Signal Detection Using Compressed Samples

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I. Introduction

We study the performance of a signal detector that uses only compressed samples of data. Let the *known* signal $\mathbf{x}_{16\times1}$ be a vector of 16 elements and $\mathbf{X}_{16\times1} = \mathbf{D}_{16\times16}\mathbf{x}$ be the transform of $\mathbf{x}$, $\mathbf{D}$ the transformation matrix, and $\mathbf{X}$ is sparse, containing only 4 non-zero elements. We construct a matrix $\mathbf{A}$ so that $\mathbf{A}_{4\times16}\mathbf{X}$ will contain only those 4 non-zero elements. For example

$$\mathbf{X} = [4 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0]^T$$

and

$$\mathbf{A} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \end{bmatrix}$$

so that

$$\mathbf{AX} = [4\ 2\ 3\ 2]^T$$

Let the data vector be

$$\mathbf{y}/s = \mathbf{x} + \mathbf{w} \quad \text{signal present}$$

$$\mathbf{y}/o = \mathbf{w} \quad \text{no signal}$$

where $\mathbf{w}$ is a vector containing zero mean Gaussian random variables as elements.

For the standard matched filter, the detection variable is

$$\text{MF: } \Lambda = \mathbf{y}^T \mathbf{x}$$

In compressive sensing, we do not use $\mathbf{y}$, but a reduced dimension of $\mathbf{y}$, i.e.,

$$\mathbf{r}_{4\times1} = \mathbf{R}_{4\times16}\mathbf{y}_{16\times1}$$

and the detection statistic is

$$\text{CS: } \gamma = \mathbf{r}^T{}_{1\times4}\mathbf{A}_{4\times16}\mathbf{X}_{16\times1}$$

II. Comparison between MF and CS Detection

How do we choose $\mathbf{R}$?

In compressive sensing literature, $\mathbf{R}$ is a full rank matrix containing random elements. But for detection, we want

$$\mathbf{r} = \mathbf{R}\mathbf{y} = \mathbf{R}(\mathbf{x} + \mathbf{w}) = \mathbf{R}\mathbf{x} + \mathbf{R}\mathbf{w}$$

i.e., we want

$$\mathbf{R}\mathbf{x} = \mathbf{A}\mathbf{X}$$

or

$$\mathbf{R}\mathbf{D}^{-1}\mathbf{X} = \mathbf{A}\mathbf{X}$$

so that

$$\mathbf{R} = \mathbf{A}\mathbf{D}$$

Then when the signal is present

$$\begin{aligned}\mathbf{r}/s &= \mathbf{ADy} = \mathbf{AD}(\mathbf{x}+\mathbf{w}) \\ &= \mathbf{ADx} + \mathbf{ADw} \\ &= \mathbf{AX} + \mathbf{AW}\end{aligned}$$

where

$$\mathbf{W} = \mathbf{Dw}.$$

Hence

$$\gamma / s = \mathbf{r}^T \mathbf{AX} = \mathbf{X}^T \mathbf{A}^T \mathbf{AX} + \mathbf{W}^T \mathbf{A}^T \mathbf{AX} = \mathbf{X}^T \mathbf{X} + \mathbf{W}^T \mathbf{X}$$

Also

$$\begin{aligned}\Lambda / s &= \mathbf{y}^T \mathbf{x} = (\mathbf{x}^T + \mathbf{w}^T)\mathbf{x} \\ &= \left[(\mathbf{D}^{-1}\mathbf{X})^T + (\mathbf{D}^{-1}\mathbf{W})^T\right]\mathbf{D}^{-1}\mathbf{X} \\ &= \mathbf{X}^T (\mathbf{D}^{-1})^T \mathbf{D}^{-1}\mathbf{X} + \mathbf{W}^T (\mathbf{D}^{-1})^T \mathbf{D}^{-1}\mathbf{X}\end{aligned}$$

Now if $\mathbf{D}$ is orthogonal, i.e.,

$$\mathbf{D}^T = \mathbf{D}^{-1}$$

Then

$$\Lambda = \gamma$$

That is CS detection = MF detection.

Note that if $\mathbf{D}$ is the Fourier Transform Matrix, then $\mathbf{D}^T = \mathbf{D}^{-1}$ and since

$$\mathbf{Dx} = \mathbf{X}$$

$$\mathbf{X}^T \mathbf{X} = \mathbf{x}^T \mathbf{D}^T \mathbf{Dx} = \mathbf{x}^T \mathbf{x}$$

which is the discrete version of the Parseval Theorem.

III. Simulation Studies

We consider 3 examples to compare the detection performance of MF and CS

Example 1: The transformation matrix is orthogonal and comes from the eigenvectors of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & \ddots & 1 \\ 1 & 1 & 1 & 16 \end{bmatrix}_{16\times 16}$$

D =

-0.9523	0.1843	0.1097	0.0802	-0.0642	-0.0539	0.0467	0.0414	-0.0372	0.0337	-0.0308	0.0281	0.0256	0.0230	0.0196	0.1548
0.2560	0.9129	0.1952	0.1150	-0.0834	-0.0663	0.0554	0.0478	-0.0422	0.0377	-0.0340	0.0308	0.0278	0.0248	0.0210	0.1613
0.1128	-0.3092	0.8853	0.2029	-0.1192	-0.0861	0.0681	0.0567	-0.0487	0.0428	-0.0380	0.0340	0.0305	0.0269	0.0226	0.1683
0.0724	-0.1322	-0.3490	0.8611	-0.2087	-0.1226	0.0883	0.0696	-0.0577	0.0494	-0.0431	0.0380	0.0336	0.0294	0.0246	0.1761
0.0533	-0.0841	-0.1458	-0.3836	-0.8381	-0.2132	0.1254	0.0900	-0.0708	0.0584	-0.0498	0.0431	0.0376	0.0325	0.0269	0.1845
0.0421	-0.0616	-0.0921	-0.1569	0.4158	-0.8150	0.2165	0.1276	-0.0914	0.0716	-0.0588	0.0497	0.0426	0.0363	0.0297	0.1938
0.0349	-0.0487	-0.0673	-0.0986	0.1666	0.4470	0.7913	0.2187	-0.1291	0.0923	-0.0720	0.0587	0.0491	0.0411	0.0331	0.2041
0.0297	-0.0402	-0.0531	-0.0719	0.1042	0.1754	-0.4781	0.7661	-0.2199	0.1300	-0.0927	0.0718	0.0579	0.0473	0.0374	0.2156
0.0259	-0.0342	-0.0438	-0.0566	0.0758	0.1091	-0.1836	-0.5099	-0.7388	0.2197	-0.1301	0.0922	0.0706	0.0558	0.0430	0.2284
0.0230	-0.0298	-0.0373	-0.0466	0.0595	0.0792	-0.1136	-0.1913	0.5431	0.7087	-0.2181	0.1289	0.0905	0.0679	0.0506	0.2429
0.0206	-0.0264	-0.0324	-0.0397	0.0490	0.0621	-0.0822	-0.1177	0.1986	-0.5785	-0.6745	0.2143	0.1261	0.0868	0.0614	0.2593
0.0187	-0.0237	-0.0287	-0.0345	0.0417	0.0511	-0.0645	-0.0850	0.1215	-0.2054	0.6172	0.6345	0.2075	0.1201	0.0782	0.2781
0.0171	-0.0215	-0.0258	-0.0305	0.0362	0.0434	-0.0530	-0.0665	0.0875	-0.1249	0.2117	-0.6606	0.5858	0.1954	0.1075	0.2998
0.0158	-0.0197	-0.0234	-0.0274	0.0321	0.0378	-0.0450	-0.0547	0.0684	-0.0897	0.1278	-0.2172	-0.7114	0.5224	0.1718	0.3252
0.0146	-0.0181	-0.0214	-0.0248	0.0287	0.0334	-0.0391	-0.0464	0.0561	-0.0700	0.0915	-0.1300	-0.2213	-0.7750	0.4284	0.3553
0.0136	-0.0168	-0.0197	-0.0227	0.0261	0.0299	-0.0346	-0.0403	0.0476	-0.0574	0.0713	-0.0927	-0.1310	-0.2225	-0.8689	0.3915

$$\mathbf{X} = \begin{bmatrix} 4 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} -3.5 & 0.4 & -0.00 & -0.6 & -1.6 & -0.2 & -0.2 & -1.3 & -2.3 & 0.6 & -0.1 & -0.8 & 1.5 & 0.7 & 0.4 & 2.2 \end{bmatrix}^T$$

$$\mathbf{A} = \begin{bmatrix} 1000000000000000 \\ 0000100000000000 \\ 0000000010000000 \\ 0000000000001000 \end{bmatrix}$$

Since $\mathbf{D}^T = \mathbf{D}^{-1}$, the ROC curves of MF and CS are identical in Figure 1.

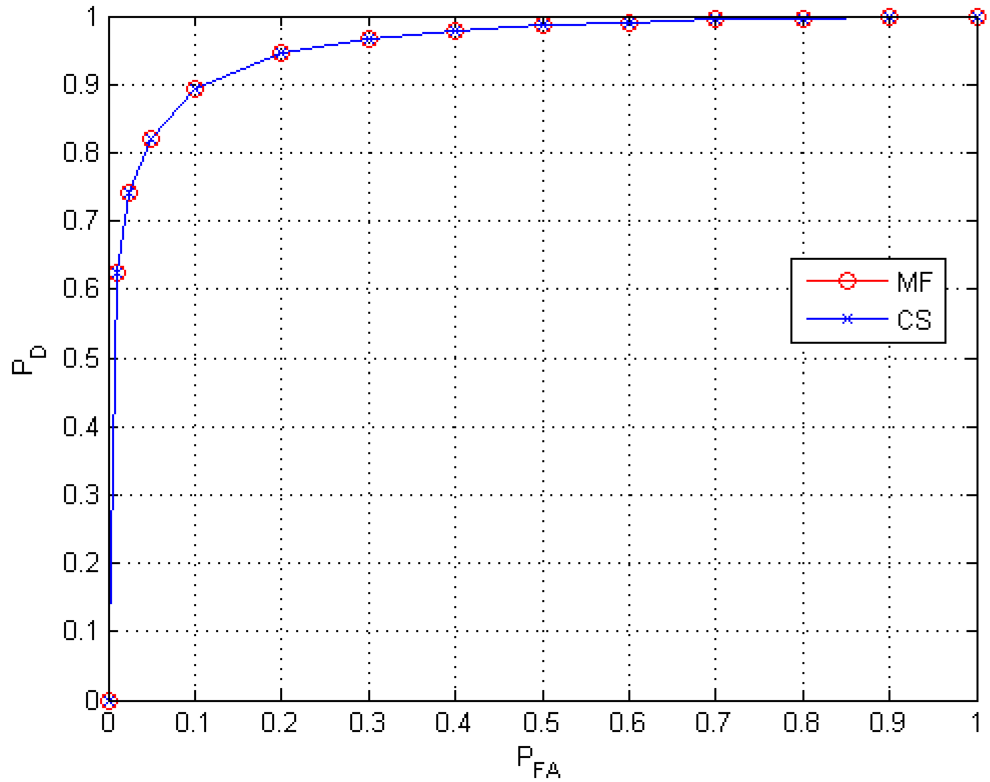

Figure 1: ROC curves for $\mathbf{D}$ based on eigenvectors.

<u>Example 2:</u> The transform matrix is the Fourier Transform Matrix

$$\mathbf{D} = \frac{1}{\sqrt{N}} \begin{bmatrix} D_{11} & D_{12} & \cdots & D_{1N} \\ D_{21} & D_{22} & \ddots & D_{2N} \\ \vdots & \cdots & D_{kl} & D_{kN} \\ D_{N1} & D_{N2} & D_{Nl} & D_{NN} \end{bmatrix}_{N \times N}$$

where

$$D_{kl} = \omega^{(k-1)\times(l-1)}, \qquad \omega = \exp\left(\frac{2\pi j}{N}\right)$$

and

$$\mathbf{X} = \begin{bmatrix} 4 & 0 & 0 & 0 & 2+2j & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 2-2j & 0 & 0 & 0 \end{bmatrix}^T$$

$$\mathbf{x} = \frac{1}{4} \times [11 \quad -3 \quad 3 \quad 5 \quad 11 \quad -3 \quad 3 \quad 5 \quad 11 \quad -3 \quad 3 \quad 5 \quad 11 \quad -3 \quad 3 \quad 5]^T$$

$$\mathbf{A} = \begin{bmatrix} 1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \end{bmatrix}$$

Again since $\mathbf{D}$ is orthogonal, the ROC curves in Figure 2 show that MF and CS give identical performance.

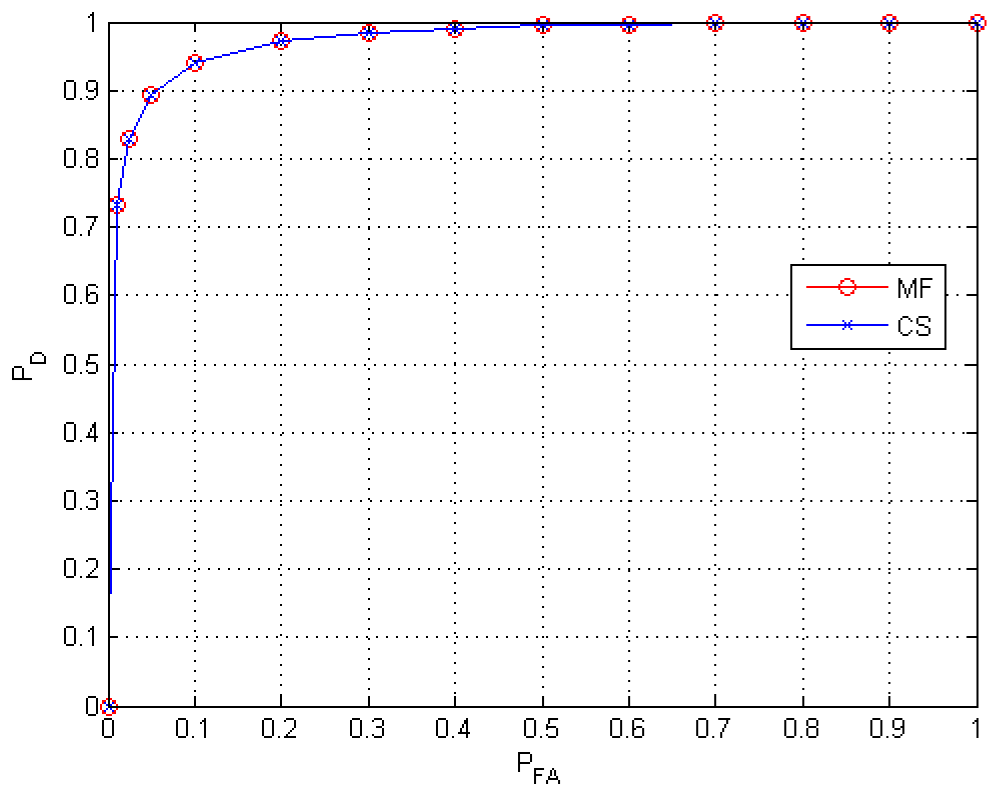

Figure 2: ROC curves for $\mathbf{D}$ based on Fourier Transform Matrix.

Example 3: The transform matrix is a random matrix of full rank. But $\mathbf{D}^T \neq \mathbf{D}^{-1}$.

Figure 3 shows that a non-orthogonal transform matrix, the performance of MF is superior to CS.

D =

0.6940	0.8463	0.8505	0.8507	0.5622	0.7423	0.4449	0.0478	0.4066	0.7691	0.4763	0.6525	0.3768	0.6418	0.0441	0.7426
0.5549	0.8850	0.7322	0.8897	0.7989	0.3829	0.6267	0.6795	0.8284	0.6330	0.1942	0.3596	0.4933	0.7731	0.9649	0.9179
0.5455	0.2173	0.6956	0.4070	0.7507	0.6221	0.6741	0.8724	0.3275	0.1088	0.7646	0.8512	0.0025	0.8112	0.4864	0.1018
0.3543	0.3860	0.8006	0.7004	0.1574	0.3115	0.1608	0.6340	0.2248	0.3469	0.1144	0.2840	0.9528	0.6356	0.7364	0.6037
0.6233	0.6859	0.0408	0.7585	0.7160	0.8916	0.3391	0.9857	0.0893	0.6437	0.2850	0.8541	0.9420	0.0915	0.6794	0.8894
0.4845	0.0873	0.4762	0.3320	0.4721	0.0311	0.5903	0.0571	0.1697	0.7595	0.8493	0.3681	0.7553	0.9516	0.1576	0.8040
0.8506	0.9370	0.6755	0.7763	0.5236	0.1726	0.7545	0.4114	0.5906	0.5287	0.3151	0.8119	0.5656	0.6631	0.3922	0.6760
0.9428	0.2756	0.8900	0.0878	0.2843	0.0668	0.7127	0.5211	0.7492	0.9900	0.8358	0.1419	0.5814	0.3611	0.6032	0.0528
0.9072	0.2553	0.3526	0.7376	0.6613	0.3001	0.1606	0.2692	0.7053	0.1042	0.3902	0.4467	0.9957	0.6224	0.0313	0.9974
0.8885	0.6231	0.6278	0.1823	0.8687	0.8177	0.7874	0.9148	0.2950	0.1732	0.0245	0.3176	0.6306	0.6262	0.4009	0.6318
0.8192	0.1985	0.2893	0.8103	0.6437	0.2407	0.9545	0.6909	0.1465	0.2715	0.7063	0.6316	0.2401	0.4472	0.3208	0.2177
0.1619	0.7403	0.9383	0.7928	0.1242	0.1791	0.4958	0.3424	0.6364	0.9069	0.4845	0.5282	0.6348	0.9500	0.3445	0.1269
0.5427	0.6494	0.4852	0.0616	0.0036	0.8095	0.4368	0.1676	0.0809	0.7952	0.4958	0.9498	0.8926	0.6742	0.7283	0.6018
0.0025	0.2381	0.2598	0.0018	0.6005	0.9111	0.0226	0.3800	0.6878	0.8999	0.3839	0.2493	0.5337	0.4543	0.0006	0.1217
0.4504	0.9150	0.7742	0.9540	0.5289	0.9698	0.1239	0.7317	0.1055	0.6540	0.3672	0.9841	0.5089	0.3966	0.6684	0.0634
0.1058	0.2948	0.0543	0.1525	0.3288	0.4906	0.7411	0.7005	0.5341	0.6649	0.3006	0.2340	0.9251	0.1416	0.4962	0.9380

$$\mathbf{X} = [4 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0 \quad 3 \quad 0 \quad 0 \quad 0 \quad 2 \quad 0 \quad 0 \quad 0]^T$$

$\mathbf{x}$ = [5.0 9.9 -8.3 -1.2 -8.8 -0.3 -9.0 11.0 0.1 0.6 9.9 -8.0 -8.9 6.7 -0.3 5.4]^T

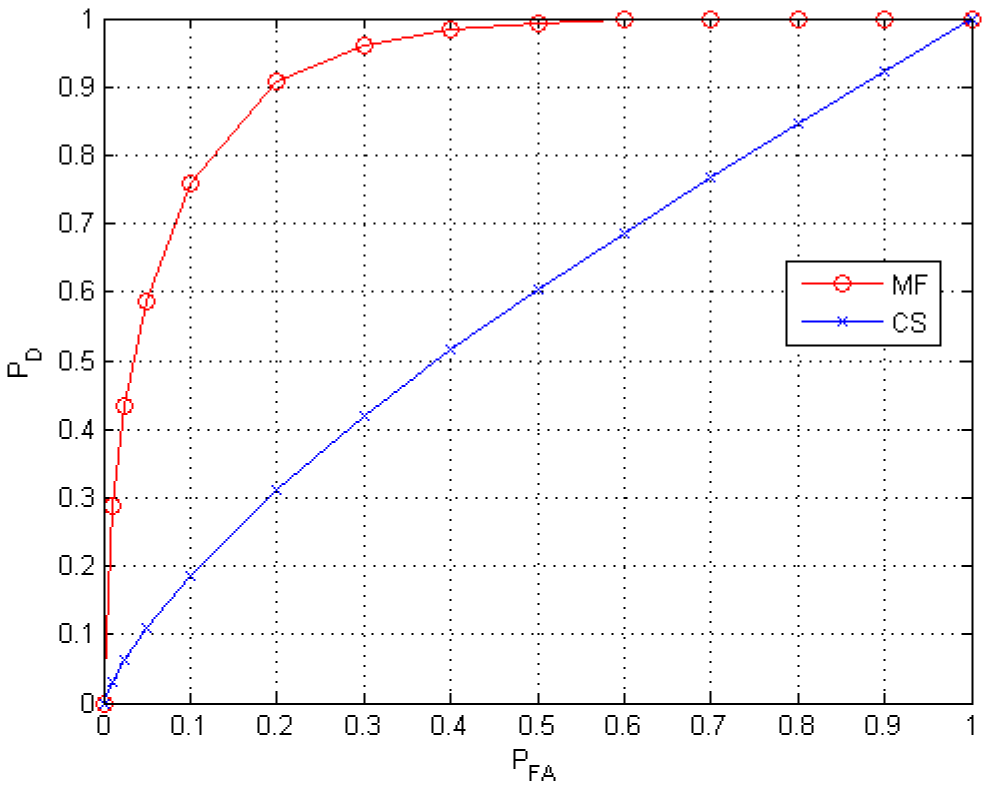

Figure 3: ROC curves for **D** based on matrix with random elements.

IV. Conclusions and Further Work

For known signal detection, i.e., a signal known in time and amplitude, if the transform of a signal is sparse, we can detect the signal using a lower dimension of compressed samples.

If the transformation matrix $\mathbf{D}$ is orthogonal, the performance of MF and CS is identical. If not, MF is superior.

Future work includes

(i) Finding a way to improve CD detection when $\mathbf{D}$ is not orthogonal.

(ii) Extend the work to unknown signals, when the detection is via cross-correlation.

The above work is not available in these two papers:

[1] J. Haupt and R. Nowak, "Compressive Sampling for Signal Detection," *ICASSP* 2007, pp. 1509-1512. http://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=4218008

[2] A. Eftekhari, J. Romberg and M. B. Wakin, "Matched Filtering From Limited Frequency Samples," *IEEE Trans. Information Theory*, 2013, pp. 3475-3496. http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=06459025