Search Strategies for Detecting Targets Exhibiting Rectangular Symmetry

Dr. Bao Nguyen
Maritime Operational Research Team

Dr. Alex Bourque
Maritime Operational Research Team

DRDC CORA TM 2012–031
February 2012

Defence R&D Canada
Centre for Operational Research and Analysis

Maritime Operational Research Team
Director of Maritime Strategy

National Defence   Défense nationale

Canada

Search Strategies for Detecting Targets Exhibiting Rectangular Symmetry

Dr. Bao U. Nguyen

Dr. Alex Bourque

Defence R&D Canada – CORA

Technical Memorandum
DRDC CORA TM 2012-031
February 2012

Principal Author

Original signed by Dr. B. U. Nguyen

Dr. B. U. Nguyen

Maritime OR Team

Approved by

Original signed by Dr. R. E. Mitchell

Dr. R. E. Mitchell

Head Maritime and ISR OR

Approved for release by

Original signed by P. Comeau

P. Comeau

Chief Scientist

Abstract

In this paper, search strategies where targets are observed at several different angles are found and proven to be critical points. Targets are assumed to exhibit rectangular symmetry and to have uniformly-distributed orientation. By rectangular symmetry, it is meant that one side of a target is the mirror image of its opposite side. Finding critical points is generally an NP-hard problem. Fortunately, symmetry principles allow analytical and intuitive solutions to be found. One such search strategy consists of choosing n angles evenly separated on the half-circle and provides a lower bound estimate for the probability of not detecting targets.

Résumé

Dans ce présent document, des stratégies de recherche liées à l'observation de cibles selon plusieurs angles différents sont identifiées et démontrées être des points critiques. Il est présumé que les cibles ont une symétrie rectangulaire et que leur orientation est uniformément distribuée. Le terme symétrie rectangulaire signifie qu'un côté du plan qui divise la cible est l'image-miroir du côté opposé. La détermination des points critiques est généralement un problème de type NP. Heureusement, les principes de la symétrie permettent de trouver des solutions analytiques et intuitives. L'une de ces stratégies de recherche consiste à choisir n angles répartis de façon uniforme sur le demi-cercle et permet de déterminer une valeur limite inférieure quant à la probabilité de ne pas détecter les cibles.

This page intentionally left blank.

Executive summary

Search Strategies for Detecting Targets Exhibiting Rectangular Symmetry

Bao U. Nguyen; Alex Bourque; DRDC CORA TM 2012-031; Defence R&D Canada – CORA; February 2012.

Introduction: In mine hunting operations it is known that the detection performance improves when a target is observed many times at different aspect angles. However, this fact is often overlooked when planning search missions. As a consequence, targets may not be detected even if the search area is entirely covered. In this paper, search strategies where targets are observed at several different angles are found and proven to be critical points of the probability of no detection.

In this paper, targets are assumed to exhibit rectangular symmetry. That is, the left hand side of a target is the mirror image of its right hand side, and the rear end is the mirror image of its front end. Many targets can be approximated with this class of symmetry including hull forms, mines and human bodies. Additionally, within the search area, their orientation is assumed to be uniformly distributed.

Results: Finding critical points is generally a hard problem. Fortunately, symmetry principles allow analytical and intuitive solutions to be found. One such search strategy is to observe the target such that the separation between two consecutive observations is a constant and equal to 180 degrees divided by the number of observations. For example, two observations separated by 90 degrees or three observations separated by 60 degrees lead to higher probability of detection.

Significance: The simplicity of the solution implies that no complicated calculations are required prior to a search as long as the target has the assumed rectangular symmetry. This fact should improve the task of planning the path of mobile sensors, such as unmanned vehicles, to search for fixed targets, as well as of deploying a fixed sensor array to monitor traffic through choke points.

Future plans: Work is currently underway to prove that no other search strategies lead to a higher probability of detecting targets and to explore at which angle a target should be observed given a prior set of observations, a situation encountered in actual missions.

Sommaire

Search Strategies for Detecting Targets Exhibiting Rectangular Symmetry

Bao U. Nguyen; Alex Bourque; DRDC CORA TM 2012-031; R & D pour la défense Canada – CORA; Février 2012.

Introduction ou contexte: Dans les opérations de chasse aux mines, il est bien connu que l'efficacité de la détection est plus grande lorsque la cible est observée à de nombreuses reprises, selon des angles d'aspect différents. Toutefois, il arrive souvent qu'on ne tienne pas compte de ce fait lors de la planification des missions de recherche. En conséquence, il est possible que les cibles ne soient pas détectées même si la zone de recherche est entièrement couverte. Dans le présent document, des stratégies de recherche sont présentés pour des situations où la zone d'intérêt est complètement couverte et où les cibles sont observées selon plusieurs angles différents; on démontre ensuite de quelles façons ces stratégies déterminent les valeurs extrêmes de la probabilité de non détection.

Dans ce document, on présume que les cibles présentent une symétrie rectangulaire. Cela signifie que le côté gauche de la cible est l'image-miroir du côté droit et que la partie arrière est l'image-miroir de la partie avant. De nombreuses cibles peuvent être intégrées à cette classe de symétrie, y compris les formes de caisses, les mines et les corps humains. De plus, dans la zone de recherche, on présume que l'orientation des cibles est uniformément distribuée.

Résultats: La détermination des points critiques est habituellement un problème difficile. Heureusement, les principes de la symétrie permettent de trouver des solutions analytiques et intuitives. L'une de ces stratégies consiste à observer la cible de façon à ce que la séparation entre deux observations consécutive est une constante et qu'elle soit égale à 180 degrés divisés par le nombre d'observations. Par exemple, deux observations séparées par 90 degrés ou trois observations séparées par 60 degrés entraînent une plus grande probabilité de détection.

Importance: La simplicité de la solution implique qu'aucun calcul compliqué n'est nécessaire avant d'entreprendre une recherche, dans la mesure où la cible respecte la symétrie rectangulaire. Ce fait devrait améliorer la tâche de planifier le parcours des capteurs mobiles ─ tels que les véhicules téléguidés K pour les recherches de cibles fixes ainsi que pour déployer un réseau de capteurs fixes pour surveiller la circulation aux points de passage obligé.

Perspectives: Des travaux actuellement en cours visent à démontrer que cette stratégie de recherche offre la meilleure probabilité de détecter des cibles et de déterminer selon quel angle une cible devrait être observée compte tenu d'un ensemble précédents d'observations, une situation que l'on rencontre dans les missions actuelles.

Table of contents

List of figures

List of tables

Acknowledgements

The authors would like to acknowledge Aaron Percival from Defence R&D Canada - Atlantic for bringing to his attention the issue of correlation effects in the detection process as well as Dr. David Maybury, as well as the two reviewers for useful comments.

1 Introduction

In mine hunting operations it is known that the detection performance improves when a target is observed many times at different aspect angles [1]-[6]. Similarly, classification algorithms [7]-[9], and fixed sensor-arrays deployed for target localization and tracking [10]-[13] benefit from multi-aspect observations. This fact is, however, often overlooked or not applied in practise. For example, the formula for the probability of detecting a target in a random search derived by Koopman is widely used yet it assumes no angular dependence [14].

In this paper, search strategies that are critical points of the overall probability of not detecting a target observed at several different angles are identified. In principle, finding critical points is a priori intractable as it is multi-dimensional in the sense that each observation is independent of one another and hence each observation angle must be considered as a separate dimension. What is more, the explicit expression for the overall probability of detection can be extremely complicated even when the probability of detection for a single observation is simple and the number of observations is small.

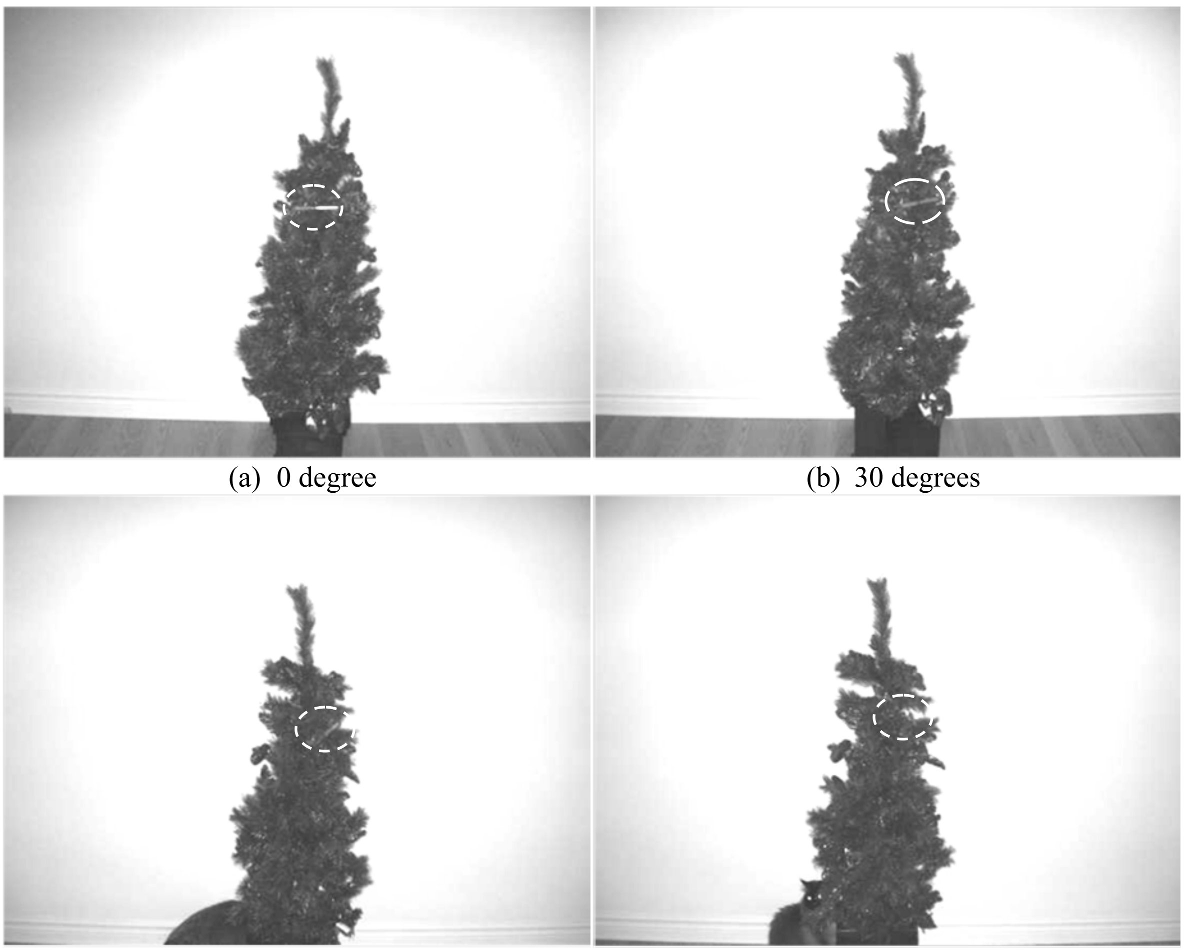
The novelty of our approach lies in the fact that this problem is solved using an elegant symmetry argument. Specifically, targets are assumed to exhibit rectangular symmetry. That is, the left hand side of a target is the mirror image of its right hand side, and the rear end is the mirror image of its front end. Many targets can be approximated with this class of symmetry including hull forms, mines and human bodies.

Using these assumptions, it is shown that observations evenly distributed on multiples of the half-circle as in Ref. [13] are a critical point of the probability of no detection. This constitutes a departure from the current literature on sensor geometry [10]-[13] as our result is derived for rectangular targets rather than for point targets. The simplicity of the solution implies that no complicated calculations are required prior to a search as long as the target has the assumed approximate symmetry. This fact should improve the task of planning the path of mobile sensors, such as unmanned vehicles, to search for fixed targets, as well as of deploying a fixed sensor array to monitor traffic through choke points.

In Section 2, the angular dependence of the detection process and the necessity of multiple observations of a target are further argued. In Section 3, assumptions and the problem are stated. Section 4 presents a set of search strategies that are critical points of the probability of no detection, while in Section 5 characterizes a sub-set of these search strategy. Section 6 provides a lower bound estimate of not detecting targets achievable using these tactics. To illustrate the main results, analytical and numerical examples are presented in Sections 7 and 8, respectively. Conclusions including future work are presented in Section 9. Annex A provides supplementary lemmata used in Sections 4 and 5, while Annex B details the simulation parameters of Section 8.

2 Motivation

In this Section, the qualitative dependence of the probability of detection as a function of angle is illustrated. A pen is put on a Christmas tree and pictures of the tree (including the pen) are taken as the tree is rotated by approximately 30 degrees each time. [1] This pen has practically the rectangular symmetry and is approximately six inches in length. From Figure 1, it is difficult to identify the pen when angle is 60 degrees; even impossible when angle is 90 degrees. It can be seen that the pen is easily identified when angle is zero degree or 30 degrees.



(a) 0 degree (b) 30 degrees (c) 60 degrees (d) 90 degrees

Figure 1: Pen observed at different observation angles.

[1] The pen is approximately six inches in length. The tree is about one meter in height. The distance between the camera and the tree is approximately 1.5 meter. A cat is shown to give an idea of the scale. A Canon Power Shot A530 digital camera is used.

The experiment above shows how detection performance is affected by the observation angle. In addition, it highlights how the probability of detection improves with multiple observations at different angles. For example, if the pen is first observed at 60 degrees, then it is difficult to identify. However, if the pen is further observed at zero degree, then it is easily identified. Because the orientation of the target is not known a priori, it follows that making observations at multiple angles is a valid tactic to improve the probability of detection.

3 Problem Statement

As shown in Section 2, the dependence of detection process on angle occurs often in search and detection operations. In general, the effectiveness of such an operation also depends on the distance between the sensor and the target. However, here, the probability of detection is assumed constant as a function of range and, hence, the focus is only on the angular dependence. For more details on the range dependence, refer to Ref. [6].

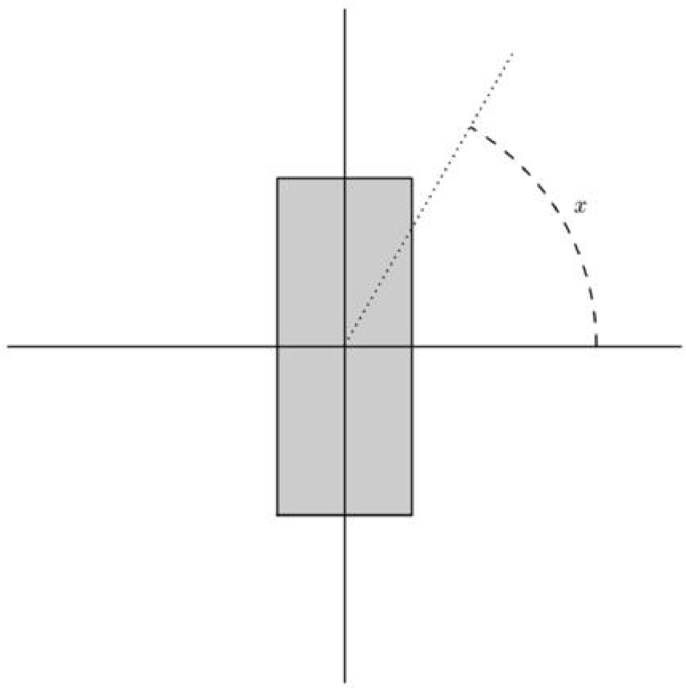


Figure 2: Cylindrical target observed at angle x.

As shown in Figure 2, the problem is modeled on a two-dimensional plane and the observation angle, x, is defined as the counter-clockwise angle measured in radian between the sensor beam and the short axis of a rectangular (positive horizontal axis) target. An observation angle of zero degree corresponds to the observation of the long side of the target, while an observation angle of $\pi/2$ degrees corresponds to the observation of the short side of the target. Targets considered will have approximate rectangular symmetries as shown in Figure 3. That is, they possess a reflection axis through their short axis (dashed line), and a rotation by 180 degrees around the centre (dot-dashed line).[2] Human bodies, canoes, ships and mines have these types of symmetries.

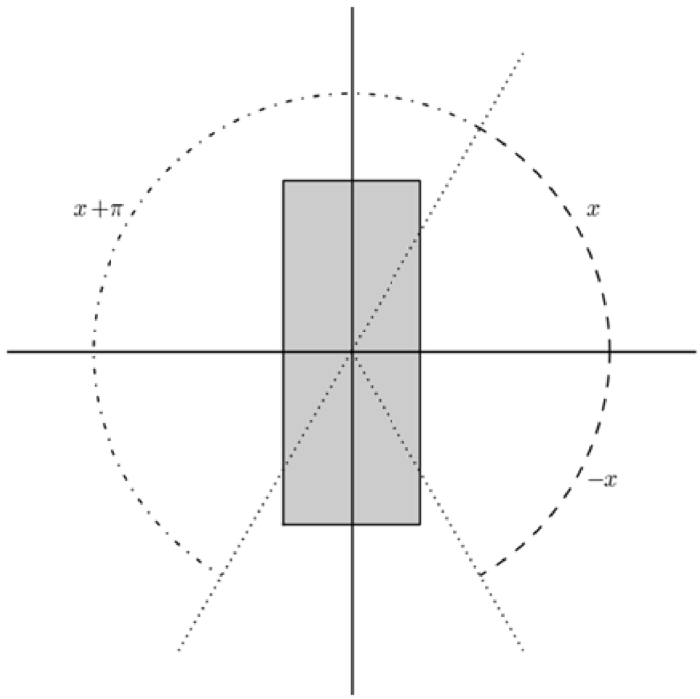


Figure 3: Symmetries of the target.

In what follows, the probability of no detection rather than the probability of detection is considered; one being the complement of the other. Define the single probability of no detection as the probability of not detecting a single target at angle x and denote this single-value real function as $g(x)$. Note that the single probability of no detection is even due to the reflection symmetry through the short axis of the target and periodic due to the rotation of 180 degrees around the target's centre. Specifically,

$$g(x) = g(-x),$$

$$g(x) = g(x + \pi).$$

[2] The composition of a reflection through the short axis followed by a rotation through the centre of the target is equivalent to a reflection through the long axis of the target (front-end/back-end mirror symmetry).

As an example consider $g(x) = \sin(x)^2$. From Figure 4, it is clear that it substantially increases if the observation angle differs from zero degrees, i.e., the single probability of not detecting the target depends significantly on the angle of observation.

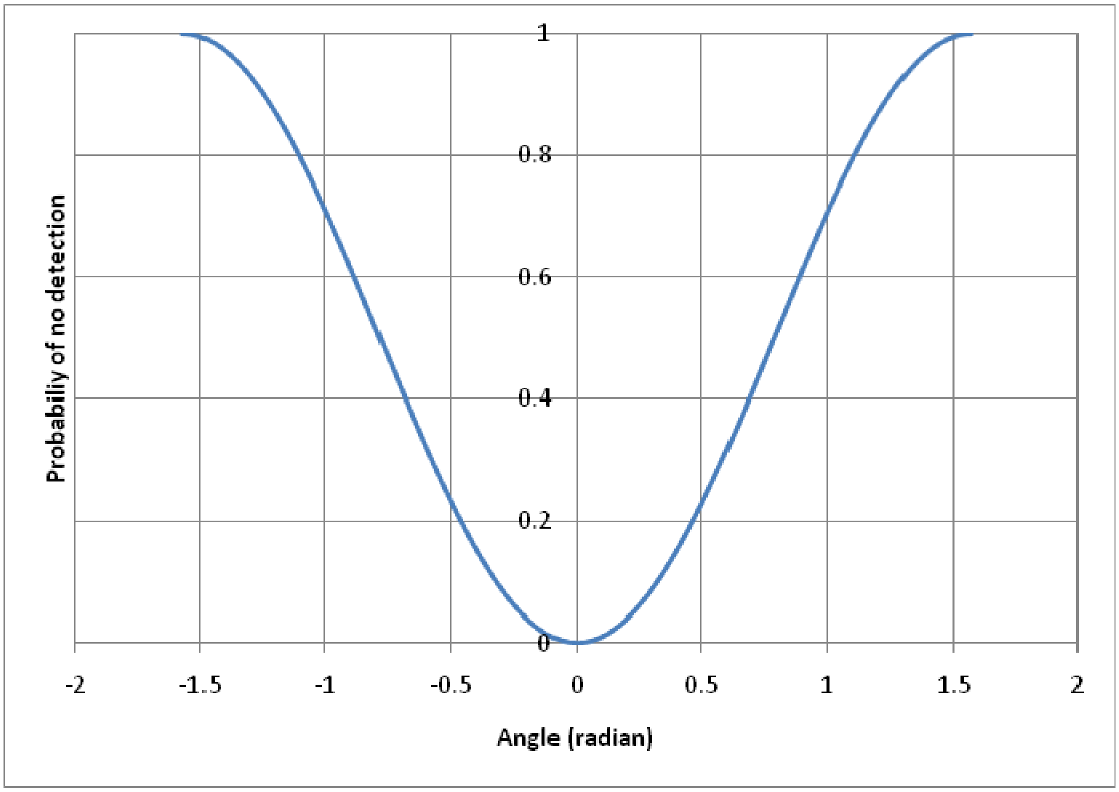


Figure 4: Probability of not detecting a single target as a function of observation angle.

Next, define the multiple probability of no detection as the probability of not detecting a single target after n observations. Let μ_i be the i-th angle at which the target is observed relative to x and $\vec{\mu} = (\mu_o, \dots, \mu_{n-1})$ be the vector of the n relative observation angles. Assume the multiple observation detection process is a Bernoulli process, i.e., all observations are independent. Then, the multi-observation probability of no detection is modeled as the product of single probabilities of no detection. In general, however, the exact value of x, i.e., the orientation of the target is unknown. To circumvent this problem, assume that the target's orientation is uniformly distributed and evaluate the average multiple probability of not detection a single target. Denote this quantity by $G(\vec{\mu})$. Then,

$$G(\vec{\mu}) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx g(x + \mu_0) g(x + \mu_1) \dots g(x + \mu_{n-1}) . \quad (1)$$

The problem is then defined as finding search strategies that are critical values of $G(\vec{\mu})$. Below such a set of search strategy is identified. From it, a lower bound of the probability of no detection is also estimated.

For simplicity, the probability of no detection is taken to mean the average multiple probability of no detection in the next sections.

4 A Set of Search Strategies

The assumption that the target's orientation is uniformly distributed implies that the probability of no detection is independent of the initial observation angle. That, along with the target's symmetry, entails that, if the observations are evenly distributed, they are a critical point of the probability of no detection.

In this section, a condition ensuring that all partial derivatives of the probability of no detection are equal to zero is derived. From this condition, a set of search strategies that are critical points of the probability of no detection is identified and used to recast the probability of no detection into a form useful in the subsequent sections.

Let us first introduce some useful notation and definitions. Let $i \in \{0, \dots, n-1\}$, $\mathcal{N}_i = \{0, \dots, n-1\} \setminus \{i\}$ and $j \in \mathcal{N}_i$. Define $\partial_i = \frac{\partial}{\partial_i}$. Denote $\vec{\mu}^*$ as a critical point of $G(\vec{\mu})$. Let a be an integer and b be a positive integer. Define the modulo operation as

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b.$$

Let m be a non-negative integer and $\mu = \frac{\pi}{n}$. Define

$$\tilde{G}(m,n) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx g(x) g(x + m\mu) \dots g(x + m(n-1)\mu),$$

Then the following hold.

Lemma 4.1: The partial derivative can be written as the integral of a product of two odd functions. That is

$$\partial_i G(\vec{\mu}) = \frac{1}{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, g'(x) \left\{ \prod_{j \in \mathcal{N}_i} g(x + \mu_j - \mu_i) - \prod_{j \in \mathcal{N}_i} g(x - \mu_j + \mu_i) \right\}.$$

Proof: Apply the partial derivative to the expression for $G(\vec{\mu})$ given by Eq. (1). Then

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, g'(x + \mu_i) \prod_{j \in \mathcal{N}_i} g(x + \mu_j).$$

Let $x \to x - \mu_i$ and note that, because the orientation of the target is uniformly distributed, Lemma A.1 applies and dictates that the integral is invariant under this change of variable. Thus,

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, g'(x) \prod_{j\in \mathcal{N}_i} g(x+\mu_j-\mu_i)\,. \tag{2}$$

Because $g(x)$ is even,

$$\partial_i G(\vec{\mu}) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, g'(x) \prod_{j\in \mathcal{N}_i} g(-x-\mu_j+\mu_i)\,.$$

Let $x \to -x$ and remark that $g'(x)$ is odd, i.e., $g'(-x) = -g'(x)$. Then

$$\partial_i G(\vec{\mu}) = -\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, g'(x) \prod_{j\in \mathcal{N}_i} g(x-\mu_j+\mu_i)\,. \tag{3}$$

And the result follows from the average of Eqs (2) and (3). ■

The next Lemma identifies a set of search strategies for which all partial derivatives are equal to zero.

Lemma 4.2: Define (m, n) to be the search strategy such that the separation between two consecutive observations is a constant and equal to $\frac{m\pi}{n}$ with m an integer. Then the search strategy (m, n) defines a subset of all possible search strategies that are critical points of $G(\vec{\mu})$.

Proof: Evaluate the product in Eq. (2) at the critical point defined by this search strategy, i.e., $\vec{\mu}^*$. Then,

$$\prod_{j\in \mathcal{N}_i} g(x+\mu_j^*-\mu_i^*).$$

Next, remark that the definition of the strategy (m, n) implies that

$$\mu_j^* - \mu_i^* = m(j-i)\mu = m[-(2i-j)+i]\mu.$$

And the definition of the modulo entails that

$$2i - j = \left\lfloor \frac{2i-j}{n} \right\rfloor n + (2i-j) \bmod n.$$

Define $\sigma_i(j) = (2i-j) \bmod n$ and recall that $g(x)$ is periodical. Then,

$$g\left(x - m\left\lfloor\frac{2i-j}{n}\right\rfloor n\mu - m\sigma_i(j)\mu + mi\mu\right) = g(x - m\sigma_i(j)\mu + mi\mu).$$

Finally, Lemma A.2 implies that the map $\sigma_i(j)$ is a bijection from the set of j to itself. Therefore,

$$\begin{aligned}\prod_{j\in \mathcal{N}_i} g\left(x+\mu_j^* - \mu_i^*\right) &= \prod_{j\in \mathcal{N}_i} g(x - m\sigma_i(j)\mu + mi\mu) \\ &= \prod_{j\in \mathcal{N}_i} g(x - mj\mu + mi\mu) \\ &= \prod_{j\in \mathcal{N}_i} g\left(x - \mu_j^* + \mu_i^*\right).\end{aligned}$$

And Lemma 4.1 then entails that $\partial_i G(\vec{\mu}^*) = 0$ for all i, which proves the claim. ■

In the next lemma, the probability of no detection for a search strategy (m, n) is cast into a form that will be useful in what follows.

Lemma 4.3: Let the search strategy be (m, n). Then the probability of no detection is equal to $\tilde{G}(m, n)$.

Proof: First, remark that only the difference between two consecutive observations is specified in Lemma 4.2. Thus, liberty exists in the choice of the absolute reference angle. Without loss of generality, take μ_0^* to be this absolute reference angle. Then,

$$\mu_i^* = \mu_0^* + mi\mu.$$

Substituting this expression into Eq. (1) yields

$$G(\vec{\mu}^*) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx g(x + \mu_0^*) g(x + \mu_0^* + m\mu) \ldots g(x + \mu_0^* + m(n-1)\mu)\,.$$

Next, let $x \rightarrow x - \mu_0^*$. Then Lemma A.1 entails that the domain of integration is invariant under such a shift of the integration variable. ■

5 Co-Prime (m, n) Strategies

In the previous section, a set of search strategies was identified: the (m, n) search strategies. In this section, it is shown that if no integer greater than one divides m and n (co-primes) then, the strategy (m, n) is equivalent to that of $(1, n)$, i.e., they yield the same probability of no detection,. This arises due to the periodicity of the probability of no detection: all observations done at an angle greater than 180 degrees can be mapped to one at an angle less than 180 degrees.

Let us first introduce some useful notation and definitions. Let $\gcd(\cdot,\cdot)$ be the greatest common divider. Let r, q and p be strictly positive integers such that $m = pq$, $n = pr$ and $p = \gcd(m, n)$. Let $i \in \{0, \dots, n-1\}$, $j \in \{0, \dots, r-1\}$ and $k \in \{1, \dots, p\}$. Define

$$h(x; m, r) = g(x)g(x + m\mu) \dots g(x + m(r-1)\mu).$$

Lemma 5.1: The following identity holds:

$$\prod_{k=1}^{p} h(x + (k-1)\mu; m, r) = h(x; 1, n).$$

Proof: Use the definition of $h(x; m, r)$ to write

$$\prod_{k=1}^{p} \prod_{j=0}^{r-1} g(x + mj\mu + (k-1)\mu).$$

The proof then follows from the periodicity of $g(x)$ and by re-arranging the product. Start by noting that the definitions of m and n and of the modulo operation imply that

$$mj = pqj = p\left(\left\lfloor \frac{qj}{r} \right\rfloor r + qj \bmod r\right) = n\left\lfloor \frac{qj}{r} \right\rfloor + p(qj \bmod r).$$

Using the right-hand most expression for mj and the periodicity of $g(x)$,

$$\prod_{k=1}^{p} \prod_{j=0}^{r-1} g(x + mj\mu + (k-1)\mu) = \prod_{k=1}^{p} \prod_{j=0}^{r-1} g(x + p(qj \bmod r)\mu + (k-1)\mu).$$

Next, from Lemma A.3, the map $\sigma(j) = qj \bmod r$ is known to be a bijection from the set of j to itself. Therefore, the product can also be written as

$$\prod_{k=1}^{p}\prod_{j=0}^{r-1} g(x + pj\mu + (k-1)\mu)\,.$$

Finally, recall that $i \in \{0, \dots, n-1\}$, $j \in \{0, \dots, r-1\}$ and $k \in \{1, \dots, p\}$ with $n = pr$. Then, from Lemma A.4, the set of (j,k) pairs can be mapped to the set of i using the bijection $\sigma(k-1, j) = (k-1) + pj$. Therefore,

$$\prod_{k=1}^{p}\prod_{j=0}^{r-1} g(x + pj\mu + (k-1)\mu) = \prod_{i=0}^{n-1} g(x + i\mu). \qquad ■$$

Lemma 5.1 implies the following corollary which identifies a subset of (m,n) search strategies with the same probability of no detection.

Corollary 5.2: Consider a search strategy (m,n) such that m and n are co-primes, i.e., $p = \gcd(m,n) = 1$. Then, $n = r$ and Lemma 5.1 implies that $h(x;m,r) = h(x;m,n) = h(x;1,n)$, which entails that $\tilde{G}(m,n) = \tilde{G}(1,n)$. Thus, for a given n, all (m,n) search strategies are equivalent if $p = \gcd(m,n) = 1$.

Example 5.1: To illustrate the corollary, consider the case where $n = 4$ and $m = 3$. Then $gcd(3,4) = 1$ and Corollary 5.2 implies that

$$g(x)g(x+\mu)g(x+2\mu)g(x+3\mu) \;=\; g(x)g(x+3\mu)g(x+6\mu)g(x+9\mu).$$

Or, equivalently,

$$g(x)g\left(x+\frac{\pi}{4}\right)g\left(x+\frac{\pi}{2}\right)\,g\left(x+\frac{3\pi}{4}\right) = g(x)g\left(x+\frac{9\pi}{4}\right)g\left(x+\frac{6\pi}{4}\right)\,g\left(x+\frac{3\pi}{4}\right)$$

Figure 5 illustrates a set of four observations. For $m = 1$, the sequence in which the target is observed is $1-2-3-4$, while it is $1-4-3-2$ for $m = 3$. It is clear that both sequences lead to the same the probability of detection since all observations are considered independent (Bernouilli process).

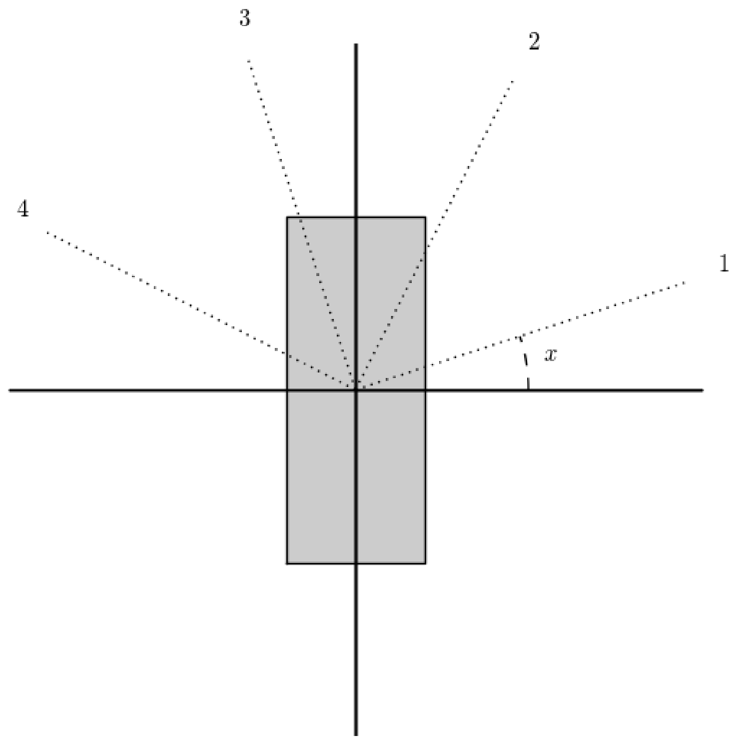


Figure 5: Four observations of a randomly-oriented target.

6 A Lower Bound Estimate for (m,n) Strategies

In the previous section, co-prime (m,n) search strategies were identified. In this section, the probability of no detection for (m,n) search strategies is shown to be minimal if m and n are co-primes. This is then taken as a lower bound estimate for the probability of no detection.

Let us first introduce some useful notation and definitions. In what follows, let $h(x) = h(x;m,r)$. Then, with $\delta_k = (p-k)\mu$, define

$$\lambda_k(x;m,r) = \begin{cases} h(x+\mu;m,r)\dots h(x+\ \delta_k;m,r), & k \in \{1,\dots,p-1\} \\ 1, & k = p. \end{cases}$$

And, with l a positive integer and (u_l, v_l) a pair of positive integers,

$$e_l = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x;m,r) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^l\right] h(x;m,r)^{u_l+v_l} + \left(\tfrac{1}{2}\right)^l h(x;m,r)^{u_l} h(x+\delta_k;m,r)^{v_l} \right\}$$

where, for $l > 0$, the transition rules are

$$(u_l, v_l) = \begin{cases} (u_{l-1} - v_{l-1}, 2v_{l-1}), & u_{l-1} \geq v_{l-1} \\ (2u_{l-1}, v_{l-1} - u_{l-1}), & v_{l-1} > u_{l-1}. \end{cases} \tag{4}$$

Let $\varphi(z)$ be the Euler's totient of the strictly positive integer z.

Lemma 6.1: The following identity holds:

$$g(x)g(x+m\mu)\dots g(x+m(n-1)\mu) = h(x;m,r)^p.$$

Proof: Write the left-hand side of the equality as

$$\prod_{i=0}^{n-1} g(x+mi\mu).$$

The proof then follows by re-arranging the product and from the periodicity of $g(x)$. Start by recalling that $i \in \{0,\dots,n-1\}$, $j \in \{0,\dots,r-1\}$ and $k \in \{1,\dots,p\}$ with $n = pr$. From Lemma A.4, the set of i can be mapped to the set of (j,k) pairs using the bijection $\sigma(j,k-1) = j + r(k-1)$. Therefore,

$$\prod_{i=0}^{n-1} g(x+mi\mu) = \prod_{k=1}^{p}\prod_{j=0}^{r-1} g(x+mj\mu+m(k-1)r\mu).$$

Next, note that the definitions of m and n imply $mr = nq$ and that the definition of μ implies $n\mu = \pi$, from which follows that $m(k-1)r\mu = (k-1)q\pi$. This equality and the periodicity of $g(x)$, then result in

$$g(x+mj\mu+m(k-1)r\mu) = g(x+mj\mu+(k-1)q\pi) = g(x+mj\mu).$$

Finally, from the definition of $h(x;m,r)$,

$$\prod_{k=1}^{p}\prod_{j=0}^{r-1} g(x+mj\mu) = \prod_{k=1}^{p} h(x;m,r) = h(x;m,r)^p . \qquad \blacksquare$$

Lemma 6.2: The following inequality holds:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_k(x;m,r)h(x;m,r)^k \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x;m,r)h(x;m,r)^{k+1} . \qquad (5)$$

Proof: For greater clarity, suppress m and r in the arguments of $\lambda_k(x;m,r)$ and $h(x;m,r)$. Because $\lambda_k(x) = \lambda_{k+1}(x)h(x+\delta_k)$ where $\delta_k = (p-k)\mu$,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_k(x)h(x)^k = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x)^k h(x+\delta_k).$$

Therefore, the inequality to prove is equivalent to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x)^k h(x+\delta_k) \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x)^{k+1} .$$

Suppress the indices of $\lambda_{k+1}(x)$ and δ_k in what follows. Next, consider the set $\{e_l\}$ with $(u_0, v_0) = (k, 1)$. The first element of which (e_0) is then equal to the left-hand side of the inequality. Next, proceed to show that one of the element of the set is less than or equal to the right-hand side of Eq. (5). Two cases are possible. To discuss these cases, introduce the two integers y and z such that $y \in \mathbb{N}$, $z \in \mathbb{N}^*\backslash\{2\}$ and $2^y z = k+1$. The first case occurs when $z = 1$. Then, since $v_0 = z = 1$, $2^l \bmod 2^y = 0$ for $l \ge y$ and Lemma A.9 states that, $(u_y, v_y) = (0, k+1)$. Therefore,

$$e_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^y \right] h(x)^{k+1} + \left(\tfrac{1}{2}\right)^y h(x+\delta)^{k+1} \right\}.$$

Since Lemma A.6 implies that $h(x+\delta)$ can be replaced with $h(x)$ in the second term, this element is also equal to the right-hand side of the inequality, i.e.

$$e_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) h(x)^{k+1}.$$

And the inequality follows since the set is partially ordered by virtue of Lemma A.8. The second case occurs when $z > 2$. Then, $2^l \bmod 2^y z > 0$ for all l and Lemma A.10 dictates that $(u_y, v_y) = (u_{y+\varphi(z)}, v_{y+\varphi(z)})$. Therefore,

$$e_y = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^y \right] h(x)^{k+1} + \left(\tfrac{1}{2}\right)^y h(x)^{u_y} h(x+\delta)^{v_y} \right\}$$

$$\mathrm{e}_{\mathrm{y}+\varphi(z)} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^{y+\varphi(z)} \right] h(x)^{k+1} + \left(\tfrac{1}{2}\right)^{\mathrm{y}+\varphi(z)} h(x)^{u_y} h(x+\delta)^{v_y} \right\}.$$

By virtue of $\{e_l\}$ being partially ordered, $e_y \le e_{y+\varphi(z)}$. Through simple algebraic manipulations, cast this inequality into

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) h(x)^{u_y} h(x+\delta)^{v_y} \le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) h(x)^{k+1}$$

Then, using this result, show that

$$\begin{aligned} e_y &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^y \right] h(x)^{k+1} + \left(\tfrac{1}{2}\right)^y h(x)^{u_y} h(x+\delta)^{v_y} \right\} \\ &\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) \left\{ \left[1 - \left(\tfrac{1}{2}\right)^y \right] h(x)^{k+1} + \left(\tfrac{1}{2}\right)^y h(x)^{k+1} \right\} \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda(x) h(x)^{k+1}. \end{aligned}$$

Therefore, the element e_y is less than or equal to the right-hand side of Eq. (5) for $z > 2$, while e_0 is equal to its left-hand side. The proof then follows as $\{e_l\}$ is a partially ordered set, i.e., $e_0 \le e_y$ by virtue of Lemma A.8.

■

A lower bound of the probability of no detection for (m, n) search strategies is now derived.

Theorem 6.3 (Lower Bound Estimate): For any (m, n) search strategies,

$$\tilde{G}(m,n) \geq \tilde{G}(1,n).$$

Proof: Recall that r, q and p are positive integers such that $m = pq$, $n = pr$, and $p = gcd(m,n)$. From the definition of $\tilde{G}(m,n)$,

$$\tilde{G}(1,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; g(x)g(x+\mu)\ldots g(x+(n-1)\mu).$$

Use the results from Lemmata 5.1 and 6.1 to write

$$\tilde{G}(1,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^{p} h(x+(k-1)\mu; m, r)\,,$$
$$\tilde{G}(m,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; h(x;m,r)^p\,.$$

Recall that $k \in \{1,\ldots,p\}$ and the definition of $\lambda_k(x)$. Lemma 6.2 states that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_k(x;m,r)h(x;m,r)^k \leq \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_{k+1}(x;m,r)h(x;m,r)^{k+1}.$$

Then,

$$\begin{aligned}\tilde{G}(1,n) &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_1(x;m,r)h(x;m,r) \leq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_2(x;m,r)h(x;m,r)^2 \leq \cdots \\ &\leq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_k(x;m,r)h(x;m,r)^k \leq \cdots \\ &\leq \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; \lambda_p(x;m,r)h(x;m,r)^p = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\; h(x;m,r)^p \\ &= \tilde{G}(m,n).\end{aligned}$$

■

Remark 6.1: Corollary 5.2 implies that all co-prime search strategies for a given n give the same probability of no detection.

7 An Analytical Example

In the previous section, a lower bound of the probability of no detection was estimated using (m, n) search strategies for a general $g(x)$. In this section, the probability of no detection is evaluated and the inequality of Theorem 6.2 is explicitly verified when $g(x) = \sin(x)^2$.

Lemma 7.1: Let $g(x) = \sin(x)^2$. Then

$$\tilde{G}(m,n) = \frac{2^p (2p-1)!!}{4^n p!}. \tag{6}$$

Proof: Recall that $j \in \{0, \dots, r-1\}$ and $k \in \{1, \dots, p\}$, as well as

$$\tilde{G}(m,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, h(x;m,r)^p = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^{p}\prod_{j=0}^{r-1} g(x + mj\mu)\,.$$

Then Lemma A.11 implies that

$$\tilde{G}(m,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^{p}\prod_{j=0}^{r-1} g\left(x + \frac{j\pi}{r}\right).$$

Letting $g(x) = \sin(x)^2$,

$$\tilde{G}(m,n) = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^{p}\prod_{j=0}^{r-1} \sin\left(x + \frac{j\pi}{r}\right)^2.$$

From Ref. [15], note that

$$\sin(rx) = 2^{r-1}\prod_{j=0}^{r-1}\sin\left(x + \frac{j\pi}{r}\right).$$

With this and since $n = pr$,

$$
\begin{aligned}
\tilde{G}(m,n) &= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \prod_{k=1}^{p} \frac{1}{4^{r-1}} \sin(rx)^2 \\
&= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx \; \frac{1}{4^{p(r-1)}} \sin(rx)^{2p} \\
&= \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} dx \; \frac{1}{4^{n-p}} \sin(rx)^{2p} .
\end{aligned}
$$

Carrying the integral by using the formula in Ref. [16] gives

$$
\tilde{G}(m,n) = \frac{1}{4^{n-p}} \frac{(2p-1)!!}{2p!!}.
$$

And Eq. (6) follows since $2p!! = 2^p p!$. ■

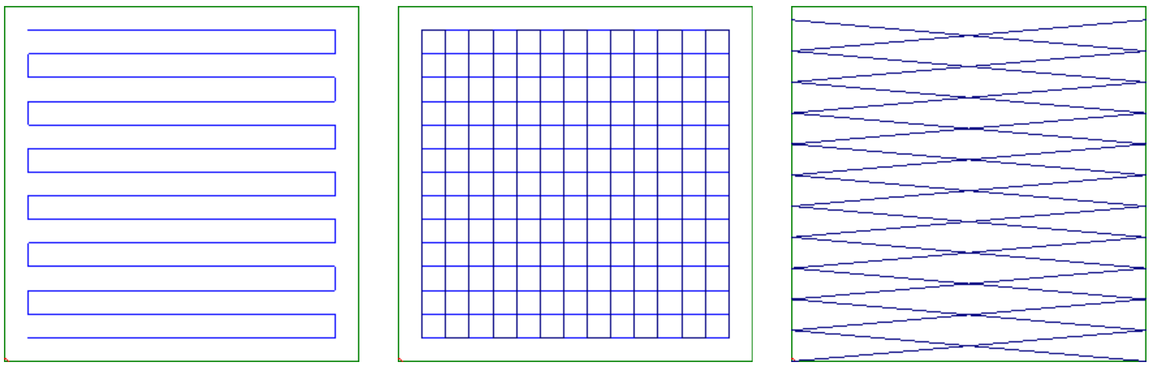
Corollary 7.2: For $p = \gcd(m,n) = 1$,

$$
\tilde{G}(m,n) = \tilde{G}(1,n) = \frac{2}{4^n}.
$$

And Theorem 6.3 follows for this case since $(2p-1)!! = 1 \times 3 \times \ldots \times (2p-1) \geq 1 \times 2 \times \ldots \times p = p!$.

8 A Numerical Example

In the previous section, the inequality of Theorem 6.2 was shown to hold for the probability of no detection evaluated with $g(x) = \sin(x)^2$. In this section, the probability of not detecting a single target is evaluated using an alternate $g(x)$ through simulation.



(a) One-pass Mowing *(b) Two-pass Mowing* *(c) Two-pass Zigzag*

Figure 6: Three Search Strategies.

Consider an area where a single mine is randomly distributed and searched for using one of the three strategies depicted in Figure 6: one-pass mowing (a); two-pass mowing (b); and two-pass zigzag (c). Next, assume that the angular dependence of the detection process is modeled using a Johnson function [5]. Then, the probability of no detection for a single target does not admit an analytical solution, and a numerical solution is sought using the Mine Inspection and Search Operation (MISO) tools.

MISO is a Monte Carlo Simulation developed to evaluate the effectiveness of Multiple Autonomous Underwater Vehicles (AUV) operations in mine defence [5]. Reader can refer to Ref. [5] for details about MISO. Figure 7 is a screenshot of MISO parameters setting for the one-pass mowing search.[3] For the two other search strategies, the # AUV is set to two.

To model the two-pass zigzag search, the Zigzag option is selected for the Sweep Pattern. However, since the distance between legs varies, targets may be imaged less than once when the distance between two legs is maximum (near the centre), or more than twice when the distance is minimal (near the edges) depending on the sensor's range selected. To provide some sense of the interplay between sensor range and coverage, the numerical results for the two-pass zigzag pattern were generated by varying the maximum range of the detection probability from 75 to 130. For the 75 setting, large central sections of the mine field are not covered, while for 130, the area is entirely covered but larger areas are imaged more than twice (which also occurs for the 75

[3] MISO version dated 26 April 2011 was used.

setting but for small areas). In all cases, 5000 Monte Carlo iterations were generated and the detection process was modeled to be independent of range.

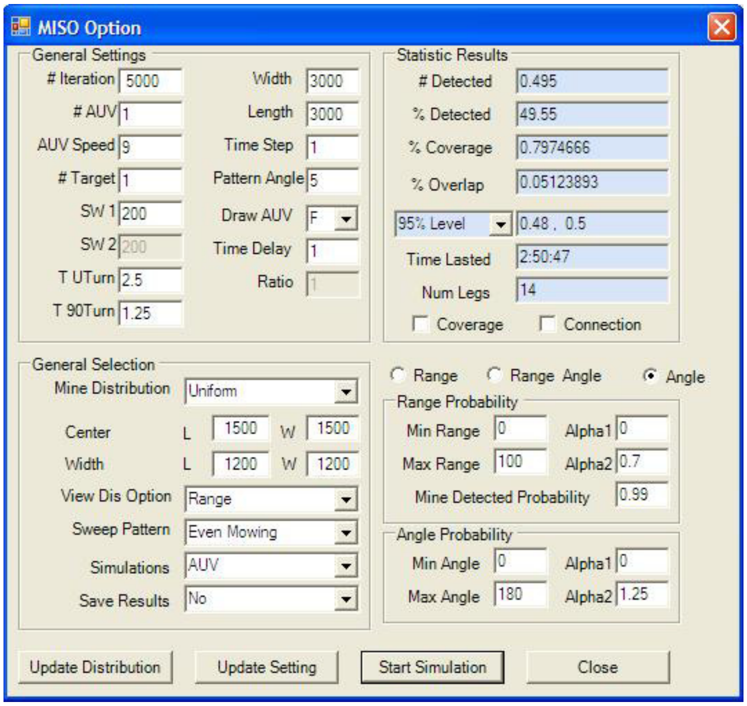


Figure 7: MISO Option Menu for one-pass mowing search.

Numerical results generated using MISO are reported in Table 1. Both two-pass search strategies are better than the one-pass mowing strategy, while the two-pass zigzag search strategy clearly underperforms compared to the two-pass mowing pattern. Note that even though the case described in Section 7 is an approximation of the one found here (for example overlap is not considered), the results for the first two entries compared well with the analytical results. There, observing a target once yields one chance out of two of not detecting targets, while observing a target twice at 90 degrees gives one chance out of eight.

Table 1: Probability of not detecting a single mine as a function of search strategies

	Search Strategies		
	One-pass Mowing	Two-pass Mowing	Two-pass Zigzag
Probability of no detection	~50%	~10%	~40-50%

9 Conclusion

In this paper, the angular dependence of the detection process which is often overlooked for search missions is explicitly accounted for by assuming that the target possesses rectangular symmetry. As a consequence of this approximate symmetry, the long side of a target is endowed with the largest cross section, which results in the highest probability of detection given that the target is observed only once. However, since the orientation of the target is in general unknown, there is likelihood that it will be imaged on the short side, i.e., the smallest cross-section. Therefore, the probability of not detecting the target may not be zero even if the search area is entirely covered. Making several observations of the target in order to increase the change of observing its long side is one way to address this problem.

Assuming that the observations are independent, a search strategy that corresponds to a critical point of the detection probability is to observe the target such that the separation between two consecutive observations is a constant and equal to a multiple of 180 degrees divided by the number of observations. The resulting tactic is simple, intuitive and robust (as no prior knowledge of the target orientation is required). For example, two observations separated by 90 degrees or three observations separated by 60 degrees will lead to a critical point of the probability of no detection.

From these search strategies leads, a lower bound of the probability of no detection is estimated. Work is currently underway to prove that the strategy $(1, n)$ leads to the global maximum of the detection probability.

A logical extension of this work is to relax the assumption that information provided by subsequent observations is uncorrelated (i.e., follows a Bernoulli process) as it entails that the probability of not detecting a target decreases with increasing number of observations even if the observations are co-linear. This is questionable as no additional information is gained.[4] Another avenue is to explore the next angle at which a target should be observed given a prior set of observations, i.e., the number of observations changes during the mission.

[4] This fact and its implications in underwater mine hunting was brought to the attention of the authors by A. Percival from Defence R&D Canada - Atlantic.

References

[1] Zerr B., Bovio, E. and Stage, B. (2001), Automatic Mine Classification Approach Based on AUV Manoeuvrability and COTS Side Scan Sonar, In *Proceedings of the Autonomous Underwater Vehicle and Ocean Modelling Networks: GOAT2 2000 Conference*, CP-46, 315-322, NATO Saclant Undersea Research Centre, La Spezia, Italy.

[2] Nguyen, B. and Hopkin, D (2005), Concepts of operations for the side scan sonar Autonomous Underwater Vehicles developed at DRDC Atlantic (U), (DRDC Atlantic TM 2005-213) Defence R&D Canada – Atlantic.

[3] Nguyen, B. and Hopkin, D. (2005), Modeling Autonomous Underwater Vehicle (AUV) Operations in Mine Hunting, In *Proceedings of Oceans 2005 – Europe Conference*, 533-538, Brest, France.

[4] Fawcett, J. A., Crawford, A., Hopkin, D., Myers, V., Couillard, M., and Zerr, B. (2008), Multi-Aspect Computer-Aided Classification of the Citadel Trial Side-Scan Sonar Images (U), (DRDC Atlantic TM 2008-029), Defence R&D Canada – Atlantic.

[5] Nguyen, B., Hopkin, D. and Yip, H. (2008), Autonomous Underwater Vehicles Conducting Mine Countermeasure Operations (U), (DRDC CORA TM 2008-42), Defence R&D Canada – CORA.

[6] Nguyen, B., Hopkin, D., and Yip, H. (2008), Autonomous Underwater Vehicles – A Transformation of Mine Counter-Measure Operations, *Defense & Security Analysis*, 24 (3), 247–266.

[7] Runkle P., Bharadwa, P. K., Couchman, L., and Carin, L. (1999), Hidden Markov Models for Multiaspect Target Classification, *IEEE Transactions on Signal Processing*, 47(7), 2035-2040.

[8] Robinson, M., Azimi-Sadjadi, M. R., and Salazar, J. (2005), Multi-Aspect Target Discrimination Using Hidden Markov Models and Neural Networks, *IEEE Transactions on Neural Networks*, 16(2), 447-459.

[9] Shihao, J. and Xuejun, L. (2005), Adaptive Multiaspect Target Classification and Detection with Hidden Markov Models, *IEEE Sensors Journal*, 5(5), 1035-1042.

[10] Martínez, S. and Bullo, F. (2006), Optimal Sensor Placement and Motion Coordination for Target Tracking, *Automatica*, 42(4), 661-668.

[11] Ash, J. N. and Moses, R. L. (2008), On Optimal anchor Node Placement in Sensor Localization by Optimization of Subspace Principal Angles, In *Proceedings of IEEE International Conference on Acoustics, Speech, and Signal Processing*, 2289-2292, Las Vegas, USA.

[12] Bishop, A. N. and Jemsfelt P. (2009), An Optimality Analysis of Sensor-Target Geometries for Signal Strength Based Lozalization, In *Proceedings of the 5th International Conference on Intelligent Sensors, Sensor Networks and Information Processing (ISSNIP'09)*, 127-132, Melbourne, Australia.

[13] Bishop, A. N., Fidan, B., Anderson, B. D. O., Doğançay, K., and Pathirana, P. N. (2010), Optimality Analysis of Sensor-Target Localization Geometries, *Automatica*, 46(3), 479-492.

[14] Koopman B. O. (1980), Law of Random Search, In *Search and Screening: General Principles with Historical Applications*, New York : Pergamon Press, p. 71-74.

[15] Gradshteyn, I. S. and Ryzhik, I. M. (1979), Table of Integrals, Series, and Products, 4^{th} ed. San Diego: Academic Press, p. 33.

[16] Zwillinger, D. (1996), Standard Mathematical Tables and Formulae, 30^{th} ed. Boca: CRC Press, p. 397.

This page intentionally left blank.

Annex A Supplementary Lemmata

Lemma A.1: Let $\omega \in \mathbb{R}$. Define the real function $h(x)$ such that

$$h(x) = h(x + \pi).$$

Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x - \omega) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x). \tag{7}$$

Proof: Let $x \to x + \omega$ on the left-hand side of Eq. (7) and break the resulting integration interval into two sub-intervals, namely $\left[-\frac{\pi}{2} - \omega, -\frac{\pi}{2}\right[$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2} - \omega\right[$. Then

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx h(x - \omega) &= \int_{-\frac{\pi}{2}-\omega}^{\frac{\pi}{2}-\omega} dx h(x) \\ &= \int_{-\frac{\pi}{2}-\omega}^{-\frac{\pi}{2}} dx h(x) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}-\omega} dx h(x)\,.\end{aligned}$$

Letting $x \to x - \pi$ in the first term of the last line and recalling that by assumption $h(x)$ is periodic, the integral becomes

$$\int_{\frac{\pi}{2}-\omega}^{\frac{\pi}{2}} dx h(x) + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}-\omega} dx h(x).$$

The sum of which is equal to the right-hand side of Eq. (7). ■

Remark A.1: Only the periodicity of the integrand was required to show that the integral is invariant under any constant shift of the integration variable.

Lemma A.2: Let $i \in \{0, \dots, n-1\}$, $\mathcal{N}_i = \{0, \dots, n-1\} \backslash \{i\}$ and $j \in \mathcal{N}_i$. Then $\sigma_i(j) = (2i - j) \bmod n$ is a bijection from $\mathcal{N}_i$ to itself and its own inverse.

Proof: Composition gives

$$\sigma_i(\sigma_i(j)) = (2i - \sigma_i(j)) \bmod n = (2i - (2i - j) \bmod n) \bmod n.$$

Use the definition of the modulo twice to write

$$\sigma_i(\sigma_i(j)) = \left(2i - (2i-j) + \left\lfloor\frac{(2i-j)}{n}\right\rfloor n\right) \bmod n$$
$$= \left(j + \left\lfloor\frac{(2i-j)}{n}\right\rfloor n\right) \bmod n$$
$$= j + \left\lfloor\frac{(2i-j)}{n}\right\rfloor n - \left\lfloor\frac{j + \left\lfloor\frac{(2i-j)}{n}\right\rfloor n}{n}\right\rfloor n.$$

Recall that $j \in \mathcal{N}_i$ and note that $\frac{j}{n} < n$. Then $\left\lfloor\frac{j}{n} + \left\lfloor\frac{(2i-j)}{n}\right\rfloor n\right\rfloor = \left\lfloor\frac{(2i-j)}{n}\right\rfloor n$ and $\sigma_i(\sigma_i(j)) = j$. ■

Lemma A.3: Let r, q be positive integers such that $\gcd(r, q) = 1$, i.e., r and q are co-primes. Let $i \in \{0, \dots, r-1\}$. Then the map $\sigma(i) = qi \bmod r$ is a bijection of the set of i to itself.

Proof: Proceed with a proof by contradiction. Assume this map is not a bijection. Then there exists a pair $u, v \in \{0, \dots, r-1\}$ such that $u \neq v$ and $qu \bmod r = qv \bmod r$. Next, assume that $u > v$ then $q(u-v) \bmod r = qw \bmod r = 0$ where $w = u - v$. This implies that $qw = ry$ with y a positive integer, as $w, q > 0$. Because q and r are co-primes, i.e., $\gcd(r, q) = 1$ then $w = rz$ with $z > 0$, which leads to a contraction as $w = u - v < r - 1$. ■

Lemma A.4: Let $u \in \{0, \dots a-1\}$, $v \in \{0, \dots, b-1\}$, and $w \in \{0, \dots, n-1\}$ where $n = ab$. Then $\sigma(u, v) = u + av$ and $\sigma^{-1}(w) = \left(w \bmod a, \left\lfloor\frac{w}{a}\right\rfloor\right)$ are bijections and inverse of each other.

Proof: Composition gives

$$\sigma(\sigma^{-1}(w)) = \sigma\left(w \bmod a, \left\lfloor\frac{w}{a}\right\rfloor\right) = w \bmod a + a\left\lfloor\frac{w}{a}\right\rfloor = w.$$

Where the last equality follows from the definition of the modulo operation. Similarly,

$$\sigma^{-1}(\sigma(u, v)) = \sigma^{-1}(u + av)$$
$$= \left((u + av) \bmod a, \left\lfloor\frac{u + av}{a}\right\rfloor\right)$$
$$= (u, v).$$

Where the last equality follows from the definition of the modulo operation and since $u < a$. Therefore, $\sigma(u, v)$ and $\sigma^{-1}(w)$ are both one-to-one, onto, and inverse of each other. ■

Lemma A.5:

$$h(x; m, r) = h(-x; m, r).$$

Proof: From the definition of $h(x;m,r)$,

$$h(x;m,r) = g(x)\prod_{j=1}^{r-1} g(x+mj\mu).$$

Because of the periodicity of $g(x)$ and $nq = mr$,

$$g(x+mj\mu) = g(x+mj\mu-nq\pi) = g(x+m(j-r)\mu)$$

Let $j \to -j + r$. Then

$$g(x)\prod_{j=1}^{r-1} g(x-mj\mu-y).$$

And the proof follows since by definition $g(x)$ is even. ■

Lemma A.6: Let a and b be positive integers. Then

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x;m,r)h(x+\delta_k;m,r)^{a+b} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x;m,r)h(x;m,r)^{a+b}.$$

Proof: For greater clarity, suppress m and r in the arguments of $\lambda_k(x;m,r)$ and $h(x;m,r)$. Consider the left-hand side of the equality. Lemma A.1 implies that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x+\delta_k)^{a+b} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x-\delta_k)h(x)^{a+b}$$

Next, let $x \to -x$. Then,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x-\delta_k)h(x)^{a+b} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(-x-\delta_k)h(-x)^{a+b}$$

The result follows since $h(x)$ is even by virtue of Lemma A.5, which implies that $h(-x)^{a+b} = h(x)^{a+b}$, and $\lambda_{k+1}(-x-\delta_k) = \lambda_{k+1}(x)$. The latter holds trivially for $k = p$ and follows for $k \in \{1, \dots, p-1\}$ from simple algebraic manipulations, namely

$$
\begin{aligned}
\lambda_{k+1}(-x-\delta_k) &= h(-x+\mu-(p-k)\mu)\dots h(-x+(p-k-1)\mu-(p-k)\mu)\\
&= h(-x-(p-k-1)\mu)\dots h(-x-\mu)\\
&= \lambda_{k+1}(x).
\end{aligned}
$$

■

Lemma A.7: Let a and b be positive integers. Then, m and r suppressed in the arguments of $\lambda_k(x;m,r)$ and $h(x;m,r)$,

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x)^a h(x+\delta_k)^b
$$

$$
\leq \begin{cases} \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)\{h(x)^{a+b}+h(x)^{a-b}h(x+\delta_k)^{2b}\}, & a\geq b \\ \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)\{h(x)^{a+b}+h(x)^{2a}h(x+\delta_k)^{b-a}\}, & a<b. \end{cases}
$$

Proof: Recall that $y^c z^c \leq \frac{1}{2}(y^{2c}+z^{2c})$ where $y,z\in\mathbb{R}^+$ and $c\in\ \mathbb{N}$. Then,

$$
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)h(x)^a h(x+\delta_k)^b
$$

$$
\leq \begin{cases} \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)\{h(x)^{a+b}+h(x)^{a-b}h(x+\delta_k)^{2b}\}, & a\geq b \\ \dfrac{1}{2}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx\, \lambda_{k+1}(x)\{h(x+\delta)^{a+b}+h(x)^{2a}h(x+\delta_k)^{b-a}\}, & a<b. \end{cases}
$$

For $a<b$, use Lemma A.6, which states that $h(x+\delta)^{a+b}$ can be replaced by $h(x)^{a+b}$ in the first term on the right-hand side of the inequality. ■

Lemma A.8: The set of e_l is partially ordered, i.e.

$$
e_0 \leq e_1 \leq \cdots \leq e_l \leq e_{l+1} \leq \cdots.
$$

Proof: Proceed by induction. Start with e_o. Then, Lemma A.7 implies that $e_0 \leq e_1$. Similarly, consider an arbitrary element and apply, again, Lemma A.7. Then, $e_l \leq e_{l+1}$. ■

Lemma A.9: Let u_0, v_0 be integers such that $v_0 > 0$ and $u_0+v_0 = 2^y z$ where $y\in\ \mathbb{N}$ and $z\in \mathbb{N}^*\backslash\{2\}$. Then

$$(u_l, v_l) = \begin{cases} (0, 2^y z), & 2^l v_0 \bmod 2^y z = 0 \\ (2^l u_0 \bmod 2^y z, 2^l v_0 \bmod 2^y z), & 2^l v_0 \bmod 2^y z > 0. \end{cases}$$

Proof: Proceed by induction. First, show that the base case holds. For $v_0 \bmod 2^y z = 0$, $v_0 = 2^y z$ since $0 < v_0 \leq 2^y z$. While for $v_0 \bmod 2^y z > 0$, $(u_0 \bmod 2^y z, v_0 \bmod 2^y z) = (u_0, v_0)$ since $0 < v_0 < 2^y z$. Next, consider the inductive step from $l-1$ to l. Start with the case where $2^{l-1} v_0 \bmod 2^y z = 0$. Then, because $2^{l-1} v_0$ is proportional to $2^y z$, the condition $2^l v_0 \bmod 2^y z = 0$ also holds. And, since $u_{l-1} < v_{l-1}$, the inductive step gives $(u_l, v_l) = (0, v_{l-1}) = (0, 2^y z)$. Continue with the second case, i.e., $2^{l-1} v_0 \bmod 2^y z > 0$. First, remark that $u_{l-1} + v_{l-1} = 2^y z$ trivially holds in the first case and is also true in the case at hand since $u_0 + v_0 = 2^y z$ implies that

$$\begin{aligned} u_{l-1} + v_{l-1} &= 2^{l-1} u_0 \bmod 2^y z + 2^{l-1} v_0 \bmod 2^y z \\ &= 2^{l-1}(2^y z - v_0) \bmod 2^y z + 2^{l-1} v_0 \bmod 2^y z \\ &= 2^{l-1}(-v_0) \bmod 2^y z + 2^{l-1} v_0 \bmod 2^y z. \end{aligned}$$

Which, since $2^{l-1} v_0$ is not proportional to $2^y z$, yields

$$u_{l-1} + v_{l-1} = -2^y z \left(\left\lfloor \frac{-2^{l-1} v_0}{2^y z} \right\rfloor + \left\lfloor \frac{2^{l-1} v_0}{2^y z} \right\rfloor \right) = 2^y z.$$

Then, because $v_{l-1} = 2^{l-1} v_0 \bmod 2^y z > 0$, $0 < u_{l-1}, v_{l-1} < 2^y z$. Contrary to the previous case, they are now three possible sub-cases: $u_{l-1} = v_{l-1}$, $u_{l-1} > v_{l-1}$ and $u_{l-1} < v_{l-1}$. First let $u_{l-1} = v_{l-1}$. Then,

$$\begin{aligned} v_{l-1} - u_{l-1} &= 2^{l-1} v_0 \bmod 2^y z - 2^{l-1} u_0 \bmod 2^y z \\ &= 2^{l-1}(v_0 - u_0) \bmod 2^y z \\ &= 2^{l-1}(2v_0 - v_0 - u_0) \bmod 2^y z \\ &= 2^l v_0 \bmod 2^y z = 0. \end{aligned}$$

And, since $u_{l-1} + v_{l-1} = 2^y z$, $u_{l-1} = v_{l-1} = 2^{y-1} z$, which leads to conclude that $(u_l, v_l) = (0, 2^y z)$. Next, let $u_{l-1} > v_{l-1}$.Then, the inductive rules entail that

$$\begin{aligned} u_l = u_{l-1} - v_{l-1} &= 2^{l-1} u_0 \bmod 2^y z - 2^{l-1} v_0 \bmod 2^y z \\ &= 2^{l-1}(u_0 - v_0) \bmod 2^y z \\ &= 2^{l-1}(2u_0 - u_0 - v_0) \bmod 2^y z \\ &= 2^l u_0 \bmod 2^y z. \end{aligned}$$

And, since $u_{l-1} > v_{l-1}$ and $u_{l-1} + v_{l-1} = 2^y z$ implies that $2^{y-1} z > v_{l-1}$ then

$$\begin{aligned} v_l = 2v_{l-1} &= 2(2^{l-1} v_0 \bmod 2^y z) \\ &= 2^l v_0 \bmod 2^y z. \end{aligned}$$

Finally, observe that this method also applies for $u_{l-1} < v_{l-1}$. ■

Lemma A.10: Let u_0, v_0 be integers such that $v_0 > 0$ and $u_0 + v_0 = 2^y z$ where $y \in \mathbb{N}$ and $z > 2$. If $2^l v_0 \bmod 2^y z > 0$ for all l, then

$$\left(u_y, v_y\right) = \left(u_{y+\varphi(z)}, v_{y+\varphi(z)}\right).$$

Proof: Let p be a positive integer. Then, Lemma A.8 states that $u_{y+p} = 2^{y+p} u_0 \bmod 2^y z$. Remark that since $2^p = \left\lfloor \frac{2^p}{z} \right\rfloor z + 2^p \bmod z$,

$$\begin{aligned} u_{y+p} &= 2^{y+p} u_0 \bmod 2^y z \\ &= [2^y u_0 (2^p \bmod z)] \bmod 2^y z. \end{aligned}$$

Let $p = \varphi(z)$ and recall that $\gcd(z, 2) = 1$. Then, $2^{\varphi(z)} \bmod z = 1$ holds and $u_{y+\varphi(z)} = u_y$ holds. This procedure leads to a similar conclusion for v_y. ■

Lemma A.11:

$$\prod_{j=0}^{r-1} g\left(x + \frac{mj\pi}{n}\right) = \prod_{j=0}^{r-1} g\left(x + \frac{j\pi}{r}\right).$$

Proof: Recall that r, q and p are positive integers such that $m = pq$, $n = pr$, and $p = \gcd(m, n)$. Then

$$g\left(x + \frac{mj\pi}{n}\right) = g\left(x + \frac{qj\pi}{r}\right).$$

Use the periodicity of $g(x)$ and the definition of the modulo to further write the left-hand side of the equality as

$$g\left(x + \frac{qj\pi}{r} - \left\lfloor \frac{qj}{r} \right\rfloor \pi\right) = g\left(x + (qj \bmod r)\frac{\pi}{r}\right).$$

Lemma A.3 implies that map $\sigma(j) = qj \bmod r$ is a bijection from the set of j to itself. ■

DOCUMENT CONTROL DATA

(Security classification of title, body of abstract and indexing annotation must be entered when the overall document is classified)

1. ORIGINATOR (The name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Centre sponsoring a contractor's report, or tasking agency, are entered in section 8.) Defence R&D Canada – CORA 101 Colonel By Drive Ottawa, Ontario K1A 0K2	2. SECURITY CLASSIFICATION (Overall security classification of the document including special warning terms if applicable.) UNCLASSIFIED (NON-CONTROLLED GOODS) DMC A REVIEW: GCEC June 2010

3. TITLE (The complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title.)

Search Strategies for Detecting Targets Exhibiting Rectangular Symmetry

4. AUTHORS (last name, followed by initials – ranks, titles, etc. not to be used)

Nguyen, B. U.; Bourque, A.

5. DATE OF PUBLICATION (Month and year of publication of document.) February 2012	6a. NO. OF PAGES (Total containing information, including Annexes, Appendices, etc.) 48	6b. NO. OF REFS (Total cited in document.) 16

7. DESCRIPTIVE NOTES (The category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

Technical Memorandum

8. SPONSORING ACTIVITY (The name of the department project office or laboratory sponsoring the research and development – include address.)

Defence R&D Canada – CORA
101 Colonel By Drive
Ottawa, Ontario K1A 0K2

9a. PROJECT OR GRANT NO. (If appropriate, the applicable research and development project or grant number under which the document was written. Please specify whether project or grant.) TIF 10bz04	9b. CONTRACT NO. (If appropriate, the applicable number under which the document was written.)
10a. ORIGINATOR'S DOCUMENT NUMBER (The official document number by which the document is identified by the originating activity. This number must be unique to this document.) DRDC CORA TM 2012-031	10b. OTHER DOCUMENT NO(s). (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

11. DOCUMENT AVAILABILITY (Any limitations on further dissemination of the document, other than those imposed by security classification.)

Unlimited

12. DOCUMENT ANNOUNCEMENT (Any limitation to the bibliographic announcement of this document. This will normally correspond to the Document Availability (11). However, where further distribution (beyond the audience specified in (11) is possible, a wider announcement audience may be selected.))

Unlimited

13. ABSTRACT (A brief and factual summary of the document. It may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall begin with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (S), (C), (R), or (U). It is not necessary to include here abstracts in both official languages unless the text is bilingual.)

In this paper, search strategies where targets are observed at several different angles are found and proven to be critical points. Targets are assumed to exhibit rectangular symmetry and to have uniformly-distributed orientation. By rectangular symmetry, it is meant that one side of a target is the mirror image of its opposite side. Finding critical points is generally an NP-hard problem. Fortunately, symmetry principles allow analytical and intuitive solutions to be found. One such search strategy consists of choosing n angles evenly separated on the half-circle and provides a lower bound estimate for the probability of not detecting targets. As no prior knowledge of the target orientation is required, such search strategies are also robust, a desirable feature in search and detection missions.

Dans ce présent document, des stratégies de recherche liées à l'observation de cibles selon plusieurs angles différents sont identifiées et démontrées être des points critiques. Il est présumé que les cibles ont une symétrie rectangulaire et que leur orientation est uniformément distribuée. Le terme symétrie rectangulaire signifie qu'un côté du plan qui divise la cible est l'image-miroir du côté opposé. La détermination des points critiques est généralement un problème de type NP. Heureusement, les principes de la symétrie permettent de trouver des solutions analytiques et intuitives. L'une de ces stratégies de recherche consiste à choisir n angles répartis de façon uniforme sur le demi-cercle et permet de déterminer une valeur limite inférieure quant à la probabilité de ne pas détecter les cibles.

14. KEYWORDS, DESCRIPTORS or IDENTIFIERS (Technically meaningful terms or short phrases that characterize a document and could be helpful in cataloguing the document. They should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location may also be included. If possible keywords should be selected from a published thesaurus, e.g. Thesaurus of Engineering and Scientific Terms (TEST) and that thesaurus identified. If it is not possible to select indexing terms which are Unclassified, the classification of each should be indicated as with the title.)

Optimization; Symmetry; Multi-Aspect; Search Strategy; Detection; Mine Hunting; Sensor Network; Sensor Placement;

Defence R&D Canada

Canada's Leader in Defence
and National Security
Science and Technology

R & D pour la défense Canada

Chef de file au Canada en matière
de science et de technologie pour
la défense et la sécurité nationale

DEFENCE R&D DÉFENSE

www.drdc-rddc.gc.ca