# A Multiple-Detection Probability Hypothesis Density Filter

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**Abstract**

Most conventional target tracking algorithms assume that one target can generate at most one detection per scan. However, in many practical target tracking applications, one target may generate multiple detections in one scan, because of multipath propagation, or high sensor resolution or some other reason. If the multiple detections from the same target can be effectively utilized, the performance of the multitarget tracking system can be improved. However, the challenge is that the uncertainty in the number of target and the many-to-one measurement set-to-target association will increase the complexity of tracking algorithms. To solve this problem, the random finite set (RFS) modeling and the random finite set statistics (FISST) are used in this paper. Without any extra approximation beyond those made in the standard probability hypothesis density (PHD) filter, a general multi-detection PHD (MD-PHD) update formulation is derived. It is also established in this paper that, with certain reasonable assumptions, the proposed MD-PHD recursion can function as a generalized extended target PHD or multisensor PHD filter. Furthermore, a Gaussian Mixture (GM) implementation of the proposed MD-PHD formulation, called the MD-GM-PHD filter, is presented. The proposed MD-GM-PHD filter is demonstrated on a simulated over-the-horizon radar (OTHR) scenario.

**Index Terms**

Target tracking, Bayesian filtering, Random finite sets, Probability hypothesis density filter, Multiple-detection tracking, Gaussian mixture, Over-the-horizon radar (OTHR).

## I. Introduction

In many multitarget tracking algorithms, such as the Multiple Hypothesis Tracker (MHT) [1], [2], the Joint Probabilistic Data Association Filter (JPDAF) [3]–[6], and algorithms based on the random finite set (RFS) method [7], [8], a common assumption is that in every scan there is at most one measurement from one target. But in many practical scenarios, one target may generate multiple detections in a single frame via different signal propagation paths, (i.e., via different measurement modes [9]). Some common examples include multitarget tracking systems using the over-the-horizon radar (OTHR) [10], where radar signals from the same target are scattered by different ionospheric layers, and the passive coherent location (PCL) system with multistatic configuration [11], where one target can generate multiple detections from different transmitters of opportunity [12]. In such situations, if the tracking algorithm is able to properly extract the information in the multiple detections, the estimation accuracy can be improved and the number of false tracks can be reduced. However, the complexity of the tracking algorithm increases significantly due to the additional measurement mode uncertainty [9]. In the literature, various techniques have been proposed to address the multiple-detection problem in OTHR [13]–[20] or PCL [12], [21] scenarios.

Recently, two multitarget tracking algorithms that are able to deal with the multi-detection problem were proposed. One is the Multiple Detection Multiple Hypothesis Tracker (MD-MHT) [22], which uses the MHT framework. The other is the Multiple-Detection Joint Probabilistic Data Association Filter (MD-JPDAF) [9], which uses the JPDA framework. Both of them can exploit the available measurement mode information to effectively track multiple targets.

In multitarget tracking, it is possible to model the set of target states and the set of observations as two RFSs and handle them using the RFS Bayesian filtering framework. One tractable Bayesian solution under the RFS framework is the probability hypotheses density (PHD) filter [7], [23]. The PHD filter is a recursion that propagates the first order statistical moment, or the intensity of the RFS of states over time [7]. It can track multiple target states and keep track of the number of targets without explicit multitarget data association between measurements and tracks. Note that the standard PHD filters [24] rely on the assumption of a single detection per target.

In [25], the multiple-detection single-target tracking problem was addressed under the RFS framework. The target-generated measurements are modeled as a mixture of one binary RFS and one Poisson RFS, which represent the single direct path measurement and other multipath measurements, respectively. Consequently, it cannot handle a general situation where the preferred (direct) path is absent, e.g., tracking using the OTHR. Moreover, there is no distinction among the various modes of detection, which will result in loss of information or in spurious tracks.

In this paper, we address the multitarget tracking problem with multiple detections within the RFS theoretical framework. A novel PHD measurement-update equation that accommodates multiple detections is derived using the finite-set statistics (FISST) theory [23]. No approximation beyond those in the standard PHD formula is made in the derivation of the new multiple detection PHD (MD-PHD) filter.

The MD-PHD filter proposed here has a more general application as well. The MD-PHD equations derived under the multiple detection assumptions can deal with multitarget tracking with multipath propagation effects, extended targets, multiple sensors, or multiple detections with multiple sensors, respectively. We also show that the proposed MD-PHD filter can be transformed into the extended target PHD [26] or reduced to the generalized multisensor PHD in [27], [28].

Furthermore, a closed form implementation of the MD-PHD filter is presented using the Gaussian Mixture (GM) representation, called the multi-detection GM-PHD (MD-GM-PHD) filter. The proposed MD-GM-PHD algorithm is demonstrated on a simulated OTHR multitarget tracking scenario. The performance of the MD-GM-PHDF is compared with the latest association based technique, MD-JPDAF [9], for a fixed number of targets, and with the conventional GM-PHDF for a dynamic number of targets. Results show the effectiveness of the new algorithm with respect to estimation accuracy and ability to handle a time-varying number of targets.

The rest of the paper is organized as follows. A brief introduction of multitarget tracking with multiple detections and the conventional PHD filter is given in Section II. The MD-PHD is described in Section III. Section IV presents the generalization of the proposed MD-PHD and the connection to other PHD formulations. The GM implementation of the proposed MD-PHD is presented in Section V. Simulation results can be found in Section VI. Conclusions are in Section VII.

## II. Background and Preliminaries

In this section, we first describe the problem of multiple-detection multitarget tracking. Then we present the mathematical preliminaries on the RFS and the PHD filter.

### A. *Multiple-Detection Multitarget Tracking*

In the multi-target tracking problem, the evolution of the objects' states and the origin of measurements are unknown. Uncertainty in target state and sensor measurement can be modelled by an RFS. Assume that there are $N_{x,k}$ targets with states $\mathbf{x}_k^1, ..., \mathbf{x}_k^{N_{x,k}}$, each taking values in a state space $\mathcal{X} \subseteq \mathbb{R}^{n_x}$, and $N_{z,k}$ measurements $\mathbf{z}_k^1, ..., \mathbf{z}_k^{N_{z,k}}$, each taking values in an observation space $\mathcal{Z} \subseteq \mathbb{R}^{n_z}$. The multi-target state and the multi-target observation are then represented by the RFS: $X_k = \{\mathbf{x}_k^1, ..., \mathbf{x}_k^{N_{x,k}}\}$ and $Z_k = \{\mathbf{z}_k^1, ..., \mathbf{z}_k^{N_{z,k}}\}$, respectively. Here, $n_x$ and $n_z$ are the dimension of the target state vector and the measurement vector, respectively.

The dynamic evolution of each target state $\mathbf{x}_k^i$ in the RFS $X_k$ is modeled by the state equation

$$\mathbf{x}_{k+1}^i = \mathbf{f}_k(\mathbf{x}_k^i) + \mathbf{v}_k \tag{1}$$

where $\mathbf{f}_k$ is a known linear or non-linear state transition function, $\mathbf{v}_k$ is a zero-mean white Gaussian process noise sequence with covariance matrix $\mathbf{Q}_k$ that models targets maneuver. Each target evolves independently.

For the multiple-detection system, we assume that there are $L$ possible measurement modes with known measurement equations, and $\mathbf{z}_k^j$, the $j$th measurement at time $k$, is given by

$$\mathbf{z}_k^j = \begin{cases} \mathbf{h}_k^1(\mathbf{x}_i) + \mathbf{w}_k^1 & \text{if mode } 1 \\ \mathbf{h}_k^2(\mathbf{x}_i) + \mathbf{w}_k^2 & \text{if mode } 2 \\ \vdots & \\ \mathbf{h}_k^L(\mathbf{x}_i) + \mathbf{w}_k^L & \text{if mode } L \\ \tilde{\mathbf{z}}_k^j & \text{if clutter} \end{cases} \tag{2}$$

where $\mathbf{h}_k^l(\cdot), l = 1, 2, ...L$ is the measurement function associated with the $l$-th measurement mode and $\mathbf{w}_k^l$ is the corresponding measurement noise, which is assumed to be zero-mean Gaussian random variable with covariance matrix $\mathbf{R}_k^l$ and is independent of the target state. Associations between target $\mathbf{x}_k^i$, measurement $\mathbf{z}_k^j$, and mode $l$

are unknown to the tracker. The false measurement $\tilde{\mathbf{z}}_k$ is assumed to be uniformly distributed in the surveillance volume of the sensors and the number of false alarms per scan is Poisson distributed with mean $\lambda$.

*Remark:* In practice, the multiple detections for a given target might be generated with unknown modes, or with a time-varying number of mode. For simplicity, we assume that the number of modes $L$ is constant and the mode information is known. The same assumptions are made in [9] and [22].

The objective of this paper is to estimate the RFS $X_k$ given the measurement RFS $Z_k$ from the sensor. We achieve this within the framework of the recursive multitarget Bayesian PHD filter [23].

### *B. Background to the PHD filter*

For RFS $\Xi$, one can define the probability density function $f_\Xi(X)$ on the finite set variable $X$, and $f_\Xi(X)$ satisfies the unit requirement

$$f(\emptyset) + \sum_{n=1}^{\infty} \frac{1}{n!} \int f_\Xi(\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}) d\mathbf{x}_1 \ldots d\mathbf{x}_n = 1 \tag{3}$$

where $n!$ counts the duplication caused by the permutation of $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$, and $f(\emptyset)$ denotes the probability of the empty set.

Based on the RFS theory, the multitarget Bayes filter can be derived. However, the multitarget Bayes filter is computationally intractable in all but the simplest scenarios since the calculation of the multitarget Bayes filter relies on the set integral, which needs to take into account all possible multitarget state finite sets [23]. To conquer this problem, the first-order multitarget statistical moment, or the PHD, has been used [24]. For a given RFS $\Xi$ with the multitarget probability density function $f_\Xi(X)$, the PHD $D_\Xi(\mathbf{x})$ can be calculated by

$$D_\Xi(\mathbf{x}) = \int \delta_X(\mathbf{x}) f_\Xi(X) \delta X \tag{4}$$

where $\delta_X(\mathbf{x}) \triangleq \sum_{\mathbf{w} \in X} \delta(\mathbf{x} - \mathbf{w})$ and $\delta(\mathbf{x})$ is the Dirac delta function, which is used to convert the finite set $X = \{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ into vectors since the first-order statistical moment is defined in the vector space.

Then, by approximating the predicted multitarget state as a Poisson point process, the PHD filter is developed in [24], where the PHD is propagated in a recursive form:

- Prediction equation:

$$D_{k+1|k}(\mathbf{x}) = b_{k+1}(\mathbf{x}) + \int F_{k+1|k}(\mathbf{x}|\mathbf{x}') D_{k|k}(\mathbf{x}') d\mathbf{x}' \tag{5}$$

  where

$$F_{k+1|k}(\mathbf{x}|\mathbf{x}') \triangleq p_S(\mathbf{x}') \mathbf{f}_{k+1|k}(\mathbf{x}|\mathbf{x}') \tag{6}$$

  and $\mathbf{f}_{k+1|k}(\mathbf{x}|\mathbf{x}')$ is the Markov transition density for the single target state; $p_S(\mathbf{x}')$ is the surviving probability for the existing targets; $b_{k+1}(\mathbf{x})$ is the PHD for the newborn targets that appear at scan $k+1$. Note that it has been assumed for simplicity that no targets are spawned from the existing ones.

- Measurement update equation:

$$D_{k+1|k+1}(\mathbf{x}) \cong L_{Z_{k+1}}(\mathbf{x}) D_{k+1|k}(\mathbf{x}) \tag{7}$$

where the pseudo-likelihood function is given by

$$L_{Z_{k+1}}(\mathbf{x}) \triangleq q_D(\mathbf{x}) + \sum_{\mathbf{z}\in Z_k} \frac{p_D(\mathbf{x})\mathbf{h}_{k+1}(\mathbf{z}|\mathbf{x})}{\lambda_{k+1}(\mathbf{z}) + \tau_{k+1}(\mathbf{z})} \tag{8}$$

and $\mathbf{h}_{k+1}(\mathbf{z}|\mathbf{x})$ denotes the sensor likelihood function; $p_D(\mathbf{x})$ is the abbreviation of $p_{D,k+1}(\mathbf{x})$ and represents the target detection probability at scan $k+1$ for the target with state $\mathbf{x}$; $q_D(\mathbf{x}) = 1 - p_D(\mathbf{x})$; $\lambda_{k+1}(\mathbf{z}_{k+1})$ represents the intensity function of clutter points; $\tau_{k+1}(\mathbf{z}) = \int p_D(\mathbf{x})\mathbf{f}_{k+1}(\mathbf{z}|\mathbf{x})D_{k+1|k}(\mathbf{x})d\mathbf{x}$ is the intensity of the measurement $\mathbf{z} \in Z_k$.

Note that above "classical" PHD filter makes the following simplifying assumptions [28]: (1) there is only a single sensor; (2) all target motions are independent; (3) measurements are conditionally independent of the target states; (4) the clutter process is Poisson; and (5) target-generated measurements are Bernoulli.

## III. Multi-detection PHD filter

In this section we derive the formula for the proposed MD-PHD recursion. The difference between the MD-PHD and the conventional single-detection PHD filter is in the measurement update equation (i.e., the corrector step). Thus, only the measurement update equation of the MD-PHD filter will be presented.

### A. Multi-detection Measurement Model

To derive the MD-PHD, the following assumptions are made.

*Assumption 1:* There are $L$ measurement modes, each with a known measurement likelihood function:

$$f^l_{k+1}(\mathbf{z}|\mathbf{x}) = f_{\mathbf{w}^l_{k+1}}(\mathbf{z} - \mathbf{h}^l_{k+1}(\mathbf{x})),\ l = 1, ..., L \tag{9}$$

where $f_{\mathbf{w}^l_{k+1}}(\cdot)$ is the probability density function of the measurement noise $\mathbf{w}^l_{k+1}$.

*Assumption 2:* There is at most one detection generated by a target through a specific mode.

*Assumption 3:* The detection probability, denoted in $p^l_D$, of each mode $l$ can be different.

Based on the above assumptions, we develop the likelihood function of multi-detection measurements for one target as follows by using the FISST formulas of [23]. The detailed derivation is given in the Appendix.

For single target state $\mathbf{x}$ and its corresponding measurement set $W$, if $W = \emptyset$, one has the probability that the target with state $\mathbf{x}$ is undetected in all modes as

$$f_{k+1}(W|\mathbf{x}) = f_{k+1}(\emptyset|\mathbf{x}) = \prod_{l=1}^{L}\left(1 - p^l_D(\mathbf{x})\right) \tag{10}$$

Suppose $W = \{\mathbf{z}_1, ..., \mathbf{z}_m\}$ with $|W| = m \leq L$, then one has

$$f_{k+1}(W|\mathbf{x}) = \prod_{l=1}^{L}\left(1 - p^l_D(\mathbf{x})\right)\sum_{\theta}\prod_{\theta(l)>0}\frac{p^l_D(\mathbf{x})\cdot\mathbf{h}^l_{k+1}(\mathbf{z}_{\theta(l)}|\mathbf{x})}{1 - p^l_D(\mathbf{x})} \tag{11}$$

where $\mathbf{h}^l_{_{k+1}}(\mathbf{z}|\mathbf{x})$ is the sensor measurement likelihood function with mode $l$, and the summation is taken over all association hypotheses $\theta : \{1, ..., L\} \to \{0, 1, ..., m\}$, which is defined in Section 10.5.4 of [23]. For instance, with

$L = 3$ and $W = \{\mathbf{z}_1, \mathbf{z}_2\}$ (i.e., $m = 2$), the corresponding likelihood function is

$$\begin{aligned} f_{k+1}(W|\mathbf{x}) = & (1 - p_D^1(\mathbf{x})) \cdot p_D^2(\mathbf{x}) \cdot p_D^3(\mathbf{x}) \cdot f_{k+1}^2(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^3(\mathbf{z}_2|\mathbf{x}) \\ & + (1 - p_D^1(\mathbf{x})) \cdot p_D^3(\mathbf{x}) \cdot p_D^2(\mathbf{x}) \cdot f_{k+1}^3(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^2(\mathbf{z}_2|\mathbf{x}) \\ & + (1 - p_D^2(\mathbf{x})) \cdot p_D^3(\mathbf{x}) \cdot p_D^1(\mathbf{x}) \cdot f_{k+1}^3(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^1(\mathbf{z}_2|\mathbf{x}) \\ & + (1 - p_D^2(\mathbf{x})) \cdot p_D^1(\mathbf{x}) \cdot p_D^3(\mathbf{x}) \cdot f_{k+1}^1(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^3(\mathbf{z}_2|\mathbf{x}) \\ & + (1 - p_D^3(\mathbf{x})) \cdot p_D^2(\mathbf{x}) \cdot p_D^1(\mathbf{x}) \cdot f_{k+1}^1(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^2(\mathbf{z}_2|\mathbf{x}) \\ & + (1 - p_D^3(\mathbf{x})) \cdot p_D^1(\mathbf{x}) \cdot p_D^2(\mathbf{x}) \cdot f_{k+1}^2(\mathbf{z}_1|\mathbf{x}) \cdot f_{k+1}^1(\mathbf{z}_2|\mathbf{x}) \end{aligned} \tag{12}$$

### *B. MD-PHD Measurement-update Equation*

In the MD-PHD measurement-update step, one has to consider the possibility that multiple elements in the current measurement set $Z_{k+1}$ might have originated from the same target. This special form of PHD measurement-update equation involves combinatorial sums taken over all partitions of $Z_{k+1}$, and its derivation is cumbersome [28]. Fortunately, the general chain rule (GCR) to simplify the derivation progress was introduced in [28]. A general PHD measurement-update expression with the following pseudo-likelihood function was provided:

$$L_{Z_{k+1}}(\mathbf{x}) = (1 - \tilde{p}_D(\mathbf{x})) + \sum_{\mathcal{P} \angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W \in \mathcal{P}} \frac{L_W(\mathbf{x})}{\kappa_W + \tau_W} \tag{13}$$

where

$$L_W(\mathbf{x}) = f_{k+1}(W|\mathbf{x}) \tag{14}$$

$$1 - \tilde{p}_D(\mathbf{x}) = L_\emptyset(\mathbf{x}) = f_{k+1}(\emptyset|\mathbf{x}) \tag{15}$$

and

$$\kappa_W = \frac{\delta \log \kappa_{k+1}}{\delta W}[0] \tag{16}$$

where $\kappa_{k+1}[g]$ is the probability generating functional (p.g.fl.) of the clutter process, which is assumed to be a Poisson point process. As a result, $\kappa_W$ is given by

$$\kappa_W = \begin{cases} -\lambda_{k+1} & \text{if } W = \emptyset \\ \kappa_{k+1}(\mathbf{z}) & \text{if } W = \{\mathbf{z}\} \\ 0 & \text{if } |W| > 1 \end{cases} \tag{17}$$

In (13), the summation is taken over all partitions $\mathcal{P}$ of the current measurement RFS $Z_{k+1}$, the notation "$\mathcal{P} \angle Z_{k+1}$" stands for "$\mathcal{P}$ partitions $Z_{k+1}$ into cell $W$", and

$$\tau_W = \int L_W(\mathbf{x}) \cdot D_{k+1|k}(\mathbf{x}) d\mathbf{x} \tag{18}$$

$$\omega_{\mathcal{P}} = \frac{\prod_{W \in \mathcal{P}} (\kappa_W + \tau_W)}{\sum_{\mathcal{P}' \angle Z_{k+1}} \prod_{W' \in \mathcal{P}'} (\kappa_{W'} + \tau_{W'})} \tag{19}$$

Note that (13) is a concise symbolic equation, which does not characterize any specific form for the probability of detection $\tilde{p}_D(\mathbf{x})$ or the target likelihood function $L_W(\mathbf{x})$. This makes it possible to advance the PHD with more general target measurement-generation models [28], e.g., the second-generation PHDs [29]. Our work also benefits from it.

Substituting (10) and (11) into (13), one gets the MD-PHD pseudo-likelihood function as

$$\begin{aligned} L_{Z_{k+1}}(\mathbf{x}) &= (1-\tilde{p}_D(\mathbf{x})) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{L_W(\mathbf{x})}{\kappa_W+\tau_W} \\ &= \left(\prod_{l=1}^{L}\left(1-p_D^l(\mathbf{x})\right)\right) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{\prod_{l=1}^{L}\left(1-p_D^l(\mathbf{x})\right)\sum_{\theta}\prod_{\theta(l)>0}\frac{p_D^l(\mathbf{x})\cdot \mathbf{h}_{k+1}^l(\mathbf{z}_{\theta(l)}|\mathbf{x})}{1-p_D^l(\mathbf{x})}}{\kappa_W+\tau_W} \end{aligned} \tag{20}$$

Note that distinct partition methods are needed for different second-generation PHDs, e.g. the extended target PHD [26] or the multisensor PHD [27]. Aside from the partition defined in [26] and [27], we define a $L_{\max}$-partition. It restricts that for each $W\in\mathcal{P}$, the total number of elements in $W$ is no larger than $L$. For example, let $Z_{k+1}=\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\}$ with $|Z_{k+1}|=3$ and $L=2$. Then, the $L_{\max}$-partitions of $Z_{k+1}$ are

$$\mathcal{P}_1=\{\{\mathbf{z}_1\},\{\mathbf{z}_2\},\{\mathbf{z}_3\}\}, \text{where } W_1=\{\mathbf{z}_1\}, W_2=\{\mathbf{z}_2\}, W_3=\{\mathbf{z}_3\}$$

$$\mathcal{P}_2=\{\{\mathbf{z}_1,\mathbf{z}_2\},\{\mathbf{z}_3\}\}, \text{where } W_1=\{\mathbf{z}_1,\mathbf{z}_2\}, W_2=\{\mathbf{z}_3\}$$

$$\mathcal{P}_3=\{\{\mathbf{z}_1,\mathbf{z}_3\},\{\mathbf{z}_2\}\}, \mathcal{P}_4=\{\{\mathbf{z}_2,\mathbf{z}_3\},\{\mathbf{z}_1\}\}, \mathcal{P}_5=\{\{\mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3\}\}$$

The summation in (20) is taken over all such $L_{\max}$-partitions, $\mathcal{P}$, of the current measurement RFS $Z_{k+1}$.

Note the following test for consistency: If the mode number $L$ is reduced to 1, the MD-likelihood function (11) will reduce to the normal single-target likelihood function $f_{k+1}(\mathbf{z}|\mathbf{x})$ with a single-detection. In addition, there will be only a singleton in the partition of the RFS $Z_{k+1}$. Thus, the MD-PHD pseudo-likelihood function (20) will reduce to the conventional single-detection PHD pseudo-likelihood function (8).

## IV. Generalizing the MD-PHD filter

The multi-detection is a general concept and can characterize broader physical measurement models. Thus, in this section, the MD-PHD is used to handle various multi-detection applications under different assumptions and measurement models.

### A. A General Multipath-aided PHD Filter

The MD-PHD proposed in Section III can be used for target tracking with multipath measurements. In a scenario with multipath measurements, there are many signal propagation paths between a target and the receiver. Among them, $L$ resolved paths, e.g., those scattered from the target and reflected at most once from a reflection point, are considered. We assume that one knows the geometry of the surface, reflection points, and the reflection coefficients. However, the target detection probability can be path-dependent. A common example of this model is the OTHR [9] [10]. Thus, we can cast the measurement mode defined in (2) to each multiple propagation path. Consequently, the uncertain association of the multipath reflection, which arises from the exploitation of the information in all paths by the algorithm, can be handled by the likelihood function (11) in the MD-PHD recursion.

### *B. A General Extended Target PHD Filter*

In target tracking with a high resolution radar, the extended nature of a target also generates multiple detections at each scan. More specifically, we assume that there are $l = 0, ..., L$ number of detections from a collection of reflection points, whose number is bigger than $L$, on the surface of the extended target. Each detection is related to a deterministic measurement model with different observability from the receiver. Then the detected measurements can be modeled by the measurement mode in (2) with a different detection probability for each mode.

In this sense, the multi-detection likelihood function (11) becomes a general extended target likelihood function and it has the same expression as the single extended target likelihood function (12.215) in [23], As a result, the proposed MD-PHD filter, defined by (11), (20) and the $L_{\text{max}}$-partitions in Section III-B, can be an extended target PHD filter.

Compared with the existing extended target PHD filter in [26], our MD-PHD filter relaxes the Poisson approximation for the target-originated measurements in [26]. Thus, it can deal with more general issues in multiple extended targets tracking, e.g., in high-resolution sensor observation scenarios where detailed measurement modes for individual reflection points on the target are known.

### *C. A General Multisensor PHD Filter*

Under the multisensor PHD filter framework [27], the measurements from each sensor are used in a centralized manner. Thus, multiple measurements from one target can be detected in the same cumulative scan via multiple sensors. In this case, each element in the measurement set will be tagged with a sensor ID. By casting each measurement mode discussed above to each sensor, we can derive a general multisensor PHD recursion.

More specifically, in order to do this, the following two assumptions are made, besides those in Subsection III-A.

*Assumption 4:* Each target-originated measurement $\mathbf{z} \in Z_{k+1}$ is generated from one of the known measurement modes.

With the above assumption, there is no more uncertainty between measurements and measurement modes. Thus, the enumerations of the associations between measurements and modes in the likelihood function (11) are eliminated, and (11) reduces to

$$\begin{aligned} f_{k+1}(W|\mathbf{x}) &= \prod_{s=1}^{S} (1 - p_D^s(\mathbf{x})) \prod_{\theta(s)>0} \frac{p_D^s(\mathbf{x}) \cdot \mathbf{h}_{k+1}^s(\mathbf{z}_{\theta(s)}|\mathbf{x})}{1 - p_D^l(\mathbf{x})} \\ &= \prod_{\theta(s)=0} (1 - p_D^s(\mathbf{x})) \prod_{\theta(s)>0} p_D^s(\mathbf{x}) \cdot \mathbf{h}_{k+1}^s(\mathbf{z}_{\theta(s)}|\mathbf{x}) \end{aligned} \tag{21}$$

where we replace the symbol of mode variable $l \in L$ with $s \in S$ to emphasize that the measurement mode here is one out of the $S$ sensors.

*Assumption 5:* The clutter is measurement-mode dependent and is statistically independent across modes.

Thus, the definition of $\kappa_W$ in (16) is modified so that sub-cell $V_s \in W, s = 1, ..., S$, denotes a measurement from one target through sensor $s$. In addition,

$$\kappa_{V_s} = \frac{\delta \log \kappa_{k+1}^s}{\delta V_s}[0],\ s = 1, ...S \tag{22}$$

where $V_s \uplus ... \uplus V_S = W$ .

Consequently, one gets the following PHD measurement-update equation:

$$\begin{aligned} \frac{D_{k+s|k+s}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} &= f_{k+1}(\emptyset|\mathbf{x}) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{L_W(\mathbf{x})}{\sum_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S} |V_{s'}|,0} + \tau_W} \\ &= \left(\prod_{s=1}^{S} \left(1 - p_D^s(\mathbf{x})\right)\right) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{\prod_{\theta(s)=0} (1 - p_D^s(\mathbf{x})) \prod_{\theta(s)>0} p_D^s(\mathbf{x}) \cdot \mathbf{h}_{k+1}^s(\mathbf{z}_{\theta(s)}|\mathbf{x})}{\sum_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S} |V_{s'}|,0} + \tau_W} \end{aligned} \tag{23}$$

where $\delta_{i,j}$ is the Kronecker delta and $\delta_{\sum_{s'=1,s'\neq s}^{S} |V_{s'}|,0}$ is one only when cells $V_1, ..., V_{s-1}, V_{s+1}..., V_S$ are $\phi$. Moreover, the $S$-ary partition [27], which confines, in each partition, the element with the same label of $s \in S$ to be one at most, should be applied on the RFS $Z_{k+1}$, and $\omega_{\mathcal{P}}$ is modified correspondingly to

$$\omega_{\mathcal{P}} = \frac{\prod_{W\in\mathcal{P}} \sum_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S} |V_{s'}|,0} + \tau_W}{\sum_{\mathcal{P}'\angle Z_{k+1}} \prod_{W'\in\mathcal{P}'} \sum_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S} |V_{s'}|,0} + \tau_{W'}} \tag{24}$$

Consequently, we get (23) as a general multisensor PHD measurement-update equation with $L$ sensors being considered. Assuming $L = 2$, (23) will reduce exactly to the multisensor PHD measurement-update equation (43) in [28] or Theorem 1 in [27] where 2 sensors were considered.

The derivation of (23) is similar to that of (20), which is based on the GCR and the FISST formulas and follows a straightforward but tedious rewriting as in [28]. We have omitted the details here for brevity.

*D. MD-PHD Filter with Multiple Sensors*

In this subsection, we extend the MD-PHD filter itself to the multisensor case.

Assume that there are $S$ sensors and that each sensor can detect the target from $L$ number of distinct propagate paths. To deal with this case within the framework of MD-PHD, we extend the definition of the measurement mode to a two dimensional form as $(l, s) \in L \times S$. For instance, a measurement mode $(2, 3)$ means that the target is detected via propagation path 2 from sensor 3. Then, the following two assumptions are made for the multisensor multi-detection PHD filer:

*Assumption 6:* Each target-originated measurement $\mathbf{z} \in Z_{k+1}$ is from a mode with a known label $s$ but an unknown label $l$.

Now, we can derive the multi-detection multisensor likelihood function, based on the multi-detection likelihood function (11), as follows:

$$f_{k+1}(W|\mathbf{x}) = \begin{cases} \sum\limits_{\theta} \prod\limits_{s=1}^{S} \left( \prod\limits_{l=1}^{L} \left(1 - p_D^{(l,s)}(\mathbf{x})\right) \prod\limits_{\theta(l,s)>0} \frac{p_D^{(l,s)}(\mathbf{x}) \cdot \mathbf{h}_{k+1}^{(l,s)}(\mathbf{z}_{\theta(l,s)}|\mathbf{x})}{1-p_D^{(l,s)}(\mathbf{x})} \right) & |W| \neq 0 \\ \prod\limits_{s=1}^{S} \prod\limits_{l=1}^{L} \left(1 - p_D^{(l,s)}(\mathbf{x})\right) & |W| = 0 \end{cases} \tag{25}$$

where the summation is taken over all association hypotheses $\theta : \{(1,1), ..., (L,S)\} \to \{0, 1, ..., m\}$. One can see that the main difference in (25) vs. (11) and (21) is that the uncertain associations between measurements and modes are enumerated within each $s$.

*Assumption 7:* The clutter is mode-dependent and statistically independent across mode $s$ but not across $l$. Thus, the clutter spatial intensity in sensor $s$, $\kappa_{V_s}$, should satisfy the following equation:

$$\kappa_{V_s} = \frac{\delta \log \kappa_{k+1}^{s}}{\delta V_s}[0],\ s = 1, ...S \tag{26}$$

where $V_s$ represents the cell that contains and only contains all measurements from sensor $s$, and $V_1 \uplus ... \uplus V_S = W$.

The resulting PHD measurement-update equation, which has the same form as (23) but now with $L_W(\mathbf{x}) = f_{k+1}(W|\mathbf{x})$ given in (25), is given by

$$\begin{aligned} \frac{D_{k+1|k+1}(\mathbf{x})}{D_{k+1|k}(\mathbf{x})} &= f_{k+1}(\emptyset|\mathbf{x}) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{L_W(\mathbf{x})}{\sum\limits_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq l}^{S}|V_{s'}|,0} + \tau_W} \\ &= \left( \prod_{s=1}^{S} \prod_{l=1}^{L} \left(1 - p_D^{(l,s)}(\mathbf{x})\right) \right) + \sum_{\mathcal{P}\angle Z_{k+1}} \omega_{\mathcal{P}} \sum_{W\in\mathcal{P}} \frac{\sum\limits_{\theta} \prod\limits_{s=1}^{S} \left( \prod\limits_{l=1}^{L} \left(1 - p_D^{(l,s)}(\mathbf{x})\right) \prod\limits_{\theta(l,s)>0} \frac{p_D^{(l,s)}(\mathbf{x}) \cdot \mathbf{h}_{k+1}^{(l,s)}(\mathbf{z}_{\theta(l,s)}|\mathbf{x})}{1-p_D^{(l,s)}(\mathbf{x})} \right)}{\sum\limits_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S}|V_{s'}|,0} + \tau_W} \end{aligned} \tag{27}$$

where

$$\omega_{\mathcal{P}} = \frac{\prod\limits_{W\in\mathcal{P}} \sum\limits_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S}|V_{s'}|,0} + \tau_W}{\sum\limits_{\mathcal{P}'\angle Z_{k+1}} \prod\limits_{W'\in\mathcal{P}'} \sum\limits_{s=1}^{S} \kappa_{V_s} \cdot \delta_{\sum_{s'=1,s'\neq s}^{S}|V_{s'}|,0} + \tau_{W'}} \tag{28}$$

In (28), the partition scheme is beyond those in (20) or (23). For each partition $W$, there are at most $L$ elements with each $s \in S$ being labeled. It means that there can be at most $L$ detections from one target observed by sensor $s$. The derivation of (27) is also based on the GCR and the FISST formulas with a procedure similar to the derivation of (20), but omitted here for brevity.

The consistency of the multisensor MD-PHD can be verified by noting that (27) will reduce to the MD-PHD equation with single sensor (20) when $S = 1$, or reduce to the single-detection multisensor PHD equation (23) when $L = 1$, respectively.

### *E. Computational Complexity of the MD-PHD*

The major computation load of the standard PHD filter arises from the measurement-update step [23], which has the computational complexity of order $O(N_{x,k} \times N_{z,k})$.

The extra calculation time for the MD-PHD filter, compared to the standard PHD filter, relies on two factors: (1) the total number of partitions of measurement sets, (2) the calculation of non-negative coefficients for each cell in partitions.

*1) The number of partitions:* The total number of partitions of a set with size $n$ (denoted by $n$-set) is given by the Bell number $B(n)$ [30]. Moreover, the number of partitions of an $n$-set with maximal $m$ elements in its subsets is (see Lemma 2.3 in [31])

$$P'_m(n) = \binom{n}{m\left[\frac{n}{m}\right]} \frac{\left(\left[\frac{n}{m}\right]m\right)!}{\left[\frac{n}{m}\right]!m!^{\left[\frac{n}{m}\right]}} \times B(n - \left[\frac{n}{m}\right]m) \tag{29}$$

where $\left[\frac{n}{m}\right]$ denotes the maxim integer not bigger than $\frac{n}{m}$, and $m\left[\frac{n}{m}\right]$ is the greatest multiple of $m$ that is less than or equal to $n$. Thus, the number of partitions of an $n$-set with at most $m$ elements in each subset is

$$B(n) - \sum_{j=m+1}^{n} P'_j(n) \tag{30}$$

TABLE I shows that the number of partitions in the MD-PHD filter is still of the same order of magnitude as that of $B(n)$.

TABLE I
NUMBER OF PARTITIONS IN MD-PHD UPDATE EQUATION ($m = 4$)

| Number of Measurements | Bell Number | Number of Partitions |
|---|---|---|
| 4 | 15 | 15 |
| 5 | 52 | 51 |
| 6 | 203 | 196 |
| 7 | 877 | 827 |
| 8 | 4140 | 3795 |
| 9 | 21147 | 18755 |
| 10 | 115975 | 99020 |

*2) Calculating the partition coefficient:* Here we need to calculate $\kappa_W$, $\tau_W$ and $L_W$ in (20) for all associations between measurement $\mathbf{z} \in W$ and $l$. For a given $W$, the total number of associations is $P^L_{|W|}$, where $P^y_x$ denotes the number of $x$-permutations of $y$ defined by

$$P^y_x = \begin{cases} \frac{y!}{(y-x)!}, & 1 \leq x \leq y \\ 1, & x = 0 \end{cases} \tag{31}$$

From the above analysis, one can see that the extra computation load required by the proposed MD-PHD filter is similar to that of the secondary-generation PHD filters [29]. Note that certain approximations, such as reducing the number of partitions by neglecting some partitions with low probability, or gating to restrict the number of the measurements and target state in the coefficient-calculation, have been suggested for the implementation of secondary-generation PHD filters [32] [33] [27]. Such techniques can be applied to the proposed MD-PHD filter as well. The computational challenges are greater with the proposed MD-PHD filter and the development of more efficient implementations is a future task.

## V. GAUSSIAN-MIXTURE IMPLEMENTATION OF THE MD-PHD FILTER

In general, the MD-PHD filter with the pesudo-ikelihood function (20) is not amenable to an analytic solution. However, the problem can be solved under the linear Gaussian assumptions. In this Section, following the derivation of the standard GM-PHDF [34], a GM-PHD recursion for targets under multiple-detection assumption, noted as MD-GM-PHDF, is proposed.

### A. MD-GM-PHD Derivation

Since the prediction update equation of the MD-PHD filter is also the same as that of the conventional PHD filter [24], the GM prediction update equation of the MD-GM-PHDF is the same as that of the conventional GM-PHDF in [34]. Thus, in this subsection we only focus on the measurement update equations of the MD-GM-PHDF.

Assume that the predicted PHD has the following GM representation:

$$D_{k+1|k}(\mathbf{x}) = \sum_{j=1}^{J_{k+1|k}} w_{k+1|k}^{(j)} \mathcal{N}(\mathbf{x}; m_{k+1|k}^{(j)}, P_{k+1|k}^{(j)}) \tag{32}$$

where $J_{k+1|k}$ is the predicted number of components, $w_{k+1|k}^{(j)}$ is the weight of the $j$-th component, $m_{k+1|k}^{(j)}$ and $P_{k+1|k}^{(j)}$ are the predicted mean and covariance of the $j$th component, respectively, and $\mathcal{N}(\mathbf{x}; m, P)$ denotes a Gaussian distribution defined over the variable $\mathbf{x}$ with mean $m$ and covariance $P$.

In [34], six assumptions were made to derive the measurement update equation of the GM-PHDF. In this paper we still adopt all of these assumptions. We further assume that the posterior intensity at scan $k$ is a GM given by

$$D_{k+1|k+1}(\mathbf{x}) = D_{k+1|k+1}^{ND}(\mathbf{x}) + \sum_{\mathcal{P}\angle Z_{k+1}} \sum_{W\in\mathcal{P}} \sum_{\theta} D_{k+1|k+1}^{D}(\mathbf{x}, \theta, W) \tag{33}$$

where the GM $D_{k+1|k+1}^{ND}(\mathbf{x})$ is used to handle the case of missing the detections in all modes. Based on (10), $D_{k+1|k+1}^{ND}(\mathbf{x})$ is given by

$$D_{k+1|k+1}^{ND}(\mathbf{x}) = \sum_{j=1}^{J_{k+1|k}} w_{k+1|k+1}^{(j)} \mathcal{N}(\mathbf{x}; m_{k+1|k+1}^{(j)}, P_{k+1|k+1}^{(j)}) \tag{34}$$

where

$$w_{k+1|k+1}^{(j)} = \left( \prod_{l=1}^{L} \left(1 - p_{D,k+1}^{l}\right) \right) \cdot w_{k+1|k}^{(j)} \tag{35}$$

and

$$m_{k+1|k+1}^{(j)} = m_{k+1|k}^{(j)}, \; P_{k+1|k+1}^{(j)} = P_{k+1|k}^{(j)} \tag{36}$$

Based on (11), $D_{k+1|k+1}^{D}(\mathbf{x}, \theta, W)$, which is used to calculate the PHD of the detected target, is given by

$$\begin{aligned} D_{k+1|k+1}^{D}(\mathbf{x}, \theta, W) &= \sum_{j=1}^{J_{k+1|k}} w_{k+1|k+1}^{(j)} \mathcal{N}(\mathbf{x}; m_{k+1|k+1}^{(j)}, P_{k+1|k+1}^{(j)}) && (37) \\ w_{k+1|k+1}^{(j)} &= \omega_{\mathcal{P}} \frac{\prod_{l=1}^{L} \left(1 - p_{D,k+1}^{l}\right) \prod_{\theta(l)>0} \frac{p_{D,k+1}^{l}}{1-p_{D,k+1}^{l}} \phi_W^{(j)}}{\kappa_W + \tau_W} w_{k+1|k}^{(j)} && (38) \\ \phi_W^{(j)} &= \mathcal{N}(\mathbf{z}_W; \mathbf{H}_W m_{k+1|k}^{(j)}, \mathbf{H}_W P_{k+1|k}^{(j)} \mathbf{H}_W^T + \mathbf{R}_W) && (39) \end{aligned}$$

The expression for the coefficient $\phi_W^{(j)}$ is calculated using

$$\mathbf{z}_W = \bigoplus_{\mathbf{z}_k \in W} \mathbf{z}_k,\ \mathbf{H}_W = \bigoplus_{\theta(l)>0} \mathbf{h}_{k+1}^l,\ \mathbf{R}_W = \textbf{blkdiag}\left(\bigoplus_{\theta(l)>0} R_{k+1}^l\right) \tag{40}$$

where the operation $\bigoplus$ denotes vertical vectorial concatenation, and $\mathbf{z}_k \in W$ denotes over all measurements $\mathbf{z}_k$ in the cell $W$. The operation **blkdiag**$(\cdot)$ denotes block diagonalization of a block vector.

In (39), to handle the nonlinear measurement function, the measurement function $\mathbf{h}_{k+1}^l$ is replaced with its corresponding Jacobian matrix $\mathbf{H}_{k+1}^l$. The partition weights, $\omega_{\mathcal{P}}$, which can be interpreted as the probability of partition $\mathcal{P}$ being true, are calculated by using (19).

The mean and covariance of the Gaussian components are updated using the standard Kalman measurement update as

$$m_{k+1|k+1}^{(j)} = m_{k+1|k}^{(j)} + \mathbf{K}(\mathbf{z}_W - \mathbf{H}_W m_{k+1|k}^{(j)}) \tag{41}$$

$$P_{k+1|k+1}^{(j)} = (\mathbf{I} - \mathbf{K}_{k+1}^{(j)} \mathbf{H}_W) P_{k+1|k}^{(j)} \tag{42}$$

$$\mathbf{K}_{k+1}^{(j)} = P_{k+1|k}^{(j)} \mathbf{H}_W^T (\mathbf{H}_W P_{k+1|k}^{(j)} \mathbf{H}_W^T + \mathbf{R}_W)^{-1} \tag{43}$$

Similar to [34], pruning and merging are used in the MD-GM-PHDF algorithm to keep the number of Gaussian components at a computationally tractable level.

### *B. MD-GM-PHD Tracker*

Because the standard GM-PHD filter [34] does not directly provide track state estimates or maintain track identities, a track display/management scheme is usually necessary to make the GM-PHDF a fully functional multitarget tracker. In this paper, the tag-based track management scheme proposed in [35] is used. The estimate of the track state is given by the weighted average of the states of all existing Gaussian components, (i.e., excluding newborn components at current scan) associated to that track. Based on the coefficients and several pre-defined threshold values, the following track management scheme is adopted. A track will be initialized when its coefficient goes beyond a threshold $th_I$. Then, an initialized (but unconfirmed) track will be confirmed when its coefficient goes beyond $th_C$. A confirmed track will be terminated once its coefficient falls below $th_{ck}$ for 3 scans. An unconfirmed track will be deleted when its coefficient is smaller than $th_{uk}$ for 2 scans.

### *C. Gaussian Component Initialization for Newborn Targets*

In this paper, a technique of placing Gaussian components for newborn targets is extended from (41) of [36], which originated from proposition 2 of [37]. The goal of the proposed method is to place the Gaussian components in a region of the state-space $\mathcal{X}$ where the likelihood values $g_k(\mathbf{z}|\mathbf{x})$ is large. The proposed method includes two steps:

1) The MD-GM-PHD of new targets, $D_{k+1}^b(\mathbf{x})$, is determined by using the posterior probability for a new-target-originated measurement $P_{k+1}(Y_i)$, which is calculated as

$$P_{k+1}(Y_i) = \frac{\tau_W}{\kappa_W + \tau_W},\ i = 1, ..., N_{z,k+1} \tag{44}$$

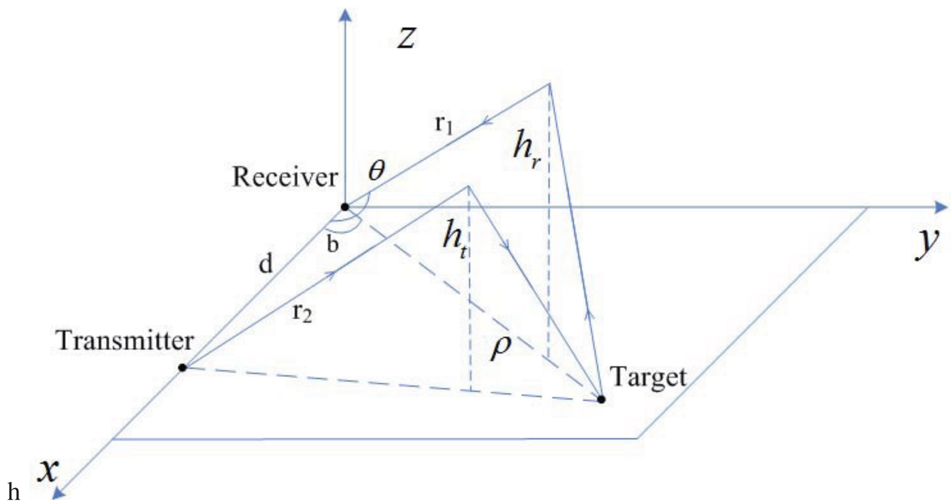


Fig. 1. OTHR transmitter-receiver-target geometry on the ground plane

where $Y_i$ denotes an $i$-th observed measurement in $Z_{k+1}$ and originates from a new target at scan $k+1$; $W$ denote all cells of partition with $Z_{k+1}$; $\kappa_W$ and $\tau_W$ are defined in (16) and (18), respectively. If $P_{k+1}(Y_i)$ for one measurement is bigger than the tuning threshold probability $\epsilon$, the corresponding measurement is determined to be new-target-originated [36].

2) Assume that there are $J$ number of new-target-originated measurements. They must be mapped to the state-space $\mathcal{X}$ to locate the newborn Gaussian components. Moreover, because of the measurement mode uncertainty, each selected measurement will initiate $L$ Gaussian components with the same tag numbers in $\mathcal{X}$, assume that there are $L$ measurement modes. Furthermore, because one new target may generate multiple detections, multiple Gaussian components with different tag number will be placed for one target, which will lead to tag ambiguity. To avoid this, the distance of any two newborn Gaussian component from different measurements with different tags is limited by a threshold $\eta$ as

$$|\mathbf{x}_{il} - \mathbf{x}_{jl'}| < \eta, \qquad i,j = 1,...,J \text{ and } l,l' = 1,...,L \qquad l \neq l' \tag{45}$$

where $\mathbf{x}_{i*}$ denotes the position of a newborn Gaussian component generated from measurement $i \in J$. Once (45) is satisfied, newborn Gaussian components with subscript $i$ and $j$ are deemed to have come from the same target, and will be merged to a single Gaussian component, i.e., with the position $\mathbf{x}_{il}$.

## VI. Simulations

### A. Multitarget Tracking in Ground Coordinates Using OTHR

In our simulations, multiple targets are tracked in ground coordinates using an OTHR. The OTHR target-radar geometry, as shown in Fig. 1, is similar to those in [9] [17], where a flat earth model is assumed for simplicity. The receiver is located at the origin and the transmitter is at a distance $d$ from it along the $x$-axis. The ray paths from the transmitter to the target and from the target to the receiver are assumed to have been reflected from two idealized ionospheric layer at virtual height $h_t$ and $h_r$, respectively. The reflection at the ionosphere is also assumed to be ideal, i.e., obeying the Snell's law. The ground range from the target to the receiver is $\rho$. Half the slant ranges from the receiver and the transmitter to target are $r_1$ and $r_2$, respectively. Azimuth on the ground plane is denoted by $b$ and the apparent azimuth $\alpha$ is given by $\alpha = \pi/2 - \theta$ [9].

The target dynamics in the ground plane is modeled using a nearly constant velocity (NCV) model [38]. The state of the target $\mathbf{x}_k$ at scan $k$ in the ground plane is $\mathbf{x}_k = [x(k), \dot{x}(k), y(k), \dot{y}(k)]^{\mathrm{T}}$, where $x(k)$, $y(k)$ and $\dot{x}(k)$, $\dot{y}(k)$, are the position and velocity of the target, along $x$ coordinate and $y$ coordinate, respectively. Then the state equation is given by

$$\mathbf{x}_{k+1} = \mathbf{F}_k \mathbf{x}_k + \mathbf{v}_k \tag{46}$$

where $\mathbf{F}_k$ is the state transition matrix, $\mathbf{v}_k$ is a zero-mean white Gaussian process noise sequence with covariance matrix $\mathbf{Q}_k$ that models maneuvers of the target. Also, $\mathbf{F}_k$ and $\mathbf{Q}_k$ are given by

$$\mathbf{F} = \mathbf{I}_2 \bigotimes \begin{bmatrix} 1 & T_k \\ 0 & 1 \end{bmatrix}, \mathbf{Q} = \mathbf{I}_2 \bigotimes \begin{bmatrix} q\frac{T_k^3}{3} & q\frac{T_k^2}{2} \\ q\frac{T_k^2}{2} & qT_k \end{bmatrix} \tag{47}$$

where $T_k$ is the time between scans $k$ and $k+1$ and $q$ is the power spectral density of the process noise. Also, $\mathbf{I}_2$ denotes a $2 \times 2$ identity matrix and $\bigotimes$ denotes the Kronecker product.

In this paper, we only consider two dominant ionospheric layers, as those in [17]. Thus, there are four possible multipath propagation modes $(EE, EF, FE, FF)$. Let vector $\mathbf{z}_k = [z_r(k), z_{\dot{r}}(k), z_\alpha(k)]$ denote the OTHR measurement at scan $k$, which consists of half the slant range, range-rate and apparent azimuth, respectively. They are related to the target state and propagation modes $l = 1, ..., L$ through following nonlinear equation:

$$\mathbf{h}^l(\mathbf{x}_k) = \begin{pmatrix} r_{1,l}(k) + r_{2,l}(k) \\ \frac{\dot{\rho}(k)}{4}\left(\frac{\rho(k) r_{1,l}(k)}{r_{2,l}(k)} + \frac{\rho(k) - d \sin b(k)}{r_{2,l}(k)}\right) \\ \sin^{-1}\left(\frac{\rho(k)\sin b(k)}{2 r_{1,l}(k)}\right) \end{pmatrix} \tag{48}$$

where $r_{1,l}(k)$ and $r_{2,l}(k)$ are defined as [17]

$$r_{1,l}(k) = \sqrt{\left(\frac{\rho(k)}{2}\right)^2 + h_{r,l}^2} \tag{49}$$

$$r_{2,l}(k) = \sqrt{\left(\frac{\rho(k)}{2}\right)^2 - d(\frac{\rho(k)}{2})\sin b(k) + \left(\frac{d}{2}\right)^2 + h_{t,l}^2} \tag{50}$$

where $\rho(k) = \sqrt{x(k)^2 + y(k)^2}$ is the ground range from the receiver and $b(k) = \tan^{-1}(y(k)/x(k))$ is the bearing to the target from the receiver. Here, $h_{r,l}$ and $h_{t,l}$ denote the heights of the transmit and receive ionospheric layers, respectively.

An extended Kalman version of the GM-MD-PHDF is implemented in this simulation. The corresponding Jacobian matrix of (48) is given in [17].

To initialize the Gaussian components for a newborn target, an inverse mapping of the OTHR measurements to ground coordinates is required. Such a mode-dependent mapping is given by [17]

$$\rho(k) = 2\sqrt{r_{1,l}(k)^2 - h_{r,l}^2} \tag{51}$$

$$\dot{\rho}(k) = \frac{4\dot{r}_r(k)}{\rho(k)/r_{1,l}(k) + (\rho(k) - d\sin(b))/r_{2,l}(k)} \tag{52}$$

$$b(k) = \sin^{-1}\left(\frac{2r_{1,l}(k)\sin(\theta(k))}{\rho(k)}\right) \tag{53}$$

where

$$r_{1,l} = \frac{r_r(k)^2 + h_{r,l}^2 - h_{t,l}^2 - (d/2)^2}{2r_r(k) - d\sin(\theta(k))} \tag{54}$$

$$r_{2,l} = r_r(k) - r_{1,l}(k) \tag{55}$$

### B. Performance Evaluation

Two Monte Carlo simulation scenarios, each with 100 trials, are used to verify the performance of the MD-GM-PHDF algorithm proposed above. In the first scenario, it is assumed that the number of targets remains constant. The proposed MD-GM-PHDF's performance is compared with that of the MD-JPDAF [9]. In the second scenario, two targets appear and disappear at varied times, and the performance of the MD-GM-PHDF is compared with that of the classic GM-PHDF.

In both scenarios, the surveillance region was $[500, 1200]$ km $\times$ $[500, 1200]$ km. The number of clutter points was assumed to be Poisson distributed with mean $\lambda = 5$ in each scan, The clutter measurements were generated by first generating uniformly distributed spurious locations in the surveillance region and then generating measurements from these locations. For the slant range rate, the clutter values were generated from a uniform probability distribution $\mathcal{U}(0.01, 0.2)$ km/s.

The power spectral density of the process noise $q$ was set to $1e{-}8$ km$^2$/s$^3$ in both $x$ and $y$ directions. The transmitter of the OTHR is located at $(1, 0)$ km and the receiver is located at the origin of the coordinate system. The scan duration of the radar was assumed to be 20s. The standard deviation of the bearing, slant range and range-rate measurement noise are $\sigma_b = 0.004$ rad, $\sigma_r = 0.1$ km, and $\sigma_{\dot{r}} = 0.005$ km/s, respectively.

For the newborn Gaussian components, the velocity components are set to zero. The initial variance of the speed in both $x$ and $y$ directions is set to $V_{max}^2/3$, where the maximum velocity $V_{max}$ is assumed to be $0.5$ km/s. The target survival probability is $0.98$ and the expected number of newborn targets is $0.03$ for both scenarios. The parameters for tracker management are $th_I = 0.3, th_C = 0.6, th_{ck} = 0.008$ and $th_{uk} = 0.1$, respectively.

*1) Simulation for the constant target number case:* In this scenario, there are two targets whose initial states are $(850, 670)$ km and $(855, 655)$ km, respectively. Their initial velocities are $(0.25, 0.2)$ km/s and $(0.25, -0.3)$ km/s, respectively. A total of 30 scans of measurements were simulated. The mode-dependent probabilities of target detection are $P_{D_{EE}} = P_{D_{FF}} = 0.7$ and $P_{D_{EF}} = P_{D_{FE}} = 0.8$.

Fig. 2 shows the average position Root Mean Squared Error (RMSE) of MD-GM-PHDF and MD-JPDAF. As shown in this figure, there is a notable performance gain with MD-GM-PHDF over MD-JPDAF. The reason is that

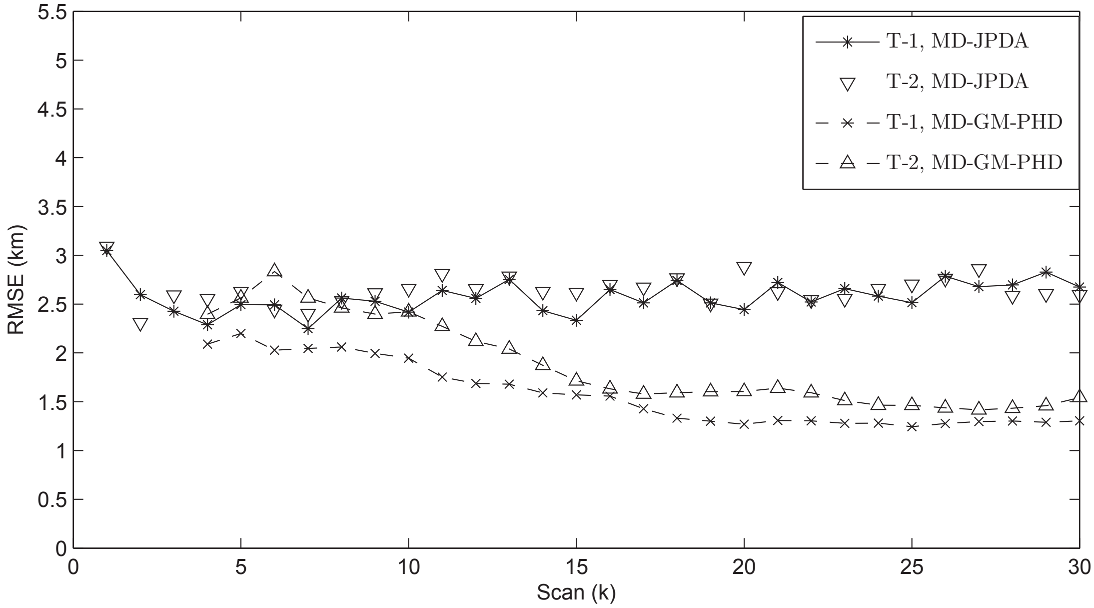


Fig. 2. Position RMSE for MD-PHD filter and MD-JPDAF with OTHR data

the MD-GM-PHDF encapsulates all information about the target by enumerating all of the association between modes and measurements.

*2) Simulation for the dynamic target number case:* In this scenario, target 1 appears at scan $k = 2$ with initial state $[850 \text{ km}, -0.25 \text{ km/s}, 670 \text{ km}, 0.2 \text{ km/s}]^{\mathrm{T}}$ and disappears at scan $k = 36$. Target 2 appears at $k = 6$ with initial state $[855 \text{ km}, 0.25 \text{ km/s}, 655 \text{ km}, -0.3 \text{ km/s}]^{\mathrm{T}}$ and disappears at $k = 46$. In each of the 100 Monte Carlo trials, 50 scans of measurements were simulated. The mode dependent target detection probabilities are set as $P_{D_{EE}} = P_{D_{FF}} = 0.4$ and $P_{D_{EF}} = P_{D_{FE}} = 0.5$.

As shown in Fig. 3, the Monte Carlo averaged estimate of the number of targets from MD-GM-PHDF accurately follows the true values after a two-scan delay. On the other hand, the GM-PHDF cannot achieve this, but it can respond target disappearances well. Fig. 4 shows the Monte Carlo average of the optimal subpattern assignment (OSPA) distance [39] with order $p = 2$ and cut-off $c = 25$. The peaks from the OSPA with MD-GM-PHDF at scan index $2, 6, 36, 46$, correspond to the emergence and disappearance of targets, respectively. The high OSPA distance of GM-PHDF illustrates that the GM-PHDF cannot track the target state correctly even at scans where target number is accurately estimated.

The results in this scenario confirm that the proposed MD-GM-PHDF can accurately estimate the dynamic target number and improve the target state estimates when compared with the GM-PHDF.

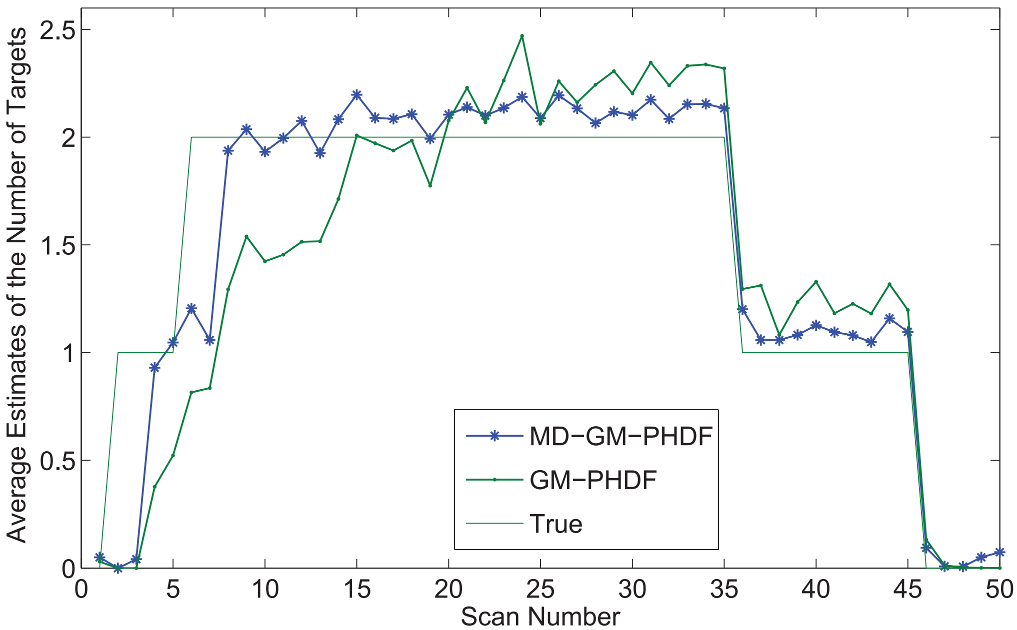


Fig. 3. Mente Carlo Average Estimates of the Number of Targets

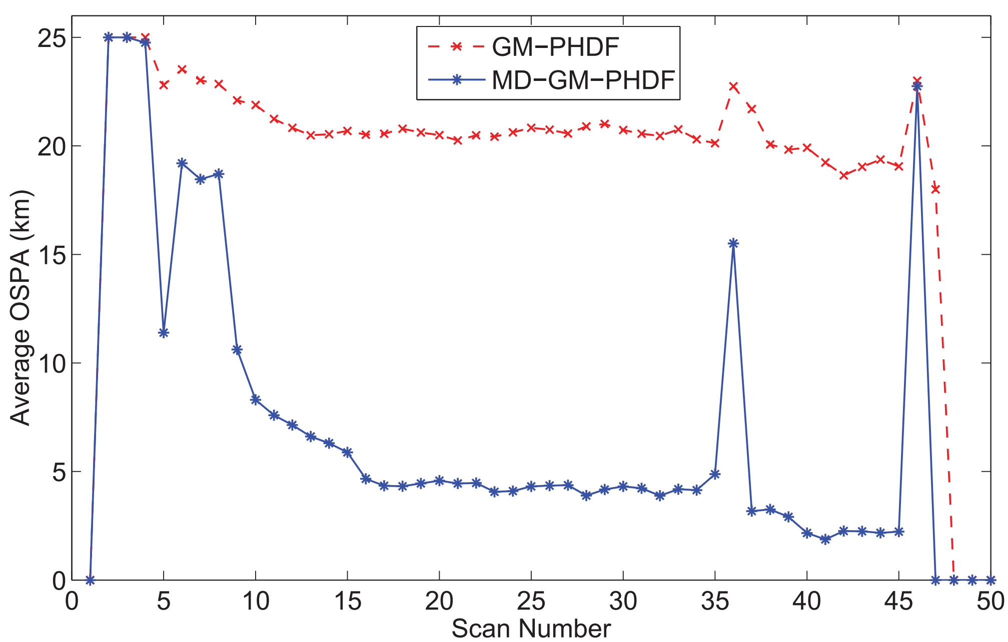


Fig. 4. Monte Carlo average of the OSPA distance

## VII. Conclusion

In this paper, the multitarget tracking problem where one target generates multiple detections was addressed under the PHD filter framework. The general MD-PHD measurement-update equation was derived without any approximation beyond those made in the standard PHD. We demonstrated that the proposed MD-PHD recursion can be a generalized solution for a wide range of multiple detection problems. It was applied as a multipath PHD, an extended target PHD, and a multisensor PHD in this paper. It was also extended to the multisensor multi-detection case. Furthermore, a Gaussian mixture implementation of the proposed MD-PHD, denoted as the MD-GM-PHDF, was presented. Its performance was demonstrated on a classical multi-detection scenario for OTHR multitarget tracking. Simulation results showed that the MD-GM-PHDF not only outperforms the recently published MD-JPDAF in a scenario with a constant number of targets but also successfully tracks multiple time-varying number of targets.

Although the proposed MD-PHD filter has high computational complexity, it is suitable for tracking a few high-value targets under difficult conditions. This is the case in typical OTHR scenarios. Thus, further work will focus on reducing the computational complexity or casting it to parallel processing systems, such as the general purpose graphical processing unit (GPGPU).

## Appendix

### Derivation of the Multi-Detection Likelihood Function of a Single-Target

Based on the formulas of FISST and the relationship between the probability density function and the belief-mass function defined in (12.1) of [23], we have

$$\begin{aligned} f_{k+1}(W|\mathbf{x}) &\triangleq \frac{\delta\beta_{k+1}}{\delta W}(\emptyset|\mathbf{x}) \\ &= \sum_{V_1\uplus\ldots\uplus V_m=W} \frac{\delta p_1}{\delta V_1}(\emptyset)\ldots\frac{\delta p_m}{\delta V_m}(\emptyset) \end{aligned} \tag{56}$$

However,

$$\frac{\delta p_l}{\delta \emptyset}(\emptyset) = p_l(\emptyset) = 1 - p_D^l(\mathbf{x}) \tag{57}$$

$$\frac{\delta p_l}{\delta \{\mathbf{z}\}}(\emptyset) = p_D^l(\mathbf{x}) \cdot f_{k+1}^l(\mathbf{z}|\mathbf{x}) \tag{58}$$

$$\frac{\delta p_l}{\delta V}(\emptyset) = 0 \text{ if } |V| > 1. \tag{59}$$

The only surviving terms in (56) are those for which $|V_1| \leq 1, ..., |V_m| \leq 1$.

If $W = \emptyset$, then

$$\begin{aligned} f_{k+1}(\emptyset|\mathbf{x}) &= \frac{\delta p_1}{\delta\emptyset}(\emptyset)\ldots\frac{\delta p_L}{\delta\emptyset}(\emptyset) = p_1(\emptyset)\ldots p_L(\emptyset) \\ &= \prod_{l=1}^{L}\left(1 - p_D^l(\mathbf{x})\right) \end{aligned} \tag{60}$$

If $W \neq \emptyset$, then

$$\begin{aligned} f_{k+1}(W|\mathbf{x}) &= f_{k+1}(\emptyset|\mathbf{x}) \cdot \sum_{V_1 \uplus \ldots \uplus V_m = W} \frac{\frac{\delta p_1}{\delta V_1}(\emptyset) \ldots \frac{\delta p_L}{\delta V_L}(\emptyset)}{\prod_{l=1}^{L}(1 - p_D^l(\mathbf{x}))} \\ &= f_{k+1}(\emptyset|\mathbf{x}) \cdot \sum_{1 \leq l_1 \neq \ldots \neq l_m \leq L} \frac{\frac{\delta p_{l_1}}{\delta \mathbf{z}_1}(\emptyset) \ldots \frac{\delta p_{l_m}}{\delta \mathbf{z}_m}(\emptyset)}{(1 - p_D^{l_1}(\mathbf{x})) \cdots (1 - p_D^{l_m}(\mathbf{x}))} \\ &= f_{k+1}(\emptyset|\mathbf{x}) \cdot \sum_{1 \leq l_1 \neq \ldots \neq l_m \leq L} \prod_{j=1}^{m} \frac{p_D^{l_j}(\mathbf{x}) \cdot f_{k+1}^{l_j}(\mathbf{z}_j|\mathbf{x})}{1 - p_D^{l_j}(\mathbf{x})} \\ &= L! \cdot f_{k+1}(\emptyset|\mathbf{x}) \cdot \sum_{1 \leq l_1 \leq \ldots \leq l_m \leq L} \prod_{j=1}^{m} \frac{p_D^{l_j}(\mathbf{x}) \cdot f_{k+1}^{l_j}(\mathbf{z}_j|\mathbf{x})}{1 - p_D^{l_j}(\mathbf{x})} \end{aligned} \tag{61}$$

where each $m$-tuple $\{l_1, ..., l_m\}$ with $1 \leq l_1 \neq ... \neq l_m \leq L$ determines a one-to-one function $\tau : \{1, ..., m\} \to \{1, ..., L\}$ by $\tau(j) = l_j$. For each $m$-tuple, define the association

$$\theta : \{1, ..., L\} \to \{0, 1, ..., m\}$$

with $\theta(l) = \tau^{-1}(l)$ if $l$ is in the image of $\tau$, and $\theta(l) = 0$ otherwise. Then the $m$-tuple is in a one-to-one correspondence with association $\theta$. It is the same with those defined in Section 10.5.4 of [23]. Now, from (61) we can get a similar form as (11) in Subsection III-B

$$\begin{aligned} f_{k+1}(W|\mathbf{x}) &= f_{k+1}(\emptyset|\mathbf{x}) \sum_{\theta} \prod_{\theta(l)>0} \frac{p_D^l(\mathbf{x}) \cdot f_{k+1}^l(\mathbf{z}_{\theta(l)}|\mathbf{x})}{1 - p_D^l(\mathbf{x})} \\ &= \prod_{l=1}^{L} \left(1 - p_D^l(\mathbf{x})\right) \sum_{\theta} \prod_{\theta(l)>0} \frac{p_D^l(\mathbf{x}) \cdot f_{k+1}^l(\mathbf{z}_{\theta(l)}|\mathbf{x})}{1 - p_D^l(\mathbf{x})} \end{aligned} \tag{62}$$

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