\sigma_t^2)\Big] \\
- \frac{1}{2\alpha^3\gamma^3}\big[\sigma_T^2\sigma_x^2\zeta\lambda^2 + \sigma_T^2\alpha^3\delta S_0(L(h)S_0 - 2\lambda) + \sigma_x^2\alpha^2\gamma^2\tau^2 \\
- 4c\sigma_T^2\alpha\gamma\tau(\alpha^2S_0 + \sigma_x^2(\alpha+\gamma)\lambda)\big] \\
- \frac{1}{2\gamma^3\gamma'^3}\big[\sigma_t^2\kappa(\lambda - L(h)S_0)^2 \\
- 2c(\sigma_T^2 + L(h)\sigma_t^2)\gamma^3(2\gamma' + \delta')(\lambda - L(h)S_0)t + 4c\sigma_t^2\gamma\gamma'\eta(\lambda - L(h)S_0)\tau \\
- L(h)(L(h)+1)\sigma_x^2\gamma^3\delta' t^2 + \gamma^3\gamma'(\gamma'+\delta')t\tau + c^2\sigma_t^2\gamma^2\gamma'^2\tau^2\big],
\end{aligned}
\tag{A.11}
$$

$$
\begin{aligned}
B = \frac{\chi}{\alpha} - \frac{c\sigma_T^2\mu}{\alpha^2} - \frac{L(h)\sigma_x^2}{c\alpha} - \frac{L(h)c\sigma_x^2(\sigma_T^2-\sigma_t^2)}{\alpha\gamma'} \\
- \frac{(\alpha S_0 + \sigma_x^2\lambda)}{\alpha^2\gamma^2}\big(\alpha\gamma\tau + c\sigma_T^2(L(h)\alpha S_0 - (\alpha+\gamma)\lambda)\big) \\
- \frac{\big(\gamma t + c\sigma_t^2(\lambda - L(h)S_0)\big)}{\gamma^2\gamma'^2}(\eta(\lambda - L(h)S_0) + c\gamma\gamma'\tau \\
- L(h)(L(h)+1)c\sigma_x^2\gamma t),
\end{aligned}
\tag{A.12}
$$

$$
\begin{aligned}
C = \frac{1}{2}\sum_{k:h_k>0}\left[\frac{S_k^2\big(\theta_{kh_k} - \breve{\theta}_k\big)^2}{\sigma_x^2 + S_k^2\sigma_\theta^2} + \ln\left(1 + \frac{\sigma_x^2}{S_k^2\sigma_\theta^2}\right)\right] + \frac{t^2}{2\sigma_t^2} + \frac{S_0^2}{2\sigma_x^2} + \frac{\mu}{2\alpha} - \frac{(\alpha S_0 + \sigma_x^2\lambda)^2}{2\sigma_x^2\alpha\gamma} \\
- \frac{\big(\gamma t + c\sigma_t^2(\lambda - L(h)S_0)\big)^2}{2\sigma_t^2\gamma\gamma'},
\end{aligned}
\tag{A.13}
$$

and

$$\alpha = \sigma_x^2 + c^2\sigma_T^2, \tag{A.14}$$

$$\beta = \sigma_x^2 - 3c^2\sigma_T^2, \tag{A.15}$$

$$\gamma = (L(h)+1)\sigma_x^2 + c^2\sigma_T^2, \tag{A.16}$$

$$\gamma' = \gamma + L(h)c^2\sigma_t^2, \tag{A.17}$$

$$\delta = (L(h)+1)\sigma_x^2 - 3c^2\sigma_T^2, \tag{A.18}$$

$$\delta' = \delta - 3L(h)c^2\sigma_t^2, \tag{A.19}$$

$$\eta = (L(h)+1)^2\sigma_x^4 - c^4\sigma_T^2(\sigma_T^2 + L(h)\sigma_t^2), \tag{A.20}$$

$$\begin{aligned}\zeta = -(L(h)+1)(L(h)+2)\sigma_x^6 + 3(L(h)^2 + 2L(h) + 2)\sigma_x^4 c^2\sigma_T^2 \\ + 9(L(h)+2)\sigma_x^2 c^4\sigma_T^4 + 10c^6\sigma_T^6,\end{aligned} \tag{A.21}$$

$$\begin{aligned}\kappa = (L(h)+1)^4\sigma_x^8 - 3(L(h)+1)^3\sigma_x^6 c^2(2\sigma_T^2 + L(h)\sigma_t^2) \\ - 12(L(h)+1)^2\sigma_x^4 c^4\sigma_T^2(\sigma_T^2 + L(h)\sigma_t^2) \\ - (L(h)+1)\sigma_x^2 c^6\sigma_T^2(\sigma_T^2 + L(h)\sigma_t^2)(2\sigma_T^2 + L(h)\sigma_t^2) \\ + 3c^8\sigma_T^4(\sigma_T^2 + L(h)\sigma_t^2)^2,\end{aligned} \tag{A.22}$$

$$\tau = \sum_{k:h_k>0} T_{kh_k}, \tag{A.23}$$

$$\upsilon = \sum_{k:h_k>0} T_{kh_k}^2, \tag{A.24}$$

$$\lambda = \sum_{k:h_k>0} \left(cT_{kh_k} - S_k\right), \tag{A.25}$$

$$\mu = \sum_{k:h_k>0} \left(cT_{kh_k} - S_k\right)^2, \tag{A.26}$$

$$\chi = \sum_{k:h_k>0} T_{kh_k}\left(cT_{kh_k} - S_k\right). \tag{A.27}$$

If we cut off the expansion in (A.10) at second order, then integrate over $\tilde{c}$, we get

$$\int \dots \int J(h) d\tilde{t}\, d\tilde{c}\, d\tilde{\mathbf{x}}_0 \prod_{k:h_k>0} d\tilde{\mathbf{x}}_k = \frac{(2\pi)^{L(h)+2} \sigma_x^{2L(h)+2} \sigma_t \exp\left(-C + \frac{B^2}{4A}\right)}{\sqrt{2A}\left(1 + \frac{\sigma_x^2}{c^2\sigma_T^2}\right)^{\frac{L(h)}{2}} \sqrt{1 + L(h)\left(\frac{\sigma_x^2 + c^2\sigma_t^2}{\sigma_x^2 + c^2\sigma_T^2}\right)}}, \tag{A.28}$$

which is exactly as given in (25), with

$$I(h) = \frac{1}{\sqrt{A}} \exp\left(-C + \frac{B^2}{4A}\right). \tag{A.29}$$

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A method is presented for localizing a single target with multistatic active sonar, from a single ping, by means of a numerical search for the highest peak in a probability distribution function. That function is constructed using Bayesian principles. The method takes into account errors in the locations of the transmitter and the receivers, errors in the estimate of the local speed of sound, errors in time and bearing measurements, an imperfect probability of detection, and the appearance of false contacts due to noise. The efficacy of the method is demonstrated by means of simulated multistatic data. Because of the false contacts and the imperfect probability of detection, it is inevitable that the resulting localization will sometimes be very inaccurate. But there is a tendency for multiple receivers to corroborate each other, leading to frequent successful localization in spite of these difficulties. The distribution of the error in localization over many runs is similar in form to the corresponding distribution in the case of perfect detection and no false contacts, except for the appearance of a long tail in the distribution. However, the computational load is very heavy. A streamlined version of the probability distribution function, in which some of the uncertainties are neglected, speeds up the computation by several orders of magnitude with the localization performance being degraded only slightly. The single-ping localization method presented here appears to be a promising foundation for future approaches to the tracking of underwater targets by multistatic active sonar.

Le présent document porte sur la méthode de localisation d'une cible unique, à partir d'une seule impulsion et au moyen d'un sonar actif multistatique, par la recherche numérique de la crête la plus élevée selon une fonction de distribution des probabilités. Cette fonction est fondée sur les principes de Bayes. La méthode tient compte des erreurs de localisation de l'émetteur et des récepteurs, des erreurs d'estimation de la vitesse locale du son, des erreurs de mesure du temps et du gisement, d'une probabilité de détection imparfaite et de l'apparition de faux contacts causés par le bruit. On a démontré l'efficacité de la méthode au moyen de données multistatiques simulées. En raison des faux contacts et de la probabilité de détection imparfaite, il est inévitable que la localisation qui en découle soit parfois très imprécise. Les nombreux récepteurs tendent toutefois à se corroborer et la localisation est donc souvent réussie malgré ces difficultés. De plus, la distribution des erreurs de localisation sur plusieurs passages est de forme analogue à la distribution correspondante dans le cas d'une détection parfaite sans faux contact, sauf pour l'apparition d'une longue queue de distribution. La charge de calcul est toutefois très lourde. Une version simplifiée de la fonction de distribution des probabilités, dans laquelle on ne tient pas compte de certaines incertitudes, permet d'accélérer le calcul de plusieurs ordres de grandeur et ne réduit que légèrement les performances de localisation. Par ailleurs, la méthode de localisation à une seule impulsion présentée ici semble être une assise prometteuse pour les approches futures de la poursuite de cibles sous-marines au moyen d'un sonar actif multistatique.

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localization; multistatic active sonar; sensor fusion