Spline Probability Hypothesis Density Filter for Nonlinear Maneuvering Target Tracking

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***Abstract*—The Probability Hypothesis Density (PHD) filter is an efficient algorithm for multitarget tracking in the presence of nonlinearities and/or non-Gaussian noise. The Sequential Monte Carlo (SMC) and Gaussian Mixture (GM) techniques are commonly used to implement the PHD filter. Recently, a new implementation of the PHD filter using B-splines with the capability to model any arbitrary density functions using only a few knots was proposed. The Spline PHD (SPHD) filter was found to be more robust than the SMC-PHD filter since it does not suffer from degeneracy and it was better than the GM-PHD implementation in terms of estimation accuracy, albeit with a higher computational complexity. In this paper, we propose a Multiple Model (MM) extension to the SPHD filter to track multiple maneuvering targets. Simulation results are presented to demonstrate the effectiveness of the new filter.**

***Index Terms*—Maneuvering target tracking, Nonlinear filtering, Probability Hypothesis Density filter, Spline filter, Spline Probability Hypothesis Density filter**

I. Introduction

The Spline Probability Hypothesis Density (SPHD) filter [20] is one of the latest implementations of the Probability Hypothesis Density (PHD) filter. For non-maneuvering multitarget tracking problems, the SPHD filter [20] can be an effective alternative to the Sequential Monte Carlo Probability Hypothesis Density (SMC-PHD) [18], Gaussian Mixture Probability Hypothesis Density (GM-PHD) [25], Gaussian Mixture Particle Probability Hypothesis Density (GMP-PHD) [5,26], Gaussian Mixture Unscented Sequential Monte Carlo Probability Hypothesis Density (GM-USMC-PHD) [27], Gaussian Mixture Sequential Monte Carlo Probability Hypothesis Density (GM-SMC-PHD) [19], and the Auxiliary Particle Probability Hypothesis Density (AP-PHD) [3] filters. The SPHD filter offers continuous estimates of the probability hypothesis density of the multitarget state for any system model [11,20] and avoids degeneracy by modeling the PHD in continuous space. The nonlinearity of the state evolution or measurement model is naturally handled by the SPHD filter [20]. The SPHD filter is not limited to Gaussian systems. To extend the application of the SPHD filter to maneuvering multitarget tracking problems, a multimodal version, called the Multiple Model (MM) Spline Probability Hypothesis Density (MM-SPHD) filter, is derived in this paper.

This paper presents the MM-SPHD filter derivations with details on the estimation of maneuvering multitarget state and the extraction of corresponding individual target states. The multitarget multidimensional system state transition model of MM-SPHD filter is represented by tensor products of splines. The corresponding analytical state prediction and posterior density equations are derived. A nonlinear example is used to validate the performance of the MM-SPHD filter vs. other multiple model PHD implementations. Simulation results reveal that the MM-SPHD filter works efficiently and increased measurement noise levels do not destabilize it whereas other MM implementation suffer at higher noise levels. The MM-SPHD filter can maintain highly accurate tracks by taking advantage of dynamic knot movement [20], but at the expense of higher computational complexity.

The structure of this paper is as follows. Sections II and III briefly review the introduction to PHD filters and the multiple model PHD filters, respectively. The B-spline is reviewed in Section IV. The proposed MM-SPHD filter formulations are presented in Section V for the multidimensional case with tensor products and two new knot selection schemes. Simulation results and conclusions are given in Sections VI and VII, respectively.

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II. Introduction to PHD Filter

Maneuvering multitarget tracking algorithms face target motion model uncertainties and to overcome these uncertainties, most multitarget filters adopt multiple model filtering techniques [12,18]. Thus, multiple model filters run each filter in their mode set using the same measurement assuming that the target state evolves according to one of r models in its mode set at any time and fuses the output of those filters to find an overall estimate [12].

Let ϑ_k be the number of targets at time k in multitarget state space $\mathcal{E}_s$. Then the multimodal multitarget state at time k can be written as

$$X_k \quad = \quad \{\mathbf{x}_{1,k}^{\mathbf{M}_k}, \ldots, \mathbf{x}_{\vartheta_k,k}^{\mathbf{M}_k}\} \in \mathcal{E}_s \tag{1}$$

where $\mathbf{x}_{l,k}^{\mathbf{M}_k}$ denotes the mode-dependent l-th target state vector at time k and $l \in \{1, \ldots, \vartheta_k\}$. Note that the order in which the multitarget states are listed has no significance in the Random Finite Set (RFS) multitarget model formulation. In the above, $\mathbf{M}_k \in \{1, \ldots, \mathsf{r}\}$ is the mode index parameter, where r is the number of possible models, and the mode index parameter is governed by an underlaying Markov process with mode transition probability

$$\pi_{\mathfrak{p}\mathfrak{q}} = P(\mathbf{M}_k = \mathfrak{q} | \mathbf{M}_{k-1} = \mathfrak{p}) \quad \mathfrak{p}, \mathfrak{q} = 1, 2, \ldots, \mathsf{r} \tag{2}$$

The mode transition probability $\pi_{\mathfrak{p}\mathfrak{q}}$ can be assumed time-invariant and independent of the multitarget state. The state of the l-th target is given by

$$\mathbf{x}_{l,k} \quad = \quad f_{k,\mathbf{M}_k}(\mathbf{x}_{l,k-1}, \nu_{k,\mathbf{M}_k}, \mathbf{M}_k) \tag{3}$$

where $\mathbf{x}_{l,k}$ denotes the l-th target state at time k, $\nu_{k,\mathbf{M}_k}$ is the mode-dependent iid process noise sequence with known statistics and $f_{k,\mathbf{M}_k}(\cdot)$ is the mode-dependent nonlinear system transition function.

Let $Z^{(k)} = \{Z_0, Z_1, \ldots, Z_k\} \in \mathcal{E}_o$ be the cumulative sets of measurements from time 0 to time k and assume that η_k denotes the number of target-originated measurements at time k. Measurements also consist of observations generated by the false alarm process and assume ϖ_k denotes the number of false measurements at time k. Then the set of measurements at time k in observation space $\mathcal{E}_o$ is given by

$$Z_k \quad = \quad \{\mathbf{z}_{1,k}, \ldots, \mathbf{z}_{\eta_k,k}\} \bigcup \{\mathbf{c}_{1,k}, \ldots, \mathbf{c}_{\varpi_k,k}\} \in \mathcal{E}_o \tag{4}$$

where the l-th target-originated measurement is given by

$$\mathbf{z}_{l,k} \quad = \quad h_{k,\mathbf{M}_k}(\mathbf{x}_{l,k}, \omega_{k,\mathbf{M}_k}, \mathbf{M}_k) \tag{5}$$

and $\omega_{k,\mathbf{M}_k}$ denotes the mode-dependent iid measurement noise with known statistics and $h_{k,\mathbf{M}_k}$ is a mode-dependent nonlinear function. The false measurements $\mathbf{c}_{i,k}$ are assumed to be uniformly distributed and their number ϖ_k is Poisson-distributed. Let $P_{\mathfrak{d},k}$ denote the probability of detection, thus the probability of $Z_k(\mathbf{x}_{i,k}^{\mathbf{M}_k}) = \mathbf{z}_{i,k} = \emptyset$ (i.e., the i-th target is

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not detected) is $1-P_{\mathfrak{d},k}$. The average number of measurement is $\beth_k = P_{\mathfrak{d},k} \times \eta_k + \varpi_k^{'}$, where $\varpi_k^{'}$ is the average number of false alarms.

There are ϑ_k targets in the state space $\mathcal{E}_s$ at time k and these targets can continue to exist, spawn new targets or terminate. In addition, new targets are born independently of already-existing targets. The number of targets and their states are unknown and, with maneuvering targets, the dynamic model of a target at any time is also unknown. That is, there are two unknown discrete random variables (i.e., number of targets and mode index of each target) and a continuous random variable (i.e., target state of each target) to be estimated at each time. From the observation space $\mathcal{E}_o$, $\beth_k$ measurements are received at time k. The origins of the measurements are not known, and thus the order in which they appear bears no significance. The measurements can also originate from clutter and false alarms.

At time k the dynamic models of all targets and the dimension of the multitarget state X_k are unknown and time-varying. In the absence of model uncertainty the randomness in the set can be characterized by modeling the multitarget states and multitarget measurements as random finite set Ξ_k and $\aleph_k$ respectively. Given the realization X_{k-1} of Ξ_{k-1} at time k-1, the multitarget states at time k can be modeled by the RFS as

$$\Xi_k = S_k(X_{k-1}) \cup B_k(X_{k-1}) \cup \Gamma_k \quad (6)$$

where $S_k(X_{k-1})$ denotes the surviving targets and $B_k(X_{k-1})$ denotes the spawned targets. In addition, Γ_k denotes the new-born targets and these newborn targets are born independently from the surviving targets. Similarly, given a realization of X_k of $\aleph_k$, the multitarget measurement can be modeled by the RFS as

$$\aleph_k = \Phi_k(X_k) \cup C_k \quad (7)$$

where $\Phi_k(X_k)$ denotes the RFS measurement generated by X_k and C_k denotes measurement generated by clutter.

Let $p_{k-1|k-1}(X_{k-1}|Z^{(k-1)})$ denote the multitarget prior density of the system dynamic model at time $k-1$. Then prediction and update steps of the optimal multitarget Bayes filter recursion are given by [1]

$$\begin{aligned} p_{k|k-1}(X_k|Z^{(k-1)}) = & \int p_{k|k-1}(X_k|X_{k-1}) \\ & \cdot p_{k-1|k-1}(X_{k-1}|Z^{(k-1)}) \\ & \cdot \mu_s(dX_{k-1}) \end{aligned} \quad (8)$$

$$p_{k|k}(X_k|Z^{(k)}) = \frac{g_k(Z_k|X_k)p_{k|k-1}(X_k|Z^{(k-1)})}{\int g_k(Z_k|X_k)p_{k|k-1}(X_k|Z^{(k-1)})\mu_s(dX_k)} \quad (9)$$

respectively, where $p_{k|k-1}(X_k|X_{k-1})$ is the multitarget dynamic model transition density, $g_k(Z_k|X_k)$ denotes the multitarget likelihood and μ_s takes the place of Lebesque measure. The posterior density $p_{k|k}(X_k|Z^{(k)})$ can be determined using (9).

For the multiple model PHD (MM-PHD) filter, with the additional model uncertainty in $\mathbf{M}_k \in \{1, \ldots, \mathsf{r}\}$, we cannot define the model probability over X_k. The MM-PHD filter in [14] is not only able to provide maximum a posterior (MAP) estimate of the total number of targets, but also able to provide the posterior PHD function and the expected a posterior (EAP) estimates of the number of targets for each dynamic models. It should be noted that the single target state space for different dynamic models may be different, so the dimension of $\mathbf{x}$ depends on the type of dynamic model represented by $\mathbf{M}$.

III. PHD Filter for Maneuvering Targets

A cycle of the recursive MM-PHD filter algorithm can be described in three stages: mixing, prediction and update [18].

A. MM-PHD mixing

In this stage each mode-matched filter is fed with a different density that is a combination of the previous mode-dependent densities. Let the initial density be

$$\tilde{D}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)})$$

and the initial density fed to the PHD filter that is matched to motion model $\mathfrak{q}$ is calculated based on the Markovian model transition probability matrix $\pi_{\mathfrak{pq}}$ and model-dependent prior density $D_{k-1|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}|Z^{(k-1)})$ as [18]

$$\begin{aligned} &\tilde{D}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)}) = \\ &\sum_{\mathfrak{p}=1}^{\mathsf{r}} D_{k-1|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}|Z^{(k-1)})\pi_{\mathfrak{pq}} \quad \mathfrak{q} = 1, \ldots, \mathsf{r} \end{aligned} \quad (10)$$

where r denotes number of filters. Target spawning, birth and disappearance are not considered at the mixing stage but they are considered in the subsequent prediction stage. The densities $\tilde{D}_{k|k-1}(\cdot)$ and $D_{k-1|k-1}(\cdot)$ in (10) are similar to probability densities except that they do not integrate to unity. The mixing described in (10) is similar to the total probability theorem.

B. MM-PHD prediction

Assume that each target evolves and generates observations independently of one another. A target can continue to survive or disappear from the scene, can be spawned by already existing targets, and also new targets can be born in the scene independently from the already-existing targets. Once the initial density for the PHD filter matched to target model $\mathfrak{q}$ is calculated, the mode-dependent predicted density is calculated as

$$\begin{aligned} D_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)}) = & D_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \\ & + D_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \\ & + D_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \end{aligned} \quad (11)$$

where the mode-dependent predicted density of existing targets is expressed as follows:

$$\begin{aligned} D_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = & \int P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) \\ & \cdot p_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) \\ & \cdot \tilde{D}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)}) \\ & \cdot d\mathbf{x}_{k-1} \end{aligned} \quad (12)$$

where $p_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p})$ denotes a mode-dependent single Markov transition density of state of those existing individual targets and $P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q})$ denotes the mode-dependent survival probability of existing targets that accounts for the event that a target with state $\mathbf{x}_{k-1}$ at time step $k-1$ will survive at time step k. The mode-dependent predicted density of spawned targets can be expressed as

$$\begin{aligned} D_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = & \int \beta_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) \\ & \cdot \tilde{D}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)})d\mathbf{x}_{k-1} \end{aligned}$$

where $\beta_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p})$ denotes the mode-dependent PHD of the new targets spawned by existing targets. The PHD of the mode-dependent likelihood function $\beta_{\mathbf{s},k|k-1}(\mathbf{X}_k|\mathbf{M}_k = \mathfrak{q})$, which is the mode-dependent likelihood that a group of new targets with state set X_k will be spawned at time step k by a single target that had state $\mathbf{x}_{k-1}$ at time step $k-1$.

Appearance of completely new targets is also described by mode-dependent $\beta_{\mathbf{nb},k}(X_k, \mathbf{M}_k = \mathfrak{q})$, where the equation is the mode-dependent likelihood that new targets with state set X_k will enter the scene at time step k and its PHD is $\beta_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$. The mode-dependent predicted newborn target density $D_{\mathbf{nb},k}(.)$ depends on the system model. The mode integral of mode-dependent predicted PHD $D_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)})$ over a region gives the expected/predicted number of targets in that region.

C. MM-PHD update

The predicted density can be corrected with the available measurements $Z_k \in Z^{(k)}$ from observation space $\mathcal{E}_o$ at time step k to get the updated density with the assumption that no target generates more than one measurement. Each measurement is generated by no more than a single target and all measurements are conditionally independent of target state. The number of false alarms is Poisson distributed with average rate of λ_k and the probability density of the spatial distribution of false alarm is $C_k(\mathbf{z}_k)$ with the assumptions of standard multimodal multitarget measurement model from Section II. The detection probability of a target with state $\mathbf{x}_k$ at time step k is $P_{\mathfrak{d}}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ and the mode-dependent updated PHD at scan k can be determined as (for $\mathfrak{q} = 1, \ldots, \mathfrak{r}$)

$$D_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k)}) \cong (1 - P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})) \cdot D_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)}) + \sum_{\mathbf{z}_k \in Z_k} \frac{\phi_k(\mathbf{z}_k|Z^{(k-1)}, \mathbf{M}_k = \mathfrak{q})}{\lambda_k C_k(\mathbf{z}_k) + \int \phi_k(\mathbf{z}_k|Z^{(k-1)}, \mathbf{M}_k = \mathfrak{q}) d\mathbf{x}_k}$$

where the function $\phi_k(.)$ is given as

$$\phi_k(\mathbf{z}_k|Z^{(k-1)}, \mathbf{M}_k = \mathfrak{q}) = P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot p_{k|k}(\mathbf{z}_k|\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot D_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k-1)})$$

The expected number of targets can be determined by taking the integral of the mode-dependent updated PHD $D_{k|k}(.)$ as

$$\hat{N}_{k|k}^{\mathbf{M}_k=\mathfrak{q}} = \int D_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k)}) d\mathbf{x}_k \quad (13)$$

and the total number of estimated targets as

$$\hat{N}_{k|k} = \sum_{\mathfrak{q}=1}^{\mathfrak{r}} \hat{N}_{k|k}^{\mathbf{M}_k=\mathfrak{q}} \quad (14)$$

IV. B-SPLINES

A brief background on B-spline is provided in this section and for further details readers can refer to [4]. A p-th order B-spline curve $\mathbf{C}(x)$ of a certain variable x (e.g., multitarget state) is defined as

$$\mathbf{C}(x) = \sum_{i=1}^{n_s} \mathbb{P}_i B_{i,p,\mathbf{t}}(x) \quad 2 \le p \le n_s, \quad (15)$$

where $\mathbb{P}_i$ is the i-th control point of the n_s point control polygon vertices and $B_{i,p,\mathbf{t}}(x)$ are the B-spline blending functions (basis functions), which are basically polynomials of degree $p-1$. The order p can be chosen from 2 to n_s and the continuity of the curve can be kept by selecting $p \ge 3$. The knot denoted by $\mathbf{t}$ is a $1 \times \tau$ vector and $\mathbf{t}$ is a non-decreasing sequence of real numbers, where $\mathbf{t} = \{t_1, \ldots, t_\tau\}$, i.e., $t_i \le t_{i+1}$, $i = 1, \ldots, \tau$. The knot vector relates the parameter x to the control points. This relationship, together with the location of the control points, provides control over the shape of the curve.

The i-th basis function is defined by the recursion formula from [6] as

$$B_{i,p}(x) = \frac{(x - t_i)B_{i,p-1}(x)}{t_{i+p-1} - t_i} + \frac{(t_{i+p} - x)B_{i-1,p-1}(x)}{t_{i+p} - t_{i+1}} \quad (16)$$

where, $t_i \leqslant x < t_{i+p}$ and

$$B_{i,1}(x) = \begin{cases} 1 & \text{if } t_i \leqslant x < t_{i+1}, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

The t_is in (16) are elements of a knot vector. From (16), it is clear that the basis function $B_{i,p,\mathbf{t}}(x)$ is non-zero in the interval $[t_i, t_{i+p}]$. The basis function spans the knots $t_i, \ldots, t_{i+p}$. Note that when knots are not repeated, B-spline is zero at the end-knots t_i and t_{i+p}, i.e.,

$$B_{i,p}(x = t_i) = 0 \qquad B_{i,p}(x = t_{i+p}) = 0$$

In B-splines, we can have repeated knots (i.e., $t_i = t_{i+1} = \ldots$) and $B_{i,p}$ can have the form $\frac{0}{0}$. Hence, we assume $\frac{0}{0} = 0$ to incorporate repeated knots. For any value of the parameter, x, the sum of the basis functions is one, i.e.,

$$\sum_{i=1}^{n_s} B_{i,p}(x) = 1, \quad (18)$$

the B-spline curve lies within the convex hull defined by its control polygon. Therefore, the entire curve lies within the union of all such convex hulls formed by taking p successive defining polygon vertices. The curve is affine invariant and follows the shape of the defining polygon.

Unidimensional splines can be extended to multidimensional ones through the use of tensor product spline construction [4]. A spline subspace $B_{i_j,p_j,t_j}(x_j)$ is defined for each dimension where x_j denotes the variable in the j-th dimension. Thus, the spline representation of a multidimensional function $\mathbf{C}(x_1, \ldots, x_m)$ is given as

$$\mathbf{C}(x_1, \ldots, x_m) = \sum_{i_1}^{n_s} \ldots \sum_{i_m}^{n_s} \mathbb{P}_{i_1,\ldots,i_m} B_{i_1,p_1,\mathbf{t}_1}(x_1) \ldots \cdot B_{i_m,p_m,\mathbf{t}_m}(x_m) \quad (19)$$

V. MM-SPHD FILTERING

The MM-SPHD filter implementation is based on the SPHD filter's extension to multiple model estimation. This section derives the MM-SPHD filter for the multidimensional multitarget state space models. Assume that a multidimensional multitarget system state at time k is denoted as $X_k = \{\mathbf{x}_{1,k}, \ldots, \mathbf{x}_{\vartheta_k,k}\}$ where each target has multidimensional state $\mathbf{x}_k = [\mathbf{x}_k^1, \ldots, \mathbf{x}_k^n]'$ and n denotes the number of dimensions.

A. MM-SPHD mixing

The MM-SPHD filter derivations follow Section III. Let the initial MM-SPHD be

$$\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) = \sum_{i_1}^{n_s} \ldots \sum_{i_n}^{n_s} \mathbb{P}_{i_1,\ldots,i_n} \cdot B_{i_1,p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_k = \mathfrak{q}) \ldots \cdot B_{i_n,p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_k = \mathfrak{q}) \quad (20)$$

where $\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q})$ denotes the $\mathfrak{q}$-th mode-dependent initial multitarget multidimensional MM-SPHD and $\mathbf{M}_k \in \{1, \ldots, \mathfrak{r}\}$ is the model mode index at time k, where $\mathfrak{r}$ denotes the total number of models. The number of dimensions is denoted by n and $\mathfrak{i} = i_1, \ldots, i_n$. The number of knots for all dimension is the same at τ. The n dimensional knot $\mathbf{t}_{k-1} = \{\mathbf{t}_{k-1}^1, \ldots, \mathbf{t}_{k-1}^n\}$, is a $n \times \tau$ array. Each row vector of $\mathbf{t}_{k-1}$ consists of a set of prior knots $\mathbf{t}_{k-1}^l = \{t_{1,k-1}^l, \ldots, t_{\tau,k-1}^l\}$ where $l = 1, \ldots, n$. The n dimensional control point set or coefficient matrix is denoted by $\mathbb{P}_\mathfrak{i}$ and n_s denotes the number of control points. The number of control points for all dimensions is the same. Note that the number of knots must be greater than the number of control points.

The initial MM-SPHD $\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q})$ is fed to the MM-SPHD filter, which is matched to multitarget model $\mathfrak{q}$. The initial MM-SPHD $\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q})$ can be calculated on the basis of Markovian model transition probability matrix $\pi_{\mathfrak{pq}}$ and model-dependent multitarget multidimensional prior MM-SPHD $\mathbf{B}_{k-1|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p})$, i.e.,

$$\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) = \sum_{\mathfrak{p}=1}^{\mathfrak{r}} \mathbf{B}_{k-1|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) \cdot \pi_{\mathfrak{pq}} \qquad \mathfrak{q} \in \{1, \ldots, \mathfrak{r}\} \tag{21}$$

where the prior MM-SPHD of the $\mathfrak{p}$-th dynamic system can be determined as

$$\mathbf{B}_{k-1|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) = \sum_{g_1}^{n_s} \ldots \sum_{g_n}^{n_s} \mathbb{P}_{g_1,\ldots,g_n} \cdot B_{g_1,p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_{k-1} = \mathfrak{p}) \ldots \cdot B_{g_n,p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_{k-1} = \mathfrak{p}) \tag{22}$$

where $\mathfrak{p} = \{1, \ldots, \mathfrak{r}\}$. For all $\mathfrak{r}$ system models, prior MM-SPHD $\mathbf{B}_{k-1|k-1}(\cdot)$ are summed together with scaling by the corresponding mode probability $\pi_{\mathfrak{pq}}$ to determine the initial MM-SPHD $\tilde{\mathbf{B}}_{k|k-1}(.)$ as in (21). The prior number of expected targets is the integral of $\tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q})$ over the region of state space $\mathcal{E}_s$ for the $\mathfrak{q}$-th model evaluated as

$$\hat{N}_{k-1|k-1} = \int_{\mathcal{E}_s} \tilde{\mathbf{B}}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) d\mathbf{x}_{k-1} \tag{23}$$

where $\mathfrak{q} = \{1, \ldots, \mathfrak{r}\}$ and using (23) the prior number of expected targets can be determined for all the models. Then the overall prior number of expected targets can be determined by summing over all $\hat{N}_{k-1|k-1}$ models. Target spawning, birth and disappearance are not considered at the mixing stage, but they are considered in the prediction stage.

B. MM-SPHD prediction

The spline representation of the mode-dependent multitarget state transition density $p_{k|k-1}$ is a $2n$ dimensional function determined using system model (3) as

$$\begin{aligned} &p_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q} | \mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) = \\ &\quad \sum_{j_1} \ldots \sum_{j_{2n}} \mathbb{P}_{j_1,\ldots,j_{2n}} B_{j_1,\mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ &\quad \cdot B_{j_n,\mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \\ &\quad \cdot B_{j_{n+1},p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_{k-1} = \mathfrak{p}) \ldots \\ &\quad \cdot B_{j_{2n},p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_{k-1} = \mathfrak{p}) \end{aligned} \tag{24}$$

where $\mathfrak{q}, \mathfrak{p} \in \{1, \ldots, \mathfrak{r}\}$, $\mathfrak{j} = \{j_1, \ldots, j_{2n}\}$ and $\mathbf{t}_k$ denotes $n \times \tau$ knot array at k and it consists of row vectors $\mathbf{t}_k^1, \ldots, \mathbf{t}_k^n$. Each row vector of $\mathbf{t}_k$ consists of a set of predicted knots $\mathbf{t}_k^l = \{t_{1,k}^l, \ldots, t_{\tau,k}^l\}$ where $l = 1, \ldots, n$. The predicted knot selection of a multidimensional system is much more challenging. A suboptimal but computationally efficient method as described in [20] is used here to find the mode-dependent predicted knots for the multidimensional spline. The mode-dependent coefficients or control points of spline transition density $\mathbb{P}_{j_1,\ldots,j_{2n}}$ can be determined as described in [20].

Once the initial spline density for the MM-SPHD filter that is matched to target model $\mathfrak{q}$ is calculated, the mode-dependent spline predicted density can be calculated using (11) as

$$\begin{aligned} \mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) &= \mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \\ &+ \mathbf{B}_{\mathbf{s},k|k-1}(X_k, \mathbf{M}_k = \mathfrak{q}) \\ &+ \mathbf{B}_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \end{aligned} \tag{25}$$

The predicted MM-SPHD for the existing targets can be determined as follows [18]:

$$\begin{aligned} D_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = \int & P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot p_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q} | \mathbf{x}_{k-1}, \mathbf{M}_{k-1} = \mathfrak{p}) \\ &\cdot \tilde{D}_{k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q} | Z^{(k-1)}) \\ &\cdot d\mathbf{x}_{k-1} \end{aligned} \tag{26}$$

and the spline predicted MM-SPHD

$$\begin{aligned} &\mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = \\ &\int P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) \sum_{j_1}^{n_s} \ldots \sum_{j_{2n}}^{n_s} \mathbb{P}_{j_1,\ldots,j_{2n}} \\ &\cdot B_{j_1,p,\mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots B_{j_n,p,\mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot B_{j_{n+1},p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_{k-1} = \mathfrak{p}) \ldots \\ &\cdot B_{j_{2n},p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_{k-1} = \mathfrak{p}) \sum_{i_1}^{n_s} \ldots \sum_{i_n}^{n_s} \mathbb{P}_{i_1,\ldots,i_n} \\ &\cdot B_{i_1,p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ &\cdot B_{i_n,p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_k = \mathfrak{q}) \; d\mathbf{x}_{k-1}^1 \ldots d\mathbf{x}_{k-1}^n \\ &= \sum_{j_1}^{n_s} \ldots \sum_{j_{2n}}^{n_s} \mathbb{P}_{j_1,\ldots,j_{2n}} B_{j_1,p,\mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ &\cdot B_{j_n,p,\mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \sum_{i_1}^{n_s} \ldots \sum_{i_n}^{n_s} \mathbb{P}_{i_1,\ldots,i_n} \\ &\cdot \int P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot B_{j_{n+1},p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_{k-1} = \mathfrak{p}) \ldots \\ &\cdot B_{j_{2n},p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_{k-1} = \mathfrak{p}) \\ &\cdot B_{i_1,p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ &\cdot B_{i_n,p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot d\mathbf{x}_{k-1}^1 \ldots d\mathbf{x}_{k-1}^n \end{aligned} \tag{27}$$

where the third equality follows from the property that the order of summation and integration of spline is interchangeable [4].

Define two $2n$ dimensional matrices W and C, and one n dimensional matrix ξ as follows:

$$\begin{aligned} W_{j_{n+1},\ldots,j_{2n},i_1,\ldots,i_n} = \int & P_{\mathfrak{s},k|k-1}(\mathbf{x}_{k-1}, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot B_{j_{n+1},p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_{k-1} = \mathfrak{p}) \ldots \\ &\cdot B_{j_{2n},p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_{k-1} = \mathfrak{p}) \\ &\cdot B_{i_1,p,\mathbf{t}_{k-1}^1}(\mathbf{x}_{k-1}^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ &\cdot B_{i_n,p,\mathbf{t}_{k-1}^n}(\mathbf{x}_{k-1}^n, \mathbf{M}_k = \mathfrak{q}) \\ &\cdot d\mathbf{x}_{k-1}^1 \ldots d\mathbf{x}_{k-1}^n \end{aligned} \tag{28}$$

Let $C_{\mathfrak{j}} = \mathbb{P}_{\mathfrak{j}}$ and $\xi_{\mathfrak{i}} = \mathbb{P}_{\mathfrak{i}}$. Using (26), (28) and (30) with additional manipulations, it can be shown that

$$\mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = \sum_{j_1} \ldots \sum_{j_n} \mathbb{P}_{j_1,\ldots,j_n} \cdot B_{j_1,p,\mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots \cdot B_{j_n,p,\mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \tag{29}$$

where $\mathbb{P}_{j_1,\ldots,j_n}$ is given by

$$\mathbb{P}_{j_1,\ldots,j_n} = \sum_{i_1} \ldots \sum_{i_n} \xi_{i_1,\ldots,i_n} \sum_{j_{n+1}} \ldots \sum_{j_{2n}} \cdot C_{j_1,\ldots,j_{2n}} W_{j_{n+1},\ldots,j_{2n},i_1,\ldots,i_n} \tag{30}$$

where $\mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ denotes the $\mathfrak{q}$-th mode-dependent existing targets' predicted SPHD.

A similar approach as described for $\mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ can be applied to determine the mode-dependent spawned targets' predicted SPHD $\mathbf{B}_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$. The mode-dependent SPHD of new targets $\mathbf{B}_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ can be determined as described in [20] as follows:

First, the mode-dependent posterior probability for an observed measurement that originates from a new target $P_k(Y_i, \mathbf{M}_k = \mathfrak{q})$ is determined as

$$P_k(Y_i, \mathbf{M}_k = \mathfrak{q}) = \frac{\mathbf{B}_{Y,k}(\mathbf{z}_i, \mathbf{M}_k = \mathfrak{q})}{\mathbf{B}_{\lambda,k}(\mathbf{z}_i) + \sum_{\mathfrak{q}=1}^{\mathfrak{r}} \mathbf{B}_{Y,k}(\mathbf{z}_i, \mathbf{M}_k = \mathfrak{q})}, \quad \mathbf{z}_i \in Z_k, \ i = 1, \ldots, \beth_k$$

and

$$\mathbf{B}_{Y,k}(\mathbf{z}_i, \mathbf{M}_k = \mathfrak{q}) = \int P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot \mathbf{B}_{l,k}(\mathbf{z}_i|\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot \mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot d\mathbf{x}_k \quad (31)$$

where Y_i denotes the i-th observed measurement that originates from a new target at time k and $\beth_k$ denotes the total number of measurements at time k. In the above, $\mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \approx \mathbf{B}_{\mathbf{c},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) + \mathbf{B}_{\mathbf{s},k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ and $\mathbf{B}_{l,k}(\cdot)$ denotes the spline likelihood density and could be determined using the measurement model in (5). The spline uniform clutter density is denoted by $\mathbf{B}_{\lambda,k}(\cdot)$ and $P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ denotes the mode-dependent probability of detection.

The mode dependent $P_k(Y_i, \mathbf{M}_k = \mathfrak{q})$ values are determined for each measurement and it is compared with a tuning threshold probability ϵ. That is,

$$N_{\mathbf{nb},i} = \begin{cases} 1 & \text{if } P_k(Y_i, \mathbf{M}_k = \mathfrak{q}) \leqslant \epsilon, \\ 0 & \text{otherwise.} \end{cases} \quad (32)$$

If the number of newborn targets $N_{\mathbf{nb},i}$ is 1 for a specific measurement index i, then a newborn target SPHD can be added as

$$\mathbf{B}_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = \sum_{i=1}^{N_{\mathbf{nb}}} \mathbf{B}_{\mathbf{nb},k,i}(\mathbf{z}_i) \quad (33)$$

where $\mathbf{B}_{\mathbf{nb},k,i}$ is the SPHD of a newborn target with mean $\mathbf{z}_i$ and variance of measurement noise. The total number of newborn targets per scan is denoted by $N_{\mathbf{nb}}$ and $\mathbf{B}_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ denotes the cumulative sum of all the SPHD values of newborn targets at scan k. Overall, if for the i-th measurement, $\mathbf{z}_i$, $P_k(Y_i, \mathbf{M}_k = \mathfrak{q}) = 1$ then each element of $\mathbf{z}_i$ can be considered as the mean of a newborn target state in its respective dimension with the variance of measurement noise. A newborn target can be added using Gaussian distribution with corresponding mean and variance from each state element of that newborn target. The mode-dependent MM-SPHD of newborn targets, $\mathbf{B}_{\mathbf{nb},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$, depends on system model $\mathfrak{q}$. The mode integral of mode-dependent MM-SPHD $\mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ over a region gives the expected/predicted number of targets in that region.

C. MM-SPHD update

Note that the MM-SPHD filter provides the PHD estimates in a continuous space in state. These predicted MM-SPHD $\mathbf{B}_{k|k-1}(\cdot)$ at any point over the interval $[t_{1,k}, t_{\tau,k}]$ can be determined using (25). Then, the interval where $\mathbf{B}_{k|k-1}(\cdot)$ is significant could be found. Using the measurement model equation (5), the value for the likelihood density function $\mathbf{B}_{l,k}(\mathbf{z}_k|\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ can be evaluated for the same interval. The updated posterior MM-SPHD can be determined as [13] (for $\mathfrak{q} = 1, \ldots, \mathfrak{r}$)

$$\begin{aligned} D_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}|Z^{(k)}) = & \\ & (1 - P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}))\mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \\ & + \sum_{\mathbf{z_k} \in Z_k} \frac{\mathbf{B}_\phi(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})}{\mathbf{B}_\lambda(\mathbf{z_k}) + \int \mathbf{B}_\phi(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) d\mathbf{x_k}} \\ & = \sum_{\iota_1, \ldots, \iota_n} \mathbb{P}_{\iota_1, \ldots, \iota_n} B_{\iota_1, p, \mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots \\ & \cdot B_{\iota_n, p, \mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \end{aligned} \quad (34)$$

where $\mathbf{B}_\phi(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ can be evaluated as follows:

$$\mathbf{B}_\phi(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = P_{\mathfrak{d},k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot \mathbf{B}_{l,k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \cdot \mathbf{B}_{k|k-1}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) \quad (35)$$

Then, the updated MM-SPHD can be further simplified as

$$\mathbf{B}_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) = \sum_{\iota_1, \ldots, \iota_n} \mathbb{P}_{\iota_1, \ldots, \iota_n} \cdot B_{\iota_1, p, \mathbf{t}_k^1}(\mathbf{x}_k^1, \mathbf{M}_k = \mathfrak{q}) \ldots \cdot B_{\iota_n, p, \mathbf{t}_k^n}(\mathbf{x}_k^n, \mathbf{M}_k = \mathfrak{q}) \quad (36)$$

where $\mathbf{t}_k$ denotes the set of posterior knots and (36) ensures that the spline posterior density is only evaluated over the interval where it is significant. Once the significant region is obtained, a simple way of selecting the knots for the posterior intensity is to uniformly distribute the knots over this significant region [11]. The expected number of targets from model $\mathfrak{q}$ can be determined by taking the integral of mode dependent MM-SPHD updated equation $\mathbf{B}_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q})$ as

$$\hat{N}_{k|k} = \int_{\mathcal{E}_s} \mathbf{B}_{k|k}(\mathbf{x}_k, \mathbf{M}_k = \mathfrak{q}) d\mathbf{x}_k \quad (37)$$

The total number of targets can be found by summing up the integrals of the updated mode-dependent MM-SPHD values. Mode probability can be updated for a particular model by integrating the mode dependent updated MM-SPHD, which is then divided it by the total expected number of targets [18].

VI. Simulation Results

In this section a nonlinear maneuvering multitarget tracking example is presented to validate the performance of the MM-SPHD filter. The selected example is a multidimensional one dealing with a bearing-only ground target tracking problem, which arises in many practical applications such as submarine tracking or airborne surveillance using a passive radar [20]. Note that a standard radar tracking problem, where range and azimuth measurements are available for tracking can be converted into a linear problem. Also, the bearing only tracking problem is inherently ill conditioned [15, 22] and is better suited for comparing nonlinear target tracking algorithms.

As shown in Figure 1, a sensor is on an aircraft with

$$x_p(k) = \bar{x}_p(k) + \Delta x_p(k) \qquad k = 0, 1, \ldots, 40 \quad (38)$$
$$y_p(k) = \bar{y}_p(k) + \Delta y_p(k) \qquad k = 0, 1, \ldots, 40 \quad (39)$$

where $\bar{x}_p(k)$ and $\bar{y}_p(k)$ are the average platform position coordinates, k is the time index and the perturbations $\Delta x_p(k)$ and $\Delta y_p(k)$ are assumed to be mutually independent zero-mean Gaussian white noise sequences with variances $\sigma^2_{\Delta x_p} = 1$ and $\sigma^2_{\Delta y_p} = 1$, respectively. Note that this problem has been used to compare nonlinear filtering tracking algorithms before [2, 10, 20]. The average unperturbed platform motion is assumed to be horizontal with a constant velocity. Its coordinates are given by

$$\bar{x}_p(k) = 100k * T \qquad \text{(m)} \tag{40}$$
$$\bar{y}_p(k) = 10000 \qquad \text{(m)} \tag{41}$$

where the sampling time $T = 10$s. A system with three models is considered here to demonstrate the MM-SPHD. In the second and third models, a time-varying control term is added. The three system models are

$$\mathbf{x}_i^1(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}_i^1(k-1) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \nu_{1,k}, \tag{42}$$

$$\mathbf{x}_i^2(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}_i^2(k-1) + \begin{bmatrix} -T/2 \\ -T/500 \end{bmatrix} (k-1) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \nu_{2,k} \tag{43}$$

and

$$\mathbf{x}_i^3(k) = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \mathbf{x}_i^3(k-1) + \begin{bmatrix} T/2 \\ T/500 \end{bmatrix} (k-1) + \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \nu_{3,k} \tag{44}$$

where the target state is

$$\mathbf{x}_i(k) = \begin{bmatrix} x_i^1(k) \\ x_i^2(k) \end{bmatrix} \qquad i = 1, 2, 3, 4, 5, 6 \tag{45}$$

and x_i^1 denotes the position in meters while x_i^2 denotes the velocity in m/s of the i-th target and $\nu_{1,k}$, $\nu_{2,k}$, and $\nu_{3,k}$ are all zero-mean white Gaussian random variables with standard deviation $\sigma_{\nu_{1,k}} = 0.05$ m/s^2, $\sigma_{\nu_{2,k}} = 0.08$ m/s^2 and $\sigma_{\nu_{3,k}} = 0.07$ m/s^2, respectively.

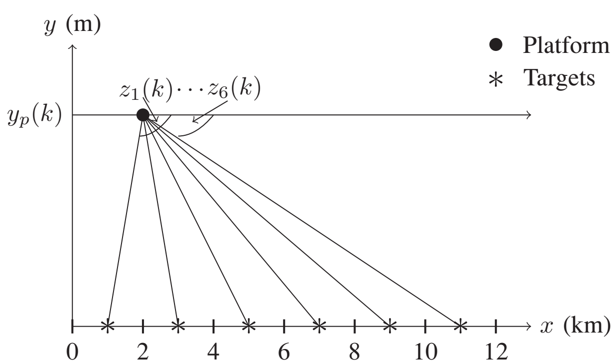

Fig. 1. Motion of the platform and the six targets

In this example, six maneuvering targets are traveling with initial states

$$[\mathbf{x}_1(k)\ \mathbf{x}_2(k)\ \mathbf{x}_3(k)\ \mathbf{x}_4(k)\ \mathbf{x}_5(k)\ \mathbf{x}_6(k)] = \begin{bmatrix} 1000 & -1000 & 17000 & -17000 & 10000 & -10000 \\ 40 & -40 & -50 & 50 & 50 & -50 \end{bmatrix} \tag{46}$$

and the start and end times of the six targets are (1,40), (1,40), (16,38), (16,38), (3,33) and (3,33), respectively.

Target 1 moves for the first 140s at a nearly constant velocity with an initial velocity of 40 m/s, then moves in a positive direction for the next 190s and, finally moves at a nearly constant velocity for the last 60s. Target 2 moves for the first 140s at a nearly constant velocity with an initial velocity of -40 m/s, then moves in the negative direction for 190s and, finally moves at a nearly constant velocity for the last 60s. Target 3 moves 90s in the negative direction with an initial velocity of -50 m/s, then moves at a nearly constant velocity for 50s and for the last 90s moves in a positive direction. Target 4 moves 40s in the positive direction with an initial velocity of 50 m/s, then moves at a nearly constant velocity for last 180s. Target 5 moves for the first 70s at a nearly constant velocity with an initial velocity of 50 m/s, then moves in a positive direction for the next 140s and, finally moves at a nearly constant velocity for the last 90s. Target 6 moves for the first 70s at a nearly constant velocity with an initial velocity of -50 m/s, then moves in the negative direction for 140s and, finally moves at a nearly constant velocity for the last 90s.

Targets move along the X-axis and these six targets, which have a probability of survival $P_{\mathfrak{s},k} = 0.98$, appear and disappear at specific times. The Markovian model transition probability matrices π_{pq} for the six targets are

$$\pi_{pq} \quad = \quad \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 2/5 & 3/5 & 0.0 \\ 2/5 & 0.0 & 3/5 \end{bmatrix}, \tag{47}$$

and the initial model probabilities for the models are 0.33.

Each target is detected with probability $P_{\mathfrak{d},k} = 0.95$ and the target-originated measurements follow the observation model

$$z_i(k) = h[x_p(k), y_p(k), x_i^1(k)] + \omega(k) \qquad i = 1, 2, 3, 4, 5, 6 \tag{48}$$

where

$$h[.] = \tan^{-1} \frac{y_p(k)}{x_i^1(k) - x_p(k)} \qquad i = 1, 2, 3, 4, 5, 6 \tag{49}$$

is the angle between the X-axis and the line of sight from the sensor to the targets. The sensor noise $\omega(k)$ is zero-mean white Gaussian with $\sigma_\omega = 2°$. The sensor noise is assumed independent of the sensor platform perturbations. The received measurements include false alarms. The clutter is modeled as uniformly distributed in the measurement space with average false alarm rate $\lambda_k = 10^{-4}$ (rad)$^{-1}$ over the whole surveillance region $[0, \pi]$ rad.

For tracking multiple targets, an MM-SPHD filter of order 3 is used with 20 knots for position and 10 knots for velocity. At scan $k = 0$, all measurements are used to initialize newborn targets as described in Section V-B. The probability of target spawning is assumed to be zero and the probability of spontaneous target birth is 0.01.

The PHD filter does not provide a mechanism to get the target state estimates directly. One solution is to identify the local maxima of the MM-SPHD surface. The K-means clustering algorithm [23] is used here for state extraction. An alternative is the Expectation-Maximization (EM) based peak extraction approach [24]. The targets are associated to tracks using global nearest-neighbor assignment [17] based on the mean of each target cluster. As shown in Figure 2, all six targets appear and disappear at various times during the surveillance interval. Also shown in Figure 2 are the average of the estimated trajectories. As shown in Figure 3, the MM-SPHD filter estimated the velocities of all targets accurately. The mean velocity of newborn targets is selected randomly from a uniform distribution in the interval $[-100, 100]$ m/s and the standard deviation is assumed to be 0.4 m/s.

The normalized estimation error squared (NEES) [2] and optimal subpattern assignment (OSPA) [21] are used as performance metrics for the example.

The OSPA [21] metric measures the miss-distance between a set of true targets and a set of estimated tracks as a combination of localisation error and cardinality error [9]. Let $X = \{x_1, \cdots, x_{\acute{n}}\}$ and $Y = \{y_1, \cdots, y_{\acute{m}}\}$ be two finite sets. Here, X denotes true finite set of targets and Y denotes the estimated finite set of tracks. The OSPA metric is defined as

$$\bar{d}_{\acute{p}}^{(\acute{c})}(X, Y) = \begin{cases} 0 & \text{if } \acute{m} = \acute{n} = 0 \\ \Psi(X, Y) & \text{if } \acute{m} \leq \acute{n}. \\ d^{(\acute{c})}(X, Y) & \text{if } \acute{m} > \acute{n} \end{cases} \tag{50}$$

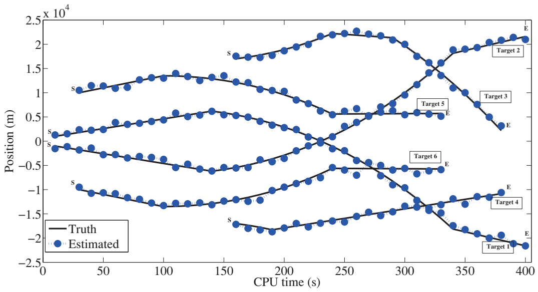

Fig. 2. True position vs. average of estimated positions from 1000 runs ($\sigma_\omega = 2^\circ$, **S**: start, **E**: end).

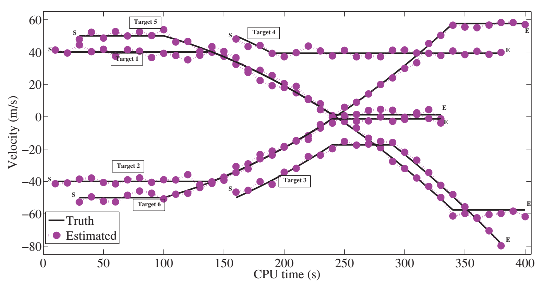

Fig. 3. True velocity vs. average of estimated velocities from 1000 runs ($\sigma_\omega = 2^\circ$, **S**: start, **E**: end).

where

$$\Psi(X,Y) \triangleq \left(\frac{1}{\acute{n}}\left(\min_{\pi\in\Pi_{\acute{n}}}\sum_{i=1}^{\acute{m}} d^{(\acute{c})}(x_i, y_{\pi(i)}) + \acute{c}^{\acute{p}}(\acute{n}-\acute{m})\right)\right)^{\frac{1}{\acute{p}}}$$

and the base distance between x and y denoted by $d^{\acute{c}}(x,y) = \min(\acute{c}, \|x-y\|)$, $\Pi_{\acute{n}}$ is the set of permutations with length $\acute{m}$ on the set of $\{1,\cdots,\acute{n}\}$ where $\acute{n} = \|X\|$ and $\acute{m} = \|Y\|$. In the simulations $\acute{p} = 10$ and $\acute{c} = 100$.

The MM-SPHD filter performance is evaluated along with those of multiple model based GM-USMC-PHD, the GM-SMC-PHD and the AP-PHD filters.

In the MM-GM-USMC-PHD filter implementation, the IS function approximated in the form of Gaussian mixture that is a sum of Gaussian components and the maximum number of Gaussian terms $= 100$. The number of samples per GM component or target is set to 2500. The newborn target initialization, resampling and state extraction follow [27]. Note that the GM-USMC-PHD filter does not need resampling because the process to manage GM for the multitarget state extraction and component deletion enables the algorithm to have the same effect as resampling. The Unscented Information Filter (UIF) is the information form of the unscented Kalman filter (UKF) [2]. The UIF is used to compute the mean and the covariance of Gaussian components.

The GM implementation of the MM-GM-SMC-PHD filter is with the EKF for filtering, pruning parameters of elimination threshold $T_p = 10^{-5}$, merging threshold $T_m = 4$m and maximum number of Gaussian terms 100. The SMC implementation of the MM-GM-SMC-PHD uses the transition density to sample particles. Particles are initialized around measurements [7] and 2500 particles are used per existing targets and 50 particles are used for each newborn target. An estimate of the number of targets is determined by summing up all the weights of the particles. The estimation of the number of targets and their state extraction carried as in [19].

The MM-AP-PHD filter uses 2500 particles per existing target, while the number particles per newborn target is set to 100. The initialization of the newborn targets is driven by the measurements. The current measurements are associated with the highest bidder if the bid is at least equals 0.4. The Auxiliary Importance Sampling (AIS) [3] process starts with the selection of the measurements that are well described by the targets' states extracted from the estimated PHD and this is achieved using auction algorithm [3]. The state extraction is determined as in [3].

In order to facilitate a fair comparisons, we ran all methods with the same multiple model strategy [18]. All PHD filters are initialized with Gaussian distribution with mean [1000 m, 40 m/s] and standard deviation $\sigma_{\nu_{1,k}} = 0.05$ m/s^2 representing target 1 and the constant-velocity model is used. At the beginning of the scenario, all MM-PHD filters assume that there is only one track corresponding to target 1 within the surveillance region.

The overall filter accuracy performance metric, the OSPA [21], is computed for each filter over 1000 Monte Carlo runs for measurement noise standard deviation levels σ_ω=2° and σ_ω=4°. The OSPA metric measures the combination of both localization and cardinality errors. The average OSPA values are plotted in Figures 4 and 5.

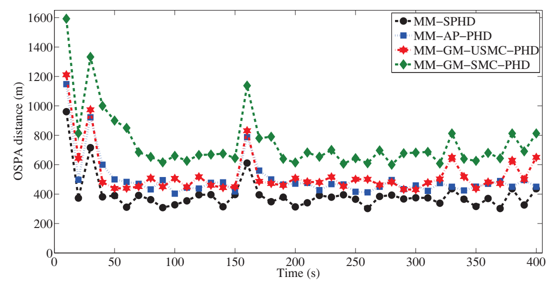

Fig. 4. OSPA distance (m) averaged over 1000 Monte Carlo runs ($\sigma_\omega = 2^\circ$, $\acute{c} = 10$, $\acute{p} = 100$).

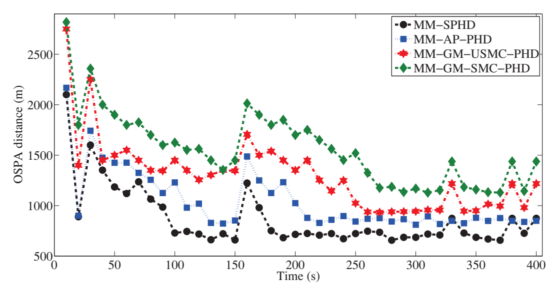

Fig. 5. OSPA distance (m) averaged over 1000 Monte Carlo runs ($\sigma_\omega = 4^\circ$, $\acute{c} = 10$, $\acute{p} = 100$).

The key observation is that the MM-SPHD filter with a few knots performed the best of in terms of OSPA for both measurement noise levels. As shown in Figure 4 and 5, high values of OSPA distance occur when new targets are born around time indices $k = 1, 3, 16$. Also targets disappear with small OSPA peaks at time indices $k = 33, 38, 40$. As shown in Figure 5, as the nonlinearity increase with increasing measurement noise levels the GM-based MM-PHD filter perform poorly.

Figure 6 reveals the consistency of the MM-SPHD, MM-GM-USMC-PHD, MM-AP-PHD and the MM-GM-SMC-PHD filters in terms of normalized estimation error squared (NEES) compared with the 95% confident-region of the $\mathcal{X}^2$ distributions [2] when the measurement noise standard deviation is

$\sigma_\omega = 2°$. To illustrate the degeneracy resistance capability of the proposed MM-SPHD filter, the standard deviation of the measurement noise is reduced to σ_ω=0.02°. The model parameters for the filters remain unchanged but with the correct measurement noise level. This scenario causes the particle-based PHD filters to become degenerative. It can be observed from Figure 7 that the MM-SPHD filter is able to provide efficient results with the same 10 velocity knots and 20 position knots. Note that using the Regularized Particle Filter (RPF) [7, 8] can avoid the degeneracy problem caused by sampling and resampling. However, the RPF has the disadvantage is that the samples are no longer guaranteed to asymptotically approximate the posterior [16].

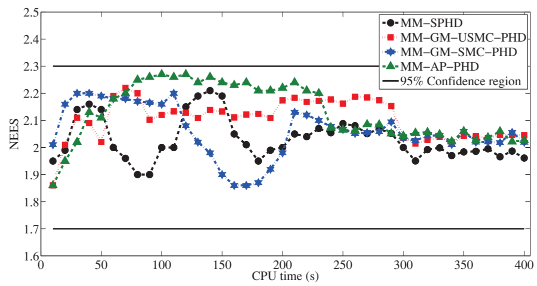

Fig. 6. NEES comparison from 1000 Monte Carlo runs ($\sigma_\omega = 2°$).

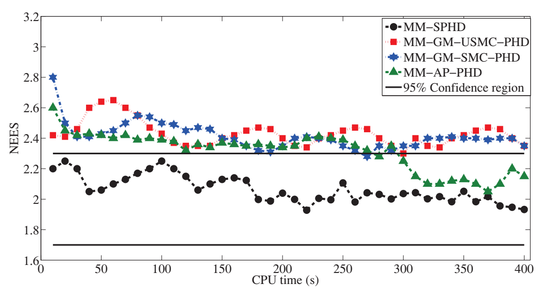

Fig. 7. NEES comparison from 1000 Monte Carlo runs ($\sigma_\omega = 0.02°$).

VII. Conclusions

In this paper, a Multiple Model Spline Probability Hypothesis Density filter implementation was presented as an alternative implementation to the Sequential Monte Carlo and the Gaussian Mixture MM-PHD filters for maneuvering target tracking problems. The resulting algorithm can handle linear, non-linear, Gaussian, and non-Gaussian models. The MM-SPHD filter can provide continuous estimates of the probability hypothesis density function and it is relatively immune to the degeneracy problem. The MM-SPHD filter can maintain highly accurate tracks by taking advantage of dynamic knot movement, but at the expense of higher computational complexity. The MM-SPHD filter performs well with a few knots and provides continuous state estimates for any system, which leads to non-degenerative results. This new filter, which yields accurate results albeit with a higher computational load, is useful in tracking high-value maneuvering targets (e.g., missiles, ground targets) in the presence of nonlinearity or non-Gaussianity.

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