A Bayesian Method for Localization by Multistatic Active Sonar

Daniel J. Peters

Abstract— **The question of localizing a target with multistatic active sonar is re-examined from the perspective of finding a peak in a probability distribution function. The probability distribution function is constructed using straightforward Bayesian principles. Both a position estimate and a covariance matrix can be found, provided that an implementation of a numerical algorithm for finding a local maximum is available. The localization method developed herein can account for transmitter and receiver location errors, sound speed errors, time errors, and bearing errors. A Monte Carlo test is conducted in order to compare the accuracy of the proposed method to that of a more conventional method used as a baseline. In each iteration, a transmitter, several receivers, and a target are positioned randomly within a square region, and the target is localized by both methods. The proposed method is generally more accurate than the baseline method, within the range of parameters considered here. The degree of improvement over the baseline is greater with a larger region area, with a larger bearing measurement error, and with a smaller time-of-arrival measurement error, and slightly greater with a larger number of receivers.**

Index Terms—**Localization, multistatic active sonar, sensor fusion**

I. INTRODUCTION

ACTIVE sonar systems, by definition, include at least one acoustic source (transmitter) and at least one acoustic sensor (receiver). Signals from the transmitter(s) will be reflected off various objects, including targets of interest, and some of these echoes will in general be detected by the receiver(s). Such a system is

Manuscript received June 5, 2015; revised September 16, 2015 and February 8, 2016.
The author is with Defence Research and Development Canada – Atlantic Research Centre, PO Box 1012, Dartmouth, NS, B2Y 3Z7, Canada (e-mail: dan.peters@drdc-rddc.gc.ca).

said to be "multistatic" if it has more than one transmitter or more than one receiver. An active sonar system with only one transmitter and one receiver is said to be "bistatic" if the transmitter and receiver are separated, and "monostatic" if the transmitter and the receiver are co-located. The systems considered in this paper have a single transmitter but possibly multiple receivers.

It is assumed here that each receiver measures both the time of arrival (TOA) and the direction of arrival (DOA) of each received echo. One of the tasks of the system is to localize a target by means of these measurements. This paper is concerned with the localization of a target based on a single signal ("ping") from the transmitter, given the assumption that each of N receivers is detecting the echo of that ping off a given target of interest.

In reality, there may be many reflective objects, creating many echoes for a given ping. With multiple receivers, the question arises of how to determine whether a given echo at one receiver corresponds to the same target as a given echo at another receiver. This association question is certainly an important one, but it lies beyond the scope of this paper. The reader is invited to imagine either that the association question has been resolved in some way, or that some set of echoes, one at each receiver, has been grouped together for the sake of argument. In the latter case, we are concerned with where the alleged target would be, *if* the chosen grouping were correct.

In many sonar applications, the depths of the receivers, the transmitter, and even the target are small in comparison to the horizontal displacements between any two of these objects. Hence, the problem can be treated as two-dimensional. So it is here.

We also neglect the possible spatial variation of the speed of sound. In addition, the local speed of sound in the region of interest is not assumed to be known with perfect precision.

The elapsed time from the production of a ping to the arrival of the echo at a given receiver constrains the target to lie on an ellipse with the transmitter at one focus and the receiver at the other, as illustrated in figure 1. The difference in arrival times between one receiver and another constrains the target to lie on a hyperbola with the two receivers as foci, as illustrated in figure 2. Of course, these constraints are only approximate when the ellipses or the hyperbolae are computed from measured values, because of the errors in the measurements. The DOA measurement of an echo at any given receiver provides another approximate constraint, with the target lying on or near the bearing ray corresponding to that measurement. The intersection between any two of these curves – ellipses, hyperbolae, or bearing rays – is called a cross-fix.

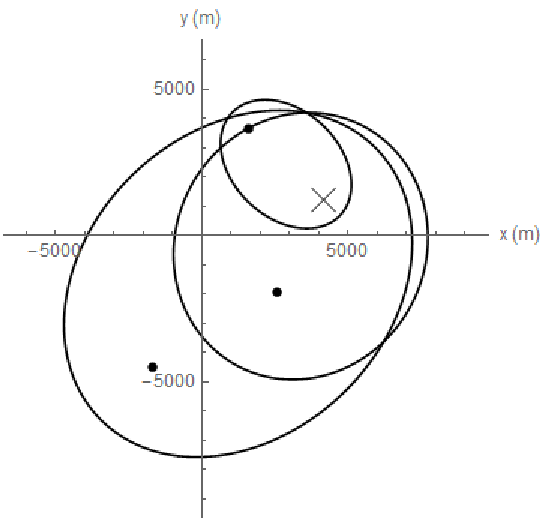


Fig. 1. An ellipse for each receiver. In each case, the receiver (a dot) is at one focus and the transmitter (an "X") is at the other.

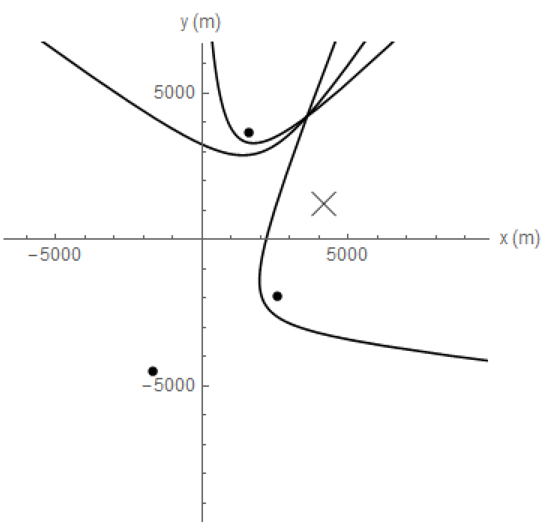


Fig. 2. One branch of a hyperbola for each pair of receivers. In each case, the two receivers are at the two foci. The geometry is the same as in figure 1.

Most existing methods for localization by multistatic active sonar are based on these cross-fixes. Cross-fixes involving only ellipses and hyperbolae are assumed in the analyses of localization errors arising from transmitter and receiver position uncertainties by McIntyre *et al.* [1], Sandys-Wunsch and Hazen [2], and

Blouin [3]. In the method presented by Coraluppi [4], the cross-fix between the ellipse and the bearing ray is found for each receiver, and a covariance matrix is computed based on a small error assumption. These estimates, one for each receiver, are then fused straightforwardly as independent measurements. Kim *et al.* [5] use a least-squares formula that is ultimately based also on the cross-fixes between the ellipse and the bearing ray for each receiver. Simakov [6] uses a Wiener filter combination of a variety of cross-fixes. In contrast to the methods listed above, some methods such as that presented by Wang *et al.* [7] are based on the received signal strength (RSS), requiring a model of propagation loss.

The purpose of this paper is to present a Bayesian localization method, and to compare its performance to that of an existing method used as a baseline. The proposed method requires a numerical search for a peak in a probability distribution function. That probability distribution function is given in (12) as a (2N+4)-fold integral (for N receivers), with the terms of the integrand given in (13)-(15). In order to make the numerical maximization practicable, the integral is evaluated approximately into a closed-form expression given in (16)-(18).

The proposed method makes use of the TOA and DOA information, but not the RSS, and therefore is most naturally compared to the existing methods that are based on cross-fixes. In this paper, Coraluppi's [4] method is taken as the baseline for performance comparison. It was deemed to be the most suitable for that role because it makes use of the full range of uncertainties considered here, as listed below.

The following quantities are assumed to be either measured or known *a priori*, within certain known error statistics, all assumed to be Gaussian and mutually uncorrelated:

- the local speed of sound in water, c, assumed to be constant over the region of interest, with uncertainty σ_c;
- the position of the transmitter, $\mathbf{x}_0 = (x_0, y_0)$, with uncertainty σ_x in each dimension (we assume no correlation between the two dimensions);
- the position of each receiver, $\mathbf{x}_k = (x_k, y_k)$ for k from 1 to N, with uncertainty σ_x in each dimension (again assuming no correlation between the two dimensions);

- the ping time t (at the transmitter), with uncertainty σ_t;
- the time-of-arrival (TOA) of the echo at each receiver, T_k for k from 1 to N, with uncertainty σ_T; and
- the direction-of-arrival (DOA) of the echo at each receiver, θ_k for k from 1 to N, with uncertainty σ_θ (each direction defined as counterclockwise from east).

The baseline method is summarized in section II. The proposed method is described in section III. The two methods are compared in section IV via a Monte Carlo study. Further discussion appears in section V.

II. Baseline Method

Our baseline for comparison is the method described by Coraluppi [4] for multistatic active sonar localization. All the equations in this section are based on that reference, with some changes in notation.

The fused estimate $\mathbf{X}_{\mathrm{BL}}$ for the target's position and the corresponding covariance matrix $\mathbf{P}_{\mathrm{BL}}$ are given by

$$\mathbf{P}_{\mathrm{BL}} = \left(\sum_k \mathbf{P}_k^{-1}\right)^{-1}$$

$$\mathbf{X}_{\mathrm{BL}} = \mathbf{P}_{\mathrm{BL}} \sum_k \mathbf{P}_k^{-1}\mathbf{X}_k \tag{1}$$

where $\mathbf{X}_k$ and $\mathbf{P}_k$ are the position estimate and its covariance matrix based on the information at the k^{th} receiver, for k from 1 to N. In the case of two receivers, (1) above is equivalent to (39) in [4] together with the un-numbered equation just before (40). For more than two receivers, (1) is the natural generalization thereof.

Let the bearing difference between the transmitter and the target, with respect to the k^{th} receiver, be denoted α_k (see figure 3), so that

$$\cos(\alpha_k + \theta_k) = \frac{x_0 - x_k}{\Delta_k}$$

$$\sin(\alpha_k + \theta_k) = \frac{y_0 - y_k}{\Delta_k} \tag{2}$$

where

$$\Delta_k = \sqrt{(x_0 - x_k)^2 + (y_0 - y_k)^2}. \tag{3}$$

Equation (2) should be regarded as a definition of α_k. In the case where $x_0 > x_k$, this means that

$$\alpha_k = \arctan\left(\frac{y_0 - y_k}{x_0 - x_k}\right) - \theta_k. \tag{4}$$

The position estimate $\mathbf{X}_k$ is given by

$$\mathbf{X}_k = \begin{pmatrix} X_k \\ Y_k \end{pmatrix} = \begin{pmatrix} x_k + r_k\cos\theta_k \\ y_k + r_k\sin\theta_k \end{pmatrix} \tag{5}$$

where

$$\begin{aligned} r_k &= a_k/b_k \\ a_k &= c^2\tau_k^2 - \Delta_k^2 \\ b_k &= 2(c\tau_k - \Delta_k\cos\alpha_k) \\ \tau_k &= T_k - t. \end{aligned} \tag{6}$$

The covariance

$$\mathbf{P}_k = \begin{pmatrix} \sigma_{X_k}^2 & \sigma_{X_kY_k} \\ \sigma_{X_kY_k} & \sigma_{Y_k}^2 \end{pmatrix} \tag{7}$$

is given by

$$\begin{aligned} \sigma_{X_k}^2 &= \sigma_x^2 + \sigma_{r_k}^2\cos^2\theta_k + r_k^2\sigma_\theta^2\sin^2\theta_k + 2\sigma_{x_kr_k}\cos\theta_k - 2r_k\sigma_{\theta_kr_k}\sin\theta_k\cos\theta_k \\ \sigma_{Y_k}^2 &= \sigma_x^2 + \sigma_{r_k}^2\sin^2\theta_k + r_k^2\sigma_\theta^2\cos^2\theta_k + 2\sigma_{y_kr_k}\sin\theta_k + 2r_k\sigma_{\theta_kr_k}\sin\theta_k\cos\theta_k \\ \sigma_{X_kY_k} &= \sigma_{x_kr_k}\sin\theta_k + \sigma_{y_kr_k}\cos\theta_k + \left(\sigma_{r_k}^2 - r_k^2\sigma_\theta^2\right)\sin\theta_k\cos\theta_k \\ &\quad + r_k\sigma_{\theta_kr_k}(\cos^2\theta_k - \sin^2\theta_k) \end{aligned} \tag{8}$$

where

$$\sigma_{\theta_kr_k} = \frac{2a_k\Delta_k}{b_k^2}\sigma_\theta^2\sin\alpha_k$$

$$\sigma_{x_k r_k} = \left[2\left(\frac{1}{b_k} - \frac{a_k \cos\alpha_k}{b_k^2 \Delta_k}\right)(x_0 - x_k) - \frac{2a_k \sin\alpha_k}{b_k^2 \Delta_k}(y_0 - y_k)\right]\sigma_x^2$$

$$\sigma_{y_k r_k} = \left[2\left(\frac{1}{b_k} - \frac{a_k \cos\alpha_k}{b_k^2 \Delta_k}\right)(y_0 - y_k) + \frac{2a_k \sin\alpha_k}{b_k^2 \Delta_k}(x_0 - x_k)\right]\sigma_x^2$$

$$\sigma_{r_k}^2 = \frac{b_k^2 \sigma_{a_k}^2 + a_k^2 \sigma_{b_k}^2 - 2a_k b_k \sigma_{a_k b_k}}{b_k^4}$$

$$\sigma_{a_k}^2 = 4(c^2 \tau_k^4 \sigma_c^2 + c^4 \tau_k^2 (\sigma_t^2 + \sigma_T^2) + 2\Delta_k^2 \sigma_x^2)$$

$$\sigma_{b_k}^2 = 4\left(\tau_k^2 \sigma_c^2 + c^2 (\sigma_t^2 + \sigma_T^2) + 2\sigma_x^2 + \Delta_k^2 \sigma_\theta^2 \sin^2\alpha_k\right)$$

$$\sigma_{a_k b_k} = 4(c\tau_k^3 \sigma_c^2 + c^3 \tau_k (\sigma_t^2 + \sigma_T^2) + 2\Delta \sigma_x^2 \cos\alpha_k). \tag{9}$$

These formulae are simplified slightly from those in [4] because of the assumption of no correlation between the two dimensions in the transmitter and receiver positions, as well as the use of a common uncertainty value for each dimension in the transmitter and receiver positions.

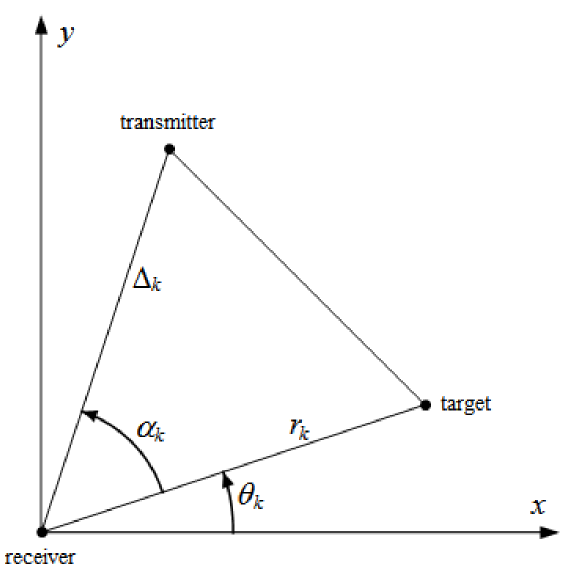


Fig. 3. Illustration of α_k, θ_k, Δ_k, and r_k.

III. Bayesian Method

Let **X** be a (two dimensional vector) position variable referring to the unknown position of the target. The method proposed here for localization of the target is to find the value of **X** that maximizes the probability density $P(\mathbf{X}|t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c)$, i.e. the probability density for **X** conditioned on the measurements. In order to construct such a probability distribution function, an assumption must be made for the prior probability distribution function $P(\mathbf{X})$. We take this prior to be minimally informative, hence uniform and unbounded, so that the posterior probability density is proportional to the corresponding likelihood density:

$$P(\mathbf{X}|t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c) \propto P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c|\mathbf{X}). \tag{10}$$

(More precisely, $P(\mathbf{X})$ is taken to be uniform within a square of side L, and we get (10) in the limit as L goes to infinity.) The likelihood density for **X** in (10) is the probability density for the measurements, under the assumption that **X** is the true target position. It can be expressed as

$$P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c|\mathbf{X}) \\ = \int \dots \int P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c|\mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) P(\tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) d\tilde{t}\, d\tilde{\mathbf{x}}_0 d\tilde{\mathbf{x}}_1 \dots d\tilde{\mathbf{x}}_N d\tilde{c} \tag{11}$$

where $\tilde{t}$, $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{x}}_k$, and $\tilde{c}$ are the assumed true values corresponding respectively to the measurements t, $\mathbf{x}_0$, $\mathbf{x}_k$, and c, and **X** is taken to be independent of those variables. Note that the assumed true values corresponding similarly to T_k and θ_k are constrained by $\tilde{t}$, $\tilde{\mathbf{x}}_0$, $\tilde{\mathbf{x}}_k$, and $\tilde{c}$, together with the target position **X**; therefore we do not introduce them as additional variables of integration. As we did with **X**, we take a uniform and unbounded prior for each of these integration variables (but see also the discussion in section V). Thus we have

$$P(\mathbf{X}|t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c) \\ \propto \int \dots \int P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c|\mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \{\tilde{\mathbf{x}}_k\}, \tilde{c}) d\tilde{t}\, d\tilde{\mathbf{x}}_0 d\tilde{\mathbf{x}}_1 \dots d\tilde{\mathbf{x}}_N d\tilde{c} \tag{12}$$

and the integrand is given (up to a constant factor) by

$$P(t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c})$$
$$\propto \exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2}+\frac{(c-\tilde{c})^2}{\sigma_c^2}+\frac{(x_0-\tilde{x}_0)^2}{\sigma_x^2}+\frac{(y_0-\tilde{y}_0)^2}{\sigma_x^2}\right.\right.$$
$$\left.\left.+\sum_k\left(\frac{(x_k-\tilde{x}_k)^2}{\sigma_x^2}+\frac{(y_k-\tilde{y}_k)^2}{\sigma_x^2}+\frac{\left(\theta_k-\tilde{\theta}_k\right)^2}{\sigma_\theta^2}+\frac{\left(T_k-\tilde{T}_k\right)^2}{\sigma_T^2}\right)\right]\right) \tag{13}$$

where $\tilde{T}_k$ is the arrival time derived from the assumed target position $\mathbf{X}=(X \quad Y)^{\mathrm{T}}$ as well as from the other assumed quantities:

$$\tilde{T}_k=\tilde{t}+\frac{\tilde{S}_0+\tilde{S}_k}{\tilde{c}}$$
$$\tilde{S}_0=\sqrt{(\tilde{x}_0-X)^2+(\tilde{y}_0-Y)^2}$$
$$\tilde{S}_k=\sqrt{(\tilde{x}_k-X)^2+(\tilde{y}_k-Y)^2} \tag{14}$$

and similarly $\tilde{\theta}_k$ is the bearing derived from the assumed target position $\mathbf{X}$ as well as from the other assumed quantities:

$$\sin\tilde{\theta}_k=\frac{Y-\tilde{y}_k}{\tilde{S}_k}$$
$$\cos\tilde{\theta}_k=\frac{X-\tilde{x}_k}{\tilde{S}_k}$$
$$\left|\theta_k-\tilde{\theta}_k\right|\le\pi. \tag{15}$$

It is desirable to find a closed-form expression for the integral in (12), so that the process of numerical peak-finding can be made practicable. An approximate expression is available, based on the assumption that the distance from the target to the nearest receiver or transmitter is large in comparison to the uncertainty in the receiver or transmitter positions. See the appendix for details. Aside from a constant factor, the integral evaluates approximately to

$$P(\mathbf{X}|t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c) \propto \frac{1}{\sqrt{A}}\exp\left(-C+\frac{B^2}{4A}\right) \tag{16}$$

where

$$\begin{aligned}
A = {} & \frac{1}{2\sigma_c^2} - \frac{\sigma_T^2\beta\mu}{2\alpha^3} - \frac{2c\sigma_T^2\chi}{\alpha^2} + \frac{\upsilon}{2\alpha} + \frac{N\sigma_x^2(3\alpha-\beta)}{4c^2\alpha^2} \\
& - \frac{N\sigma_x^2(\sigma_T^2-\sigma_t^2)}{2\alpha^2\gamma'^2}\left[(N+1)\sigma_x^4 - \sigma_x^2c^2\left((N+2)\sigma_T^2+N\sigma_t^2\right)\right. \\
& \left. - 3c^4\sigma_T^2(\sigma_T^2+N\sigma_t^2)\right] \\
& - \frac{1}{2\alpha^3\gamma^3}[\sigma_T^2\sigma_x^2\zeta\lambda^2 + \sigma_T^2\alpha^3\delta S_0(NS_0-2\lambda) + \sigma_x^2\alpha^2\gamma^2\tau^2 \\
& - 4c\sigma_T^2\alpha\gamma\tau(\alpha^2S_0+\sigma_x^2(\alpha+\gamma)\lambda)] \\
& - \frac{1}{2\gamma^3\gamma'^3}[\sigma_t^2\kappa(\lambda-NS_0)^2 - 2c(\sigma_T^2+N\sigma_t^2)\gamma^3(2\gamma'+\delta')(\lambda-NS_0)t \\
& + 4c\sigma_t^2\gamma\gamma'\eta(\lambda-NS_0)\tau - N(N+1)\sigma_x^2\gamma^3\delta't^2 + \gamma^3\gamma'(\gamma'+\delta')t\tau \\
& + c^2\sigma_t^2\gamma^2\gamma'^2\tau^2]
\end{aligned}$$

$$\begin{aligned}
B = {} & \frac{\chi}{\alpha} - \frac{c\sigma_T^2\mu}{\alpha^2} - \frac{N\sigma_x^2}{c\alpha} - \frac{Nc\sigma_x^2(\sigma_T^2-\sigma_t^2)}{\alpha\gamma'} \\
& - \frac{(\alpha S_0+\sigma_x^2\lambda)}{\alpha^2\gamma^2}\left(\alpha\gamma\tau + c\sigma_T^2(N\alpha S_0-(\alpha+\gamma)\lambda)\right) \\
& - \frac{\left(\gamma t - c\sigma_t^2(\lambda-NS_0)\right)}{\gamma^2\gamma'^2}(\eta(\lambda-NS_0)+c\gamma\gamma'\tau - N(N+1)c\sigma_x^2\gamma t)
\end{aligned}$$

$$\begin{aligned}
C = {} & \frac{1}{2}\sum_k\left[\frac{S_k^2\left(\theta_k-\breve{\theta}_k\right)^2}{\sigma_x^2+S_k^2\sigma_\theta^2} + \ln\left(1+\frac{\sigma_x^2}{S_k^2\sigma_\theta^2}\right)\right] + \frac{t^2}{2\sigma_t^2} + \frac{S_0^2}{2\sigma_x^2} + \frac{\mu}{2\alpha} - \frac{(\alpha S_0+\sigma_x^2\lambda)^2}{2\sigma_x^2\alpha\gamma} \\
& - \frac{\left(\gamma t + c\sigma_t^2(\lambda-NS_0)\right)^2}{2\sigma_t^2\gamma\gamma'}
\end{aligned} \tag{17}$$

and

$$\alpha = \sigma_x^2 + c^2\sigma_T^2$$

$$\beta = \sigma_x^2 - 3c^2\sigma_T^2$$

$$\gamma = (N+1)\sigma_x^2 + c^2\sigma_T^2$$

$$\gamma' = \gamma + Nc^2\sigma_t^2$$

$$\delta = (N+1)\sigma_x^2 - 3c^2\sigma_T^2$$

$$\delta' = \delta - 3Nc^2\sigma_t^2$$

$$\eta = (N+1)^2\sigma_x^4 - c^4\sigma_T^2(\sigma_T^2 + N\sigma_t^2)$$

$$\zeta = -(N+1)(N+2)\sigma_x^6 + 3(N^2+2N+2)\sigma_x^4c^2\sigma_T^2 + 9(N+2)\sigma_x^2c^4\sigma_T^4 + 10c^6\sigma_T^6$$

$$\kappa = (N+1)^4\sigma_x^8 - 3(N+1)^3\sigma_x^6c^2(2\sigma_T^2 + N\sigma_t^2) - 12(N+1)^2\sigma_x^4c^4\sigma_T^2(\sigma_T^2 + N\sigma_t^2) - (N+1)\sigma_x^2c^6\sigma_T^2(\sigma_T^2 + N\sigma_t^2)(2\sigma_T^2 + N\sigma_t^2) + 3c^8\sigma_T^4(\sigma_T^2 + N\sigma_t^2)^2$$

$$\tau = \sum_k T_k$$

$$\upsilon = \sum_k T_k^2$$

$$\lambda = \sum_k (cT_k - S_k)$$

$$\mu = \sum_k (cT_k - S_k)^2$$

$$\chi = \sum_k T_k(cT_k - S_k)$$

$$S_0 = \sqrt{(x_0 - X)^2 + (y_0 - Y)^2}$$

$$S_k = \sqrt{(x_k - X)^2 + (y_k - Y)^2}$$

$$\cos\breve{\theta}_k = \frac{X - x_k}{S_k}$$

$$\sin\breve{\theta}_k = \frac{Y - y_k}{S_k}$$

$$\left|\theta_k - \breve{\theta}_k\right| \leq \pi. \tag{18}$$

Localization of the target then consists of finding the peak in the expression $\exp(-C + B^2/4A)/\sqrt{A}$, or equivalently the peak in the expression $-C + B^2/4A - \frac{1}{2}\ln A$, considered as a function of $\mathbf{X}$. Let $\mathbf{X}_{\text{Bayes}}$ denote the position estimate derived in such a fashion. In the implementation used in the Monte Carlo study of the next section, local maximization was used, via the "FindMaximum" function in *Mathematica*, with $\mathbf{X}_{\text{BL}}$ used as the initial estimate.

To find the covariance matrix $\mathbf{P}_{\text{Bayes}}$, we assume that the probability distribution function is approximately Gaussian in the near vicinity of the peak, so that

$$P(\mathbf{X}|t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c) \approx K\exp\left[-\tfrac{1}{2}\left(\mathbf{X}-\mathbf{X}_{\text{Bayes}}\right)^{\text{T}}\mathbf{P}_{\text{Bayes}}^{-1}\left(\mathbf{X}-\mathbf{X}_{\text{Bayes}}\right)\right] \tag{19}$$

for some constant *K*, as long as $\left\|\mathbf{X}-\mathbf{X}_{\text{Bayes}}\right\|$ is not much greater than the width of the peak. Let *D* be a length that is characteristic of the size of the peak. In the implementation used in the next section, the value used was $D = \sqrt{\frac{1}{2}\text{trace}(\mathbf{P}_{\text{BL}})}$. If $\mathbf{e}$ is a two-dimensional unit vector, it follows from (19) that

$$\frac{P\left(\mathbf{X}=\mathbf{X}_{\text{Bayes}}+D\mathbf{e}\middle|t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c\right)}{P\left(\mathbf{X}=\mathbf{X}_{\text{Bayes}}\middle|t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c\right)} = \exp\left[-\frac{D^2}{2}\mathbf{e}^{\text{T}}\mathbf{P}_{\text{Bayes}}^{-1}\mathbf{e}\right]. \tag{20}$$

Again using a numerical local maximization algorithm, we find the maximum value of the probability distribution function restricted to a circle of radius *D* centered on the peak position $\mathbf{X}=\mathbf{X}_{\text{Bayes}}$. In other words, we find the unit vector $\mathbf{e}$ for which the quantity in (20) is maximized. Let this unit vector be denoted $\mathbf{e}_1$, and let $\mathbf{e}_2$ be a unit vector that is orthogonal to $\mathbf{e}_1$. Then the quantity in (20) is minimized at $\mathbf{e}=\mathbf{e}_2$. Moreover, $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of $\mathbf{P}_{\text{Bayes}}$. Let σ_i^2 denote the eigenvalue of $\mathbf{P}_{\text{Bayes}}$ corresponding to the eigenvector $\mathbf{e}_i$, for *i* = 1 or 2. Then we have

$$\frac{P\left(\mathbf{X} = \mathbf{X}_{\text{Bayes}} + D\mathbf{e}_i \middle| t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c\right)}{P\left(\mathbf{X} = \mathbf{X}_{\text{Bayes}} \middle| t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c\right)} = \exp\left[-\frac{D^2}{2\sigma_i^2}\right], \tag{21}$$

or in other words

$$\sigma_i^2 = \frac{D^2}{2\ln\left(\frac{P\left(\mathbf{X} = \mathbf{X}_{\text{Bayes}} \middle| t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c\right)}{P\left(\mathbf{X} = \mathbf{X}_{\text{Bayes}} + D\mathbf{e}_i \middle| t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c\right)}\right)} \tag{22}$$

for i = 1 or 2. The covariance matrix is then given by

$$\mathbf{P}_{\text{Bayes}} = \mathbf{R}\begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix}\mathbf{R}^{-1} \tag{23}$$

for some rotation matrix $\mathbf{R}$. But $\mathbf{e}^{\mathrm{T}}\mathbf{P}_{\text{Bayes}}\mathbf{e} = \sigma_i^2$ for i = 1 or 2, so $\mathbf{e}_1^{\mathrm{T}}\mathbf{R} = (1 \quad 0)$ and $\mathbf{e}_2^{\mathrm{T}}\mathbf{R} = (0 \quad 1)$, and therefore $\mathbf{R}$ is the rotation matrix whose i^{th} column is equal to $\mathbf{e}_i$, for i = 1 or 2.

IV. Monte Carlo Study

The first comparison used 100,000 iterations, each iteration having its own geometry. For each iteration, one transmitter, five receivers, and one target were each placed randomly (and independently, with a uniform distribution) within a square region of sea, 10 km on each side. The true times of arrival of the echoes at the receivers were computed for a ping at a time coordinate of 0 s and a true sound speed of 1500 m/s. Measurements were generated randomly with Gaussian errors centered on the true values. The sigmas were set to:

- $\sigma_x = 20$ m for each dimension of the transmitter position and each dimension of each receiver position;
- $\sigma_T = 0.1$ s for the TOA of the echo at each receiver;
- $\sigma_t = 0.005$ s for the ping time;
- $\sigma_\theta = 0.02$ radians for the DOA of the echo at each receiver;
- $\sigma_c = 7.5$ m/s for the local speed of sound.

The target was localized, both by the baseline method and by the proposed method. Absolute and relative errors in localization were computed as follows:

$$E_{\text{abs,BL}} = \|\mathbf{X}_{\text{BL}} - \mathbf{X}_{\text{true}}\|$$

$$E_{\text{abs,Bayes}} = \|\mathbf{X}_{\text{Bayes}} - \mathbf{X}_{\text{true}}\|$$

$$E_{\text{rel,BL}} = \sqrt{(\mathbf{X}_{\text{BL}} - \mathbf{X}_{\text{true}})^{\text{T}}\mathbf{P}_{\text{BL}}^{-1}(\mathbf{X}_{\text{BL}} - \mathbf{X}_{\text{true}})}$$

$$E_{\text{rel,Bayes}} = \sqrt{\left(\mathbf{X}_{\text{Bayes}} - \mathbf{X}_{\text{true}}\right)^{\text{T}}\mathbf{P}_{\text{Bayes}}^{-1}\left(\mathbf{X}_{\text{Bayes}} - \mathbf{X}_{\text{true}}\right)}. \tag{24}$$

The RMS values, over the 100,000 iterations, of these four metrics were respectively 150 m, 67 m, 3.0, and 1.4, showing that the proposed method has a clear advantage over the baseline method.

One of the assumptions made in reaching the result given in (16)-(18) is that $S_0 \gg \sigma_x$ and that $S_k \gg \sigma_x$ for all k. A second and a third comparison were made in which this assumption is deliberately violated. In the second comparison, the true target position is randomly generated from a Gaussian distribution, centered on one of the receivers, with a standard deviation of σ_x in each dimension. Otherwise, the second comparison was performed in the same way as the first. The third comparison is the same as the second, except that the Gaussian distribution from which the true target position was generated was centered on the transmitter. The results appear in table 1.

TABLE I
EFFECT OF VIOLATING THE ASSUMPTIONS $S_0 \gg \sigma_x$ AND $S_k \gg \sigma_x$

Target placement	RMS absolute error (baseline)	RMS absolute error (Bayesian)	RMS relative error (baseline)	RMS relative error (Bayesian)
Independent, uniform	150 m	67 m	3.0	1.4
Near receiver	130 m	55 m	3.1	1.6
Near transmitter	74 m	79 m	1.8	1.4

Additional comparisons were made, all performed in the same way as the first comparison except that one parameter was varied. The results appear in figure 4. In figure 4a, the variable parameter is the size of the square region. The side of the square is varied from 1 km to 20 km in steps of 1 km. In figure 4b, the variable parameter is the number of receivers, which is varied from 1 to 20 in steps of 1. In figures 4c through 4g, the variable parameters are the sigmas: σ_x varies from 5 m to 100 m in steps of 5 m (figure 4c), σ_T varies from 0.02 s to 0.4 s in steps of 0.02 s (figure 4d), σ_t varies from 0.001 s to 0.02 s in steps of 0.001 s (figure 4e), σ_θ varies from 0.005 radian to 0.1 radian in steps of 0.005 radian (figure 4f), and σ_c varies from 1 m/s to 20 m/s in steps of 1 m/s (figure 4g). The RMS absolute error in meters is shown for each method in the left-hand column, while the RMS relative error for each method is shown in the right-hand column. In each case the solid line represents the baseline method and the dashed line represents the Bayesian method. In each of these plots, each point represents 100,000 iterations.

The RMS relative error in the proposed method is consistently close to the ideal value $\sqrt{2}$ [8], with a few exceptions, visible in figures 4a, 4b, and 4g. Aside from the bistatic case (figure 4b), these exceptions reflect the influence of a small number of outliers, the exclusion of which would bring the error down to the ideal value. In the worst case, where the side of the square region is 19 km in figure 4a, the removal of a single outlier suffices to bring the RMS relative error below 1.5.

This empirical study was implemented in *Mathematica*, with the "FindMaximum" function being used for the local maximization required by the proposed method. The baseline estimate was used as the initial search point. A constraint was imposed on the two-dimensional search, such that the resulting estimate was not permitted to be more than 50 km from the initial search point in either dimension. There were a few occasions in which the quantity A (see (17)) reached negative values in the process of searching for a peak in $-C + B^2/4A - \frac{1}{2}\ln A$, making the method nonsensical. The solution adopted here was simply to fall back on the baseline method whenever the latter problem arose. This problem happened in one iteration when the side of the square region was 20 km (figure 4a), 51 iterations in the bistatic case (a single receiver, figure 4b), and in respectively 2, 5, 30, 54, and 118 iterations in the cases of σ_c = 16 m/s, 17 m/s, 18 m/s, 19 m/s, and 20 m/s (figure 4g).

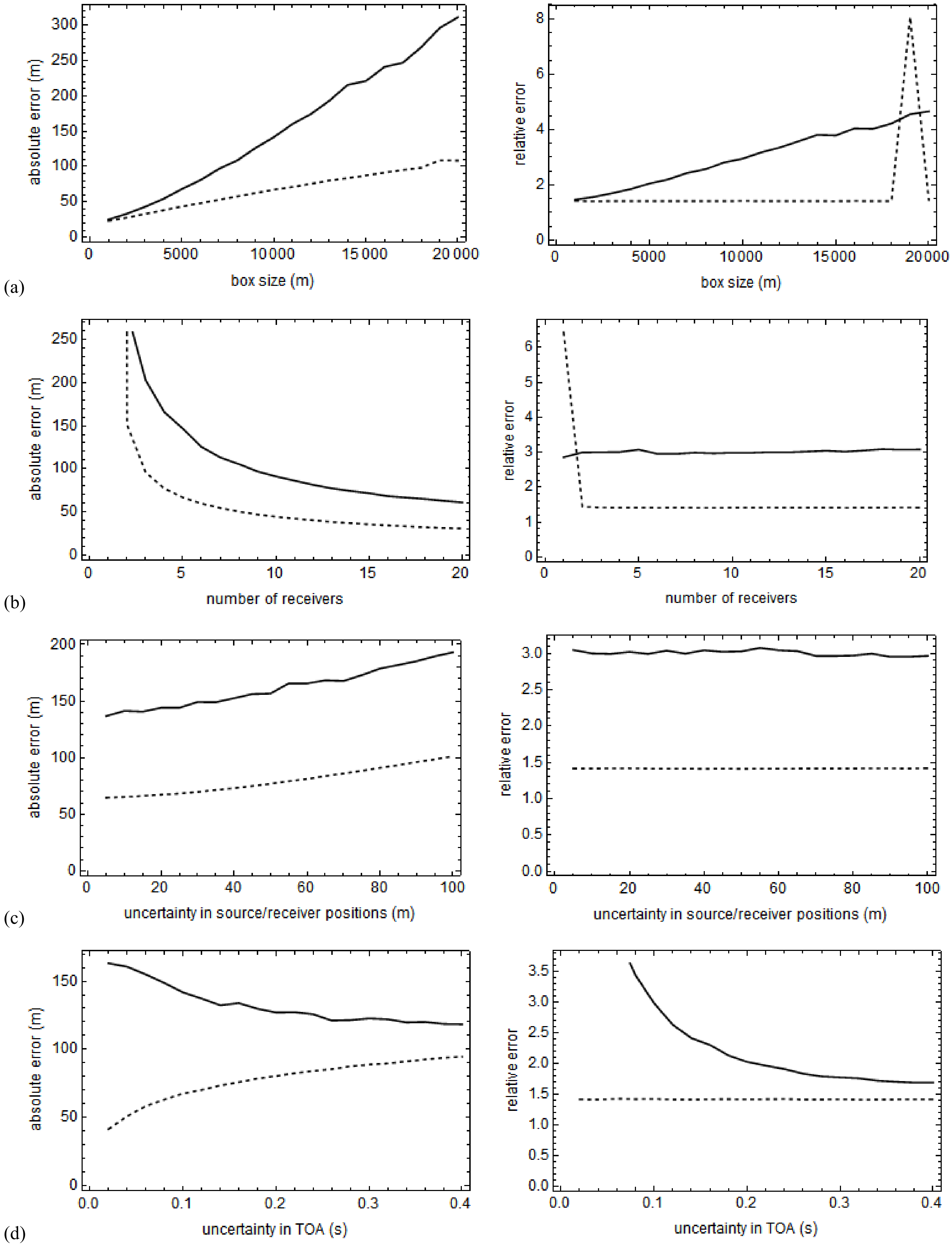
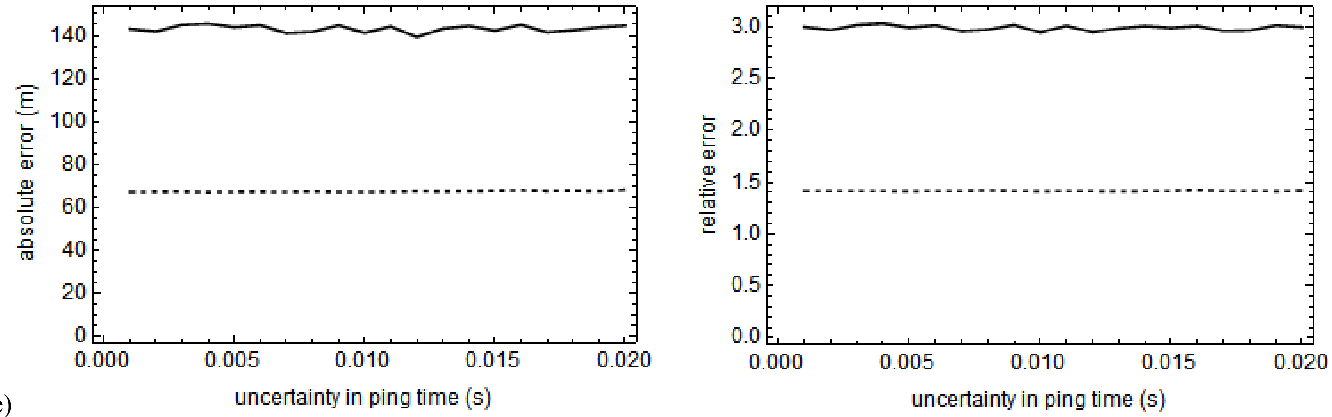
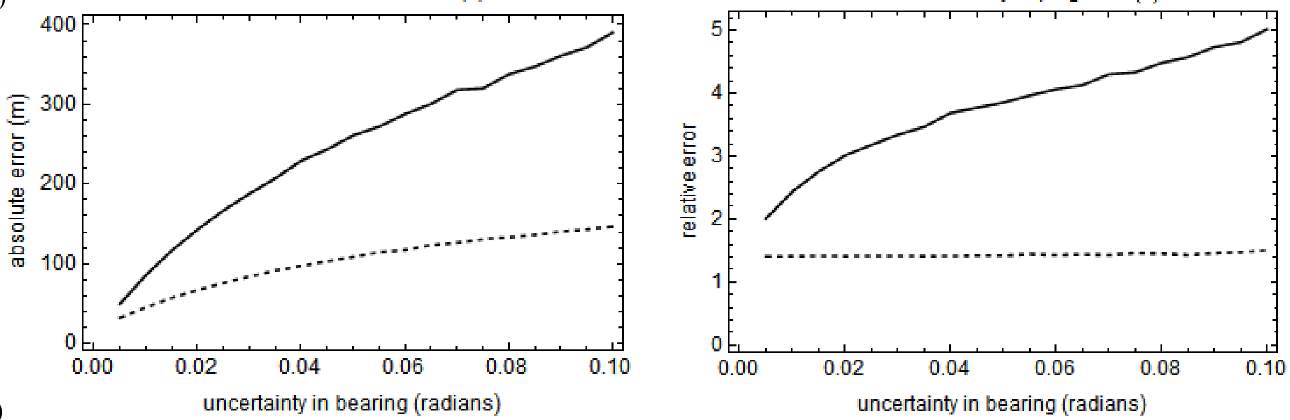


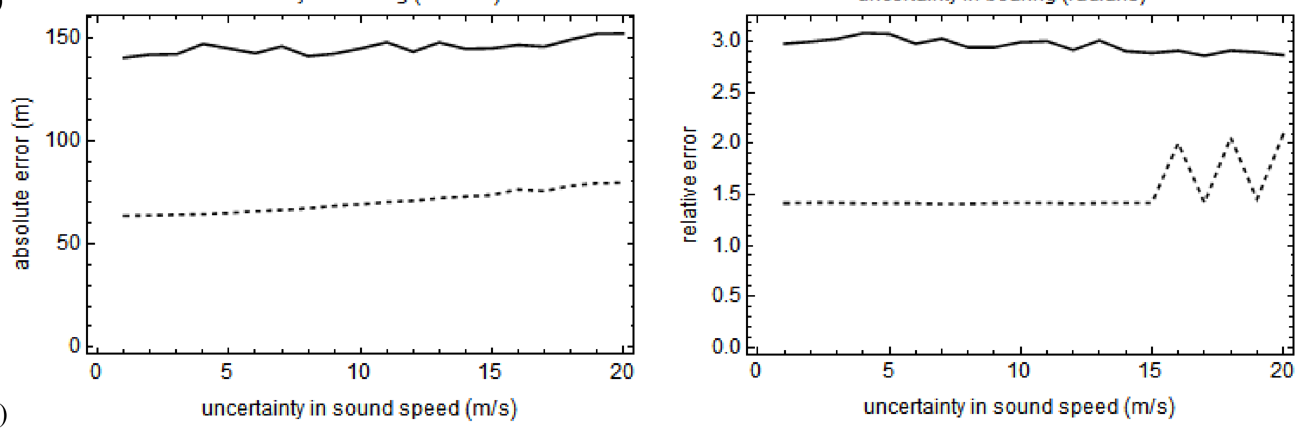
Fig. 4. RMS errors as a function of various parameters. The solid line represents the baseline method, and the dashed line represents the Bayesian method.



(e)



(f)



(g)

Fig. 4 (continued). RMS errors as a function of various parameters. The solid line represents the baseline method, and the dashed line represents the Bayesian method.

V. Discussion and Conclusions

We have presented a Bayesian method for localization by multistatic active sonar. The proposed method can account for transmitter and receiver location errors, errors in the local estimate of the speed of sound, and errors in the measurements of both time and bearing. Monte Carlo tests show a performance advantage of the proposed method over a more conventional localization method used as a baseline.

In order to account for the advantage of the proposed method over the baseline method, notice that each bistatic (single receiver) result in the baseline method is summarized by a point estimate and a covariance matrix, before those results are combined via (1) into a final estimate. Thus, there is a loss of information at an intermediate stage. No such discarding of information takes place in the proposed Bayesian method.

However, the proposed method has the disadvantage that it relies on numerical methods to find the peak in a probability distribution function. In order to make this process practicable, the function in question was approximated by a closed-form expression (see (16)-(18)). Even so, this process can take tens of milliseconds on a typical personal computer.

The main simplifying assumption used here for the integral of (12) is that $S_0 \gg \sigma_x$ and that $S_k \gg \sigma_x$ for all k, which is to say that the distance from the target to the nearest receiver or transmitter is large in comparison to the uncertainty in the receiver or transmitter positions. We investigated what would happen if this assumption were systematically violated, by running comparisons in which the true target position is placed close to the transmitter or to one of the receivers. The proposed method still compares mostly favorably to the baseline method, with only a small disadvantage in the RMS absolute error in the case where the target is close to the transmitter.

In spite of the overall favorable performance, there were a few cases where the approximations used to derive (16)-(18) led to problems. There were a few iterations in which the quantity A (see (17)) became negative, causing the peak-finding procedure to fail, as already mentioned in section IV. In addition, there were four outliers, which are manifested as the peak in the RMS relative error in figure 4a and the three peaks in the RMS relative error in figure 4g. Removal of these four outliers would be sufficient to flatten those two figures, giving the Bayesian method an RMS relative error of about 1.4 throughout the range of parameters chosen.

The degree of advantage of the proposed method over the baseline method, in the results of section IV, is greater with a larger region area, with a larger bearing measurement error, and with a smaller time measurement error, and slightly greater with a larger number of receivers, but it does not seem to depend on the transmitter/receiver position errors or on the error in the ping time. Nor does it seem to depend on the sound speed error, except that higher values of this error bring instability to the proposed method, as described in section IV.

The use of a uniform unbounded prior for $\tilde{c}$ in (11)-(12) is questionable. In the absence of a local measurement, we will still know that the speed of sound in water is somewhere in the rough vicinity of 1500 m/s, and obviously a zero or negative value is impossible. However, the form of the integrand ensures that only values of $\tilde{c}$ that are close to the estimate c will contribute.

The assumptions here (see the introduction) could be generalized. In particular, we could have different position uncertainties for each receiver and for the transmitter, along with correlations between the uncertainty in x and the uncertainty in y in each case; different uncertainties for each time measurement; and different uncertainties for each DOA measurement. The integrand given in (13) would be modified straightforwardly:

$$
\begin{aligned}
&P(t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c|\mathbf{X},\tilde{t},\tilde{\mathbf{x}}_0,\{\tilde{\mathbf{x}}_k\},\tilde{c}) \\
&\qquad \propto \exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2}+\frac{(c-\tilde{c})^2}{\sigma_c^2}+(\mathbf{x}_0-\tilde{\mathbf{x}}_0)^{\mathrm{T}}\boldsymbol{\Sigma}_0^{-1}(\mathbf{x}_0-\tilde{\mathbf{x}}_0)\right.\right. \\
&\qquad\qquad \left.\left.+\sum_k\left((\mathbf{x}_k-\tilde{\mathbf{x}}_k)^{\mathrm{T}}\boldsymbol{\Sigma}_k^{-1}(\mathbf{x}_k-\tilde{\mathbf{x}}_k)+\frac{\left(\theta_k-\tilde{\theta}_k\right)^2}{\sigma_{\theta_k}^2}+\frac{\left(T_k-\tilde{T}_k\right)^2}{\sigma_{T_k}^2}\right)\right]\right)
\end{aligned}
\tag{25}
$$

where the uncertainty in the k^{th} TOA measurement is given by σ_{T_k}, the uncertainty in the k^{th} DOA measurement is given by σ_{θ_k}, the (two dimensional) uncertainty in the position of the k^{th} receiver is expressed by the covariance matrix $\boldsymbol{\Sigma}_k$, and similarly the (two dimensional) uncertainty in the position of the transmitter is expressed by the covariance matrix $\boldsymbol{\Sigma}_0$. The corresponding integral (12) could still be evaluated with the same simplifying assumptions that were used here, but the result will be much lengthier than that of (16)-(18).

APPENDIX

We evaluate (approximately) the integral in (12), with the integrand given explicitly in (13). We first make the transformation

$$\begin{pmatrix} \tilde{x}'_k \\ \tilde{y}'_k \end{pmatrix} = \begin{pmatrix} -\cos\breve{\theta}_k & -\sin\breve{\theta}_k \\ \sin\breve{\theta}_k & -\cos\breve{\theta}_k \end{pmatrix} \begin{pmatrix} \tilde{x}_k - X \\ \tilde{y}_k - Y \end{pmatrix}$$

$$\begin{pmatrix} x'_k \\ y'_k \end{pmatrix} = \begin{pmatrix} -\cos\breve{\theta}_k & -\sin\breve{\theta}_k \\ \sin\breve{\theta}_k & -\cos\breve{\theta}_k \end{pmatrix} \begin{pmatrix} x_k - X \\ y_k - Y \end{pmatrix} \tag{26}$$

where $\breve{\theta}_k$ is given by the last three lines of (18). Thus $x'_k = S_k$ and $y'_k = 0$, and we can substitute an integral over $\tilde{x}'_k$ and $\tilde{y}'_k$ for the integral over $\tilde{x}_k$ and $\tilde{y}_k$. If we write

$$\tilde{\theta}_k = \breve{\theta}_k + \varepsilon \tag{27}$$

then to first order in ε we have $\tilde{x}'_k \approx \tilde{S}_k$ and $\tilde{y}'_k \approx \varepsilon\tilde{S}_k \approx \varepsilon S_k$. So the summand in (13) can be rewritten with the substitutions

$$(x_k - \tilde{x}_k)^2 + (y_k - \tilde{y}_k)^2 = (S_k - \tilde{x}'_k)^2 + \tilde{y}'^2_k$$

$$\left(\theta_k - \tilde{\theta}_k\right)^2 = \left(\theta_k - \breve{\theta}_k - \frac{\tilde{y}'_k}{S_k}\right)^2$$

$$\left(T_k - \tilde{T}_k\right)^2 = \left(T_k - \tilde{t} - \frac{\tilde{S}_0}{\tilde{c}} - \frac{\tilde{x}'_k}{\tilde{c}}\right)^2 \tag{28}$$

whence the 2*N* integrals over the $\tilde{x}'_k$ and $\tilde{y}'_k$ become simply Gaussian, and they result in

$$P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c | \mathbf{X}, \tilde{t}, \tilde{\mathbf{x}}_0, \tilde{c})$$

$$\propto \left(\frac{\tilde{c}^2\sigma_T^2\sigma_x^2}{\sigma_x^2 + \tilde{c}^2\sigma_T^2}\right)^{\frac{N}{2}} \left(\prod_k \sqrt{\frac{S_k^2\sigma_\theta^2\sigma_x^2}{\sigma_x^2 + S_k^2\sigma_\theta^2}}\right) \exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2} + \frac{(c-\tilde{c})^2}{\sigma_c^2} + \frac{(x_0-\tilde{x}_0)^2}{\sigma_x^2} + \frac{(y_0-\tilde{y}_0)^2}{\sigma_x^2} + \sum_k\left(\frac{S_k^2\left(\theta_k - \breve{\theta}_k\right)^2}{\sigma_x^2 + S_k^2\sigma_\theta^2} + \frac{\left(\tilde{c}(T_k - \tilde{t}) - \tilde{S}_0 - S_k\right)^2}{\sigma_x^2 + \tilde{c}^2\sigma_T^2}\right)\right]\right). \tag{29}$$

The integration over $\tilde{x}_0$ and $\tilde{y}_0$ is handled similarly: These variables are transformed (by a rotation and a translation) to $\tilde{x}'_0$ and $\tilde{y}'_0$ such that $\tilde{x}'_0 \approx \tilde{S}_0$ and $(x_0 - \tilde{x}_0)^2 + (y_0 - \tilde{y}_0)^2 = (S_0 - \tilde{x}'_0)^2 + \tilde{y}'^2_0$, and the integral results in

$$P(t, \{T_k\}, \{\theta_k\}, \mathbf{x}_0, \{\mathbf{x}_k\}, c | \mathbf{X}, \tilde{t}, \tilde{c})$$
$$\propto \sqrt{\frac{\sigma_x^2 + \tilde{c}^2\sigma_T^2}{(N+1)\sigma_x^2 + \tilde{c}^2\sigma_T^2}} \left(\frac{\tilde{c}^2\sigma_T^2\sigma_x^2}{\sigma_x^2 + \tilde{c}^2\sigma_T^2}\right)^{\frac{N}{2}} \left(\prod_k \sqrt{\frac{S_k^2\sigma_\theta^2\sigma_x^2}{\sigma_x^2 + S_k^2\sigma_\theta^2}}\right) \exp\left(-\tfrac{1}{2}\left[\frac{(t-\tilde{t})^2}{\sigma_t^2} + \frac{(c-\tilde{c})^2}{\sigma_c^2}\right.\right.$$
$$+ \frac{S_0^2}{\sigma_x^2}$$
$$+ \sum_k \left(\frac{S_k^2\left(\theta_k - \breve{\theta}_k\right)^2}{\sigma_x^2 + S_k^2\sigma_\theta^2} + \frac{(\tilde{c}(T_k - \tilde{t}) - S_k)^2}{\sigma_x^2 + \tilde{c}^2\sigma_T^2}\right)$$
$$\left.\left. - \frac{\sigma_x^2(\sigma_x^2 + \tilde{c}^2\sigma_T^2)}{(N+1)\sigma_x^2 + \tilde{c}^2\sigma_T^2}\left(\frac{S_0}{\sigma_x^2} + \frac{\sum_k(\tilde{c}(T_k - \tilde{t}) - S_k)}{\sigma_x^2 + \tilde{c}^2\sigma_T^2}\right)^2\right]\right). \tag{30}$$

The integration over $\tilde{t}$ is straightforward, as the integrand (30) is already Gaussian in that variable. Thus we get

$$P(t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c|\mathbf{X},\tilde{c})$$

$$\propto \frac{1}{\sqrt{1+N\left(\frac{\sigma_x^2+\tilde{c}^2\sigma_t^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)}}\left(\frac{\tilde{c}^2\sigma_T^2\sigma_x^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^{\frac{N}{2}}\left(\prod_k\sqrt{\frac{S_k^2\sigma_\theta^2\sigma_x^2}{\sigma_x^2+S_k^2\sigma_\theta^2}}\right)\exp\left(-\tfrac{1}{2}\left[\frac{t^2}{\sigma_t^2}\right.\right.$$

$$+\frac{(c-\tilde{c})^2}{\sigma_c^2}+\frac{S_0^2}{\sigma_x^2}$$

$$+\sum_k\left(\frac{S_k^2\left(\theta_k-\breve{\theta}_k\right)^2}{\sigma_x^2+S_k^2\sigma_\theta^2}+\frac{(\tilde{c}T_k-S_k)^2}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)$$

$$-\frac{\sigma_x^2(\sigma_x^2+\tilde{c}^2\sigma_T^2)}{(N+1)\sigma_x^2+\tilde{c}^2\sigma_T^2}\left(\frac{S_0}{\sigma_x^2}+\frac{\sum_k(\tilde{c}T_k-S_k)}{\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^2$$

$$\left.\left.-\left(\frac{\sigma_t^2((N+1)\sigma_x^2+\tilde{c}^2\sigma_T^2)}{(N+1)\sigma_x^2+\tilde{c}^2(\sigma_T^2+N\sigma_t^2)}\right)\left(\frac{t}{\sigma_t^2}+\frac{\tilde{c}(\sum_k(\tilde{c}T_k-S_k)-NS_0)}{(N+1)\sigma_x^2+\tilde{c}^2\sigma_T^2}\right)^2\right]\right). \tag{31}$$

The final integration is over $\tilde{c}$. Given the term $(c-\tilde{c})^2/\sigma_c^2$ in the argument of the exponential, most of the contribution to this integral will be in the near vicinity of *c*. This constraint suggests that a low-order expansion in powers of $(\tilde{c}-c)$ should suffice. The expression in (31) expands to

$$P(t,\{T_k\},\{\theta_k\},\mathbf{x}_0,\{\mathbf{x}_k\},c|\mathbf{X},\tilde{c}) \propto \exp(-C-B(\tilde{c}-c)-A(\tilde{c}-c)^2+O(\tilde{c}-c)^3) \tag{32}$$

where *A* and *B* and *C* are given by (17) and (18). Cutting off the expansion at second order, then integrating over $\tilde{c}$, we get the result (16).

REFERENCES

[1] M. McIntyre, J. Wang, and L. Kelly, "The effect of position uncertainty in multistatic acoustic localization," in *Proc. Conf. Inf. Decision Control*, Adelaide, Australia, Feb. 1999, pp. 347–352.

[2] M. Sandys-Wunsch and M. G. Hazen, "Multistatic localization error due to receiver positioning errors," *IEEE J. Ocean. Eng.*, vol. 27, no.2, pp. 328–334, Apr. 2002.

[3] S. Blouin, "Multistatic localization error bound due to transducers' location uncertainties," Defence R&D Canada, Scientific Report DRDC-RDDC-2014-R19, May 2014.

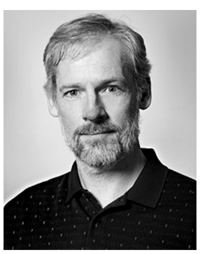
[4] S. Coraluppi, "Multistatic sonar localization," *IEEE J. Ocean. Eng.*, vol. 31, no. 4, pp. 964–974, Oct. 2006.

[5] S. Kim, B. Ku, W. Hong, and H. Ko, "Performance comparison of target localization for active sonar systems," *IEEE Trans. Aerospace Electronic Systems*, vol. 44, no. 4, pp. 1371–1380, Oct. 2008.

[6] S. Simakov, "Localization in airborne multistatic sonars," *IEEE J. Ocean. Eng.*, vol. 33, no. 3, pp. 278–288, July 2008.

[7] B. Wang, Y. Li, H. Huang, and C. Zhang, "Target Localization in underwater acoustic sensor networks," in *Proc. Conf. on Image and Signal Processing*, Sanya, China, May 2008, pp. 68–72.

[8] Y. Bar-Shalom, P. K. Willett, and X. Tian, *Tracking and Data Fusion: A Handbook of Algorithms*. Storrs, CT: YBS Publishing, 2011, pp. 94–95.



Daniel J. Peters received the Ph.D. degree in physics from the University of British Columbia, Vancouver, BC, in 1994. Since 1999, he has been with Defence Research and Development Canada – Atlantic Research Centre, in Dartmouth, NS.