Analysis of Maritime Air Defence Scenarios

Bao Nguyen and Suruz Miah
Centre for Operational Research and Analysis
Defence Research and Development Canada
Ottawa, Ontario, Canada
E-mails: {Bao.Nguyen,Suruz.Miah}@drdc-rddc.gc.ca

***Abstract*—In this paper, we examine scenarios of maritime air defence where the threats can be of different types *i.e.*, characterized by varying different single shot probabilities of hit. We develop a methodology that integrates constrained optimization calculus, dynamic programming and a genetic algorithm to determine the optimal allocation of interceptors that globally maximize the probability of raid annihilation. We call this strategy the Heterogeneous tactic. We show that the improvement in the probability of raid annihilation could exceed thirty percent for one scenario examined, relative to a Shoot-Look-Shoot tactic with fixed size salvos.**

I. Introduction

Maritime air defence usually consists of several layers such as force weapon soft-hit distraction chaff, force weapon hard-hit medium range surface-to-air missiles (MR SAMs), self-defence hard-hit short range surface-to-air missiles (SR SAMs), self-defence soft-hit seduction chaff and close-in weapon systems (CIWS).

There are up to five air defence layers, normally operating in the sequence described above. Escorts and aircraft carriers generally have SR SAM, self-defence soft-hit weapons, and CIWS. Anti-Air Warfare (AAW) ships usually have soft-hit weapons, MR SAM, and CIWS. Protected high value units sometimes have no air defence capability.

To evaluate maritime air defence systems capability and to identify potential shortfalls of air defence technologies, we present an optimal layer maritime air defence engagement model and compare its performance to existing, well–known tactics. It is common knowledge that threat missiles come with different cross sections, different sizes etc. Hence, the defence's single shot probability of hit ($SSPH$) against each type of threat will surely be different. Therefore, in this paper, we consider an optimal engagement model against a number of threats each of which is possibly associated with a different $SSPH$. However, we assume that the defence has the same number of engagement opportunities to engage those threats.

The open literature abounds with engagement tactics against missile threats e.g. [1], [2]. However, most of them assume identical threats and hence identical $SSPH$s. For example, [2] provides a globally optimal methodology that can be used to maximize the probability of raid annihilation (PRA). By deriving the closed form expressions for the optimal allocation of interceptors at each layer of defence i.e. each engagement opportunity, and combining with dynamic programming, the methodology in [2] yields the globally optimal solution in an efficient way. This efficiency is due to the assumption that all threats are identical.

Even though, in our case, the threat types are possibly different, the framework described in [2] can still be used to develop an optimal engagement tactic. Note that we consider the defence of a single platform or that of a perfectly coordinated task group. Multiple platforms defence is examined in [3] and partially coordinated defence in [4]. When the number of threats is small, five or less, dynamic programming can provide the optimal allocation in less than a minute on a standard personal computer (Intel(R) Core (TM)i7–3520M CPU 2.9GHz). However, when the number of threats is large, ten or more, we combine a genetic algorithm with dynamic programming to obtain a near optimal solution.

This paper is organized as follows. In section II, we prove that the PRA is a concave function in terms of number of interceptors allocated to the threats. Based on concavity, in section III we determine the globally optimal and continuous allocations of interceptors that maximize the PRA. Using perturbation theory, in section IV we derive approximate formulas for the optimal allocations of interceptors. With the genetic algorithm, we find the integer solutions to the interceptor allocations in Section V. We provide numerical results in Section VI. In Section VII, we summarize our findings.

II. Concavity of PRA

In general, it can be challenging to determine a globally optimal solution to a multi–variable problem or even a single variable one. But if the objective function is concave, this task becomes feasible [5]. It turns out that the PRA is a concave function of the number of interceptors allocated to incoming threats. For one defensive layer, the PRA can be written as:

$$PRA = (1 - M_1^{x_1}) \dots \left(1 - M_{a-1}^{x_{a-1}}\right)\left(1 - M_a^{x_a}\right), \quad (1)$$

where x_i, $i = 1, \dots, a$, is the number of interceptors allocated to the ith threat; M_i is the single shot probability of miss ($SSPM$) against the ith threat; a is the number of threats and $x_1 + \dots + x_a = n$, with n the number of interceptors. In this section, we will prove the concavity of the PRA. We will later use this property to find a globally optimal solution.

Theorem 1: The PRA given in Eq. (1) is concave.

Proof: If a real and continuous function f satisfies the inequality below for a particular domain of interests then f is concave in that domain [5].

$$f\left(\frac{x+y}{2}\right) \geq \frac{1}{2}\left[f(x)+f(y)\right]. \quad (2)$$

For one threat, the PRA can be written as:

$$PRA = 1 - M^n, \quad (3)$$

where M is the single shot miss probability. That is, $M = 1-K$ where K is the $SSPH$. We establish the case of one threat below:

$$1 - M^{\frac{x+y}{2}} \geq \frac{1}{2}\left\{(1-M^x)+(1-M^y)\right\} \quad (4)$$

$$2\left(1 - M^{\frac{x+y}{2}}\right) \overset{?}{\geq} \left\{(1-M^x)+(1-M^y)\right\} \quad (5)$$

$$\Rightarrow \left(M^{x/2} - M^{y/2}\right)^2 \geq 0.$$

Assuming that the condition (2) is satisfied for a threats, we will show that for $a+1$ threats:

$$\begin{aligned} &PRA = \\ &\left(1 - M_1^{\frac{x_1+y_1}{2}}\right)\left(1 - M_2^{\frac{x_2+y_2}{2}}\right)\cdots\left(1 - M_{a+1}^{\frac{x_{a+1}+y_{a+1}}{2}}\right) \\ &\geq \frac{1}{2}(1-M_1^{x_1})(1-M_2^{x_2})\cdots(1-M_{a+1}^{x_{a+1}}) + \\ &\frac{1}{2}(1-M_1^{y_1})(1-M_2^{y_2})\cdots(1-M_{a+1}^{y_{a+1}}) \quad (6) \end{aligned}$$

where

$$\frac{1}{2}\sum_{i=1}^{a+1}(x_i+y_i) = n$$

with a the number of threats and n the number of interceptors. Rearranging right–hand side of Eqn (6) we get:

$$\begin{aligned} &\left(1 - M_1^{\frac{x_1+y_1}{2}}\right)\left(1 - M_2^{\frac{x_2+y_2}{2}}\right)\cdots\left(1 - M_{a+1}^{\frac{x_{a+1}+y_{a+1}}{2}}\right) \geq \\ &\frac{1}{2}\left\{(1-M_1^{x_1})\cdot\ldots\cdot(1-M_a^{x_a}) + (1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\right\}\cdot \\ &\frac{1}{2}\left\{(1-M_{a+1}^{x_{a+1}}) + (1-M_{a+1}^{y_{a+1}})\right\} \quad (7) \end{aligned}$$

If the right hand side of Eqn (7) is greater than or equal to the right hand side of Eqn (6) then the induction is complete. That is,

$$\begin{aligned} &\frac{1}{4}\left\{(1-M_{a+1}^{x_{a+1}}) + (1-M_{a+1}^{y_{a+1}})\right\}\cdot \\ &\left\{(1-M_1^{x_1})\cdot\ldots\cdot(1-M_a^{x_a}) + (1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\right\} \\ &\overset{?}{\geq} \frac{1}{2}(1-M_1^{x_1})\cdot\ldots\cdot(1-M_{a+1}^{x_{a+1}}) + \\ &\frac{1}{2}(1-M_1^{y_1})\cdot\ldots\cdot(1-M_{a+1}^{y_{a+1}}) \quad (8) \end{aligned}$$

or

$$\begin{aligned} &\{(1-M_1^{x_1})\cdot\ldots\cdot(1-M_a^{x_a}) + \\ &(1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\}\cdot\left\{(1-M_{a+1}^{x_{a+1}}) + (1-M_{a+1}^{y_{a+1}})\right\} \\ &\overset{?}{\geq} 2(1-M_1^{x_1})\cdot\ldots\cdot(1-M_{a+1}^{x_{a+1}}) + \\ &2(1-M_1^{y_1})\cdot\ldots\cdot(1-M_{a+1}^{y_{a+1}}) \quad (9) \end{aligned}$$

or

$$\begin{aligned} &(1-M_1^{x_1})\cdot\ldots\cdot(1-M_a^{x_a})\cdot(1-M_{a+1}^{y_{a+1}}) + \\ &(1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\cdot(1-M_{a+1}^{x_{a+1}}) \\ &\overset{?}{\geq} (1-M_1^{x_1})\cdot\ldots\cdot(1-M_{a+1}^{x_{a+1}}) + \\ &(1-M_1^{y_1})\cdot\ldots\cdot(1-M_{a+1}^{y_{a+1}}) \quad (10) \end{aligned}$$

Simple algebra leads to

$$\begin{aligned} &(1-M_1^{x_1})\cdot\ldots\cdot(1-M_a^{x_a})\cdot(M_{a+1}^{x_{a+1}} - M_{a+1}^{y_{a+1}}) + \\ &(1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\cdot(M_{a+1}^{y_{a+1}} - M_{a+1}^{x_{a+1}}) \overset{?}{\geq} 0 \end{aligned}$$

Note the symmetric nature of Eqn (2). That is, if we interchange x and y then Eqn (2) remains unchanged. Therefore, we can always choose $x \leq y$. Since if not then we interchange x and y. Hence, we can choose $x_{a+1} \geq y_{a+1}$ and $x_i \leq y_i$ for $i = 1, \ldots, a$. This establishes the inequality in Eqn (6):

$$\begin{aligned} &\left\{(1-M_1^{x_1})\cdot\ldots(1-M_a^{x_a}) - (1-M_1^{y_1})\cdot\ldots\cdot(1-M_a^{y_a})\right\}\cdot \\ &\left(M_{a+1}^{x_{a+1}} - M_{a+1}^{y_{a+1}}\right) \geq 0. \quad (11) \end{aligned}$$

■

Example 1: Let the number of threats be two ($a = 2$), the number of interceptors be six ($n = 6$) , $SSPMs$ $M_1 = 0.8$, and $M_2 = 0.6$ then

$$PRA = (1-M_1^{x_1})(1-M_2^{n-x_1}) \quad (12)$$

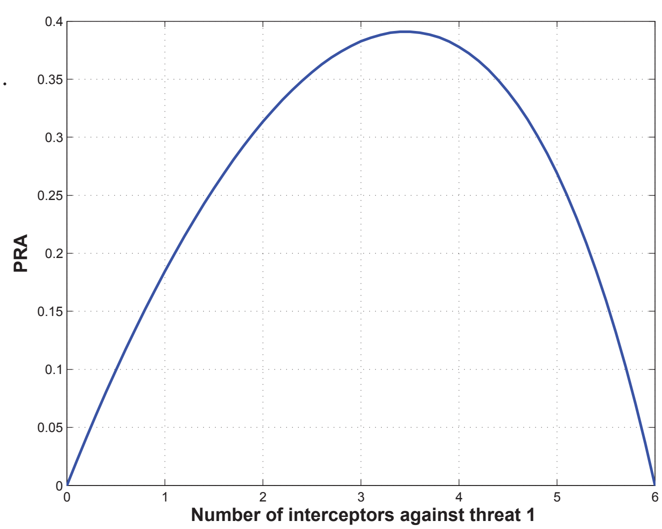

Fig. 1: PRA as a function of number of interceptors against threat 1.

Figure 1 displays the concavity of the PRA. Clearly, there is exactly one maximum. For a continuous solution, this maximum occurs at $x_1 = 3.45$ ($x_2 = n - x_1 = 2.55$) interceptors which is between three and four interceptors. For a continuous concave function, a local maximum corresponds to the global maximum [5]. To find the global maximum then requires only to search the number of interceptors against threat 1 that yield a zero derivative. For an integer solution, we can compare the PRA evaluated at integers that are in the neighbourhood of the continuous solution. Although, it is not obvious from Figure 1, the PRA at three interceptors (0.382592) is numerically greater than the one at four interceptors (0.377856). Therefore, assigning three interceptors against threat 1 (*i.e.*, $x_1 = 3$) and three interceptors against threat 2 (*i.e.*, $x_2 = n - x_1 = 3$) yields the PRA superior to 2 and 4 interceptors, respectively.

III. Continuous Optimal Solution to The PRA

In this section, we aim to maximize the PRA. This is a constrained optimization problem.

$$PRA(\vec{x}) = (1 - M_1^{x_1})(1 - M_2^{x_2}) \dots (1 - M_a^{x_a}) \quad (13)$$

where $\vec{x} = [x_1, x_2, \dots, x_a]$ denotes the row vector, $x_1 + \dots + x_a = n = T(\vec{x})$, x_i is the number of interceptors assigned to threat i ($i = 1, \dots, a$) and n is the number of interceptors to be deployed. To optimize the PRA, we make use of Lagrange multipliers [6]:

$$\frac{\partial PRA}{\partial \vec{x}} = \lambda \frac{\partial T}{\partial \vec{x}} \quad (14)$$

Performing the derivatives, we get for $i = 1, \dots, a$

$$-\ln(M_i) M_i^{x_i} \frac{PRA}{1 - M_i^{x_i}} = \lambda. \quad (15)$$

For example, with three threats, there are three equations (we have redefined λ as $-\lambda$ for convenience)

$$\begin{aligned} \ln(M_1) M_1^{x_1} PRA &= \lambda(1 - M_1^{x_1}) \\ \ln(M_2) M_2^{x_2} PRA &= \lambda(1 - M_2^{x_2}) \\ \ln(M_3) M_3^{x_3} PRA &= \lambda(1 - M_3^{x_3}) \end{aligned} \quad (16)$$

For the last equation, $x_3 = n - x_1 - x_2$, or:

$$\begin{aligned} \ln(M_3) M_3^{n - x_1 - x_2} PRA &= \lambda(1 - M_3^{n - x_1 - x_2}) \\ \ln(M_3) M_3^{n} PRA &= \lambda(M_3^{x_1 + x_2} - M_3^{n}) \end{aligned} \quad (17)$$

Dividing the second equality by the first equality of Eqn (16), we get

$$\frac{\ln(M_1) M_1^{x_1}}{\ln(M_2) M_2^{x_2}} = \frac{1 - M_1^{x_1}}{1 - M_2^{x_2}}$$

Solving for x_2, we get

$$x_2 = g_2(x_1, M_1) = \frac{1}{\ln(M_2)} \ln\left\{ \frac{\ln(M_1) M_1^{x_1}}{\ln(M_1/M_2) M_1^{x_1} + \ln(M_2)} \right\}$$

Generally,

$$x_i = g_i(x_1, M_i) = \frac{1}{\ln(M_i)} \ln\left\{ \frac{\ln(M_1) M_1^{x_1}}{\ln(M_1/M_i) M_1^{x_1} + \ln(M_i)} \right\} \quad (18)$$

for $i = 2, \dots, a-1$ and

$$x_a = n - x_1 - \dots - x_{a-1} \quad (19)$$

This means that if we could solve for x_1 then we can obtain all other x_i ($i = 2, \dots, a-1$) through the functions g_i and x_a through Eqn (19). It is indeed the case. We can divide the first equality of Eqn (17) by the first equality of Eqn (16) and simplify to obtain:

$$F(x_i) = \ln(M_1) M_1^{x_1} \left(1 - M_3^{n - x_1 - g_2(x_1, M_2)}\right) - \ln(M_3) M_3^{n - x_1 - g_2(x_1, M_2)} (1 - M_1^{x_1}) = 0 \quad (20)$$

We observe that there is only one unknown x_1 in Eqn (20). Due to the concavity of the objective function PRA, there is exactly one solution for x_1. In general, we have to solve

$$\ln(M_1) M_1^{x_1} \left(1 - M_a^{n - x_1 - g_2(x_1, M_2) - \dots - g_{a-1}(x_1, M_{a-1})}\right) - \ln(M_a) M_a^{n - x_1 - g_2(x_1, M_2) - \dots - g_{a-1}(x_1, M_{a-1})} (1 - M_1^{x_1}) = 0 \quad (21)$$

We observe that

$$g_i(0, M_1) = \frac{1}{\ln(M_i)} \ln\left\{ \frac{\ln(M_1)}{\ln(M_1)} \right\} = \frac{1}{\ln(M_i)} \ln\{1\} = 0 \quad (22)$$

Hence,

$$F(0) = \ln(M_1)(1 - M_a^n) < 0 \quad (23)$$

if $0 < M_a < 1$ and $n > 0$.
Similarly,

$$F(n) = -\ln(M_a)(1 - M_1^n) > 0 \quad (24)$$

Due to the concavity of the PRA, we know that there is only one value of x_1 that satisfies Eqn (21). In addition, we know from (23) that $F(0) < 0$ while $F(n) > 0$ (see Eqn (24)) This means that x_1 lies between zero and n. This is an ideal situation to use the bisection method to solve for x_1, [7]:

"The bisection method is one that cannot fail...The idea is simple. Over some interval the function is known to pass through zero because it changes sign. Evaluate the function at the interval's midpoint and examine its sign. Use the midpoint to replace whichever limit has the same sign. After each iteration the bounds containing the root decrease by a factor of two. If after n iterations the root is known to be within an interval of size ε_n, then after the next iteration it will be bracketed within an interval of size $\varepsilon_{n+1} = \varepsilon_n/2$ neither more nor less. Thus, we know in advance the number of iterations required to achieve a given tolerance in the solution

$n = \log_2(\varepsilon_0/\varepsilon)$ where ε_0 is the size of the initially bracketing interval, ε is the desired ending tolerance."

Example 2: We use the same scenario as the one in Example 1. That is, there are two threats, $a = 2$, six interceptors, $n = 6$; $SSPM$ against threat 1 is 80 percent, $M_1 = 0.8$, and $SSPM$ against threat 2 is 60 percent, $M_2 = 0.6$. F is plotted as a function of (number of interceptors assigned to threat 1) in Figure 2. It is seen that F is a monotonously increasing function of x_1 and F crosses the x axis exactly once at $x_1 = 3.45$ $(x_2 = n - x_1 = 2.55.)$ This corresponds to the maximum PRA in Example 1. As indicated above, the bisection methodology cannot fail for a concave function such as the PRA. Therefore, this is a very robust solution.

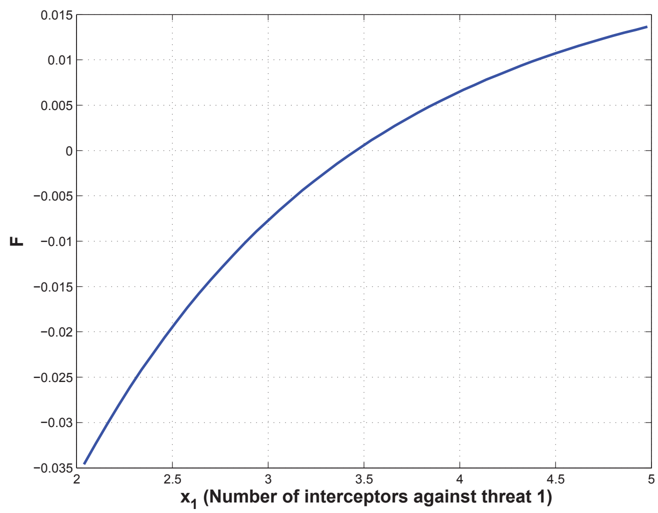

Fig. 2: F as a function of x_1 (number of interceptors against threat 1).

IV. Perturbation

The bisection methodology described in the previous section is totally robust. However, we cannot see the analytical dependency of x_1 on the $SSPM$. In this section, we make use of the perturbation theory to approximate x_1 as a function of the $SSPM$ up to the third order. That is, [8]:

$$x_1 = b_0 + \sum_i b_i\alpha_i + \frac{1}{2!}\sum_{i,j} b_{ij}\alpha_i\alpha_j + \frac{1}{3!}\sum_{i,j,k} b_{ijk}\alpha_i\alpha_j\alpha_k \quad (25)$$

where $\alpha_i = 1 - M_i/M_1$ and the $SSPM$ are sorted in decreasing orders i.e. $M_1 \geq M_2 \geq \ldots \geq M_a$ so that $0 \leq \alpha_i \leq 1$ for $i = 2, \ldots, a$.

We summarize the results below, [9], [10].

$$\beta_1 = \ln(M_1) \quad (26)$$

$$s = -\frac{1}{a\beta_1^2}\left(1 - e^{\beta_1 n/a} + \beta_1 n/a\right) \quad (27)$$

$$t = -\frac{1}{a\beta_1^2} \times \{\cdot\}$$

$$\{\cdot\} = \frac{\beta_1}{2a^2}(n^2 + na) + \beta_1^2 ns - \beta_1(a-1)s + \frac{\beta_1^3}{2}n^2s^2 + \frac{1}{2}(1 - 2s\beta_1 e^{\beta_1 n/a} - e^{\beta_1 n/a}) \quad (28)$$

$$u = -\frac{2s}{a\beta_1}\left(1 - e^{\beta_1 n/a}\right) \quad (29)$$

$$b_i = s \quad (30)$$

$$b_{ij} = \frac{-2s}{a\beta_1}\left(1 - e^{\beta_1 n/a}\right), \quad \text{for } i \neq j. \quad (31)$$

$$b_{ii} = 2t \quad (32)$$

$$b_{ijk} = \frac{-3}{a\beta_1}\left(u - e^{\beta_1 n/a}(s^2\beta_1 + u)\right) \quad (33)$$

for i, j, k distinct.

$$b_{iij} = \frac{1}{a^2\beta_1^2} \times \{\cdot\}^*$$

$$\{\cdot\}^* = 2s(2-a)\left(1 - e^{\beta_1 n/a}\right)^2 - sa\beta_1\left(1 - e^{\beta_1 n/a}\right) 3s^s a\beta_1^2 e^{\beta_1 n/a} - 2ta\beta_1\left(1 - e^{\beta_1 n/a}\right) \quad (34)$$

with i, j distinct.

$$b_{iii} = b_{iii}^{(1)} + b_{iii}^{(2)} \quad (35)$$

where

$$b_{iii}^{(1)} = -\frac{1}{a\beta_1^4} \times \{\cdot\}^{**}$$

$$\{\cdot\}^{**} = 11 - 2e^{3\beta_1 n/a} + 9\beta_1 + 2\beta_1^2 + 6s\beta_1^2 + 3s\beta_1^3 + 6t\beta_1^3 + 3e^{2\beta_1 n/a}(3 + \beta_1 + 2s\beta_1^2) \quad (36)$$

$$b_{iii}^{(2)} = \frac{1}{a\beta_1^4} \times \{\cdot\}^{***}$$

$$\{\cdot\}^{***} = e^{\beta_1 n/a}[18 + 12\beta_1 + 2(1+s)\beta_1^2 + 3(s+2t)\beta_1^3 + 3s^2\beta_1^4] - 2[3 + 3\beta_1 + \beta_1^2]\beta_1 n/a \quad (37)$$

Example 3: We use the same scenario as the one in Example 2. Eqn (25) yields $x_1 = 3.457$ hence $x_2 = 6 - x_1 = 2.543$. Note that these results are only approximations up to the third order of a Taylor series. Hence, they are slightly different from those in Example 2. The advantage of Eqn (25) is that it is a formula for x_1 unlike the purely numerical result from bisection in Section III. In addition, this formula can be used to narrow the bracketing interval for x_1 and hence improves both efficiency and accuracy of the bisection algorithm in the previous section.

V. Optimal Integer Solution to the PRA

For each engagement opportunity, we can determine the optimal interceptor allocation by first computing the continuous optimal allocation described above and examine the integer solutions surrounding the continuous solution and select the best integer solution. For more than one engagement opportunity, we use dynamic programming. Generally,

$$PRA(n,a,e) = \max_{k=0,\ldots,n} \sum_{i_1,\ldots,i_a=0,1} \{\cdot\}^{****}$$

$$\{\cdot\}^{****} = (1-M_1^{s_1})^{i_1}(M_1^{s_1})^{1-i_1} \cdot \ldots \cdot (1-M_a^{s_a})^{i_a}(M_a^{s_a})^{1-i_a} \; PRA(n-k, \vec{i}, e-1) \tag{38}$$

with boundary conditions

$$\begin{aligned} PRA(n, a=0, e) &= 1 \\ PRA(n, a, e=0) &= \delta_{a,0} \end{aligned} \tag{39}$$

and $n = s_1 + \ldots + s_a$ where s_i is the optimal integer allocation against threat i $(i = 1,\ldots,a)$ by selecting the best integer solution surrounding the continuous solution and n is the number of interceptors assigned to engagement opportunity e. The number of possible integer solutions (integer number of interceptors allocated to each type of targets) that surround the continuous solution is exponential. When the number of threats is small i.e. less than five, it is feasible to do an exhaustive search. However, when the number of threats is large i.e. more than five, we make use of the genetic algorithm to search for the best integer solution. Before we apply the genetic algorithm, we observe that we already have a good solution. This solution consists of x_1 obtained from bisection and all other x_i $(i = 2,\ldots,a)$ are obtained from Eqn (18). We take the greatest integers $\lfloor x_i \rfloor$ $(i = 1,\ldots,a)$ that are less than or equal to x_i $(i = 1,\ldots,a)$ and add one to each until the number of interceptors allocated to all the targets is equal to n. That is, $n = s_1 + \ldots + s_a$. The additional ones are in priority from target 1 to target a. Therefore, $s_i = \lfloor x_i \rfloor + 1$ or $\lfloor x_i \rfloor$. Note that if the targets are identical then $x_i = \frac{n}{a}$ for all is. Hence, we recover the results presented in [2].

The genetic algorithm that we use comes from [11]. It is the simplest version of the genetic algorithm. Below is the verbatim description of this genetic algorithm.

Given a clearly defined problem to be solved and a bit string representation for candidate solutions, a simple GA works as follows:

1) Start with a randomly generated population of n l–bit chromosomes (candidate solutions to a problem).
2) Calculate the fitness (x) of each chromosome x in the population.
3) Repeat the following steps until n offspring have been created:
   a Select a pair of parent chromosomes from the current population, the probability of selection being an increasing function of fitness. Selection is done “with replacement,” meaning that the same chromosome can be selected more than once to become a parent.
   b With probability pc (the “crossover probability” or “crossover rate”), cross over the pair at a randomly chosen point (chosen with uniform probability) to form two offspring. If no crossover takes place, form two offspring that are exact copies of their respective parents. (Note that here the crossover rate is defined to be the probability that two parents will crossover in a single point. There are also “multipoint crossover” versions of the GA in which the crossover rate for a pair of parents is the number of points at which a crossover takes place.)
   c Mutate the two offspring at each locus with probability pm (the mutation probability or mutation rate), and place the resulting chromosomes in the new population. If n is odd, one new population member can be discarded at random.
4) Replace the current population with the new population.
5) Go to step 2.

Each iteration of this process is called a generation. A GA is typically iterated for anywhere from 50 to 500 or more generations. The entire set of generations is called a *run.* At the end of a run there are often one or more highly fit chromosomes in the population. Since randomness plays a large role in each run, two runs with different random–number seeds will generally produce different detailed behaviors. GA researchers often report statistics (such as the best fitness found in a run and the generation at which the individual with that best fitness was discovered) averaged over many different runs of the GA on the same problem.

The chance that a parent is chosen depends on the fitness of that parent. We use the stochastic acceptance methodology [12]. That is, an individual j is accepted as a parent with the probability $f_j/f_{\max}$ where f_j is the fitness of individual and $f_{\max}$ is the best fitness value. A chromosome would be a vector $\vec{s}$ of size a where s_i is the number of interceptors assigned to the ith threat where $i = 1,\ldots,a$. In example 2, there are two targets, $a = 2$, The optimal chromosome would be $\vec{s} = (3\ 3)$ i.e. threat 1 is engaged with three interceptors and threat 2 is also engaged with three interceptors.

VI. Results

We consider four scenarios with five threats; two of the treats are type 1 and three are type 2. Threats of the same type have the same $SSPH$. There are also twenty interceptors. Each scenario has a fixed $SSPH$ against threats of each type. We assume $SSPH_1 = 0.2, 0.4, 0.6, 0.8$. For each scenario, $SSPH_2$ ranges from 0 to 1. We set $pc = 0.7$, $pm = 0.01$, the number of generations and the number of runs both to be 20 for our scenarios which involve a small number of threats and a small number of interceptors.

We assume that there are at most two engagement opportunities. Two types of Shoot-Look-Shoot (SLS) tactics are examined. The first tactic is called the Heterogeneous tactic where we optimize the PRA by accounting for the $SSPH$ against type 1 threats and type 2 threats as well as the number of engagement opportunities (EOs). The second tactic is the simplest one. It is independent of the $SSPH$. If there is one engagement opportunity, we engage each threat with a salvo of four interceptors. We call this the Salvo tactic. If there are two engagement opportunities, we engage each threat with a salvo of two interceptors. If a threat is missed at the first engagement opportunity, we re-engage that threat with another salvo of two interceptors. We call this the SS-L-SS tactic. The PRA for the Heterogeneous tactic is determined by Eqn (38). The PRA for the Salvo tactic and the SS-L-SS tactic are determined by [1].

In these scenarios, the Salvo tactic and the SS-L-SS tactic yield the same PRA as one would expect. Generally, Figure 3 shows that the PRA increases as a function of $SSPH_2$ as expected in the sense that if the $SSPH_2$ increases then the defence effectiveness increases and so does the PRA. Also, when $SSPH_1 = SSPH_2$, the Heterogeneous tactic and the Salvo tactic provides the same PRA. This must be true as when the $SSPH$ is the same for all threats, the threats are considered identical and the optimal tactic is to allocate four interceptors against each threat, [2].

The Heterogeneous tactic always provides a better PRA than the one of the SS-L-SS tactic. With one engagement opportunity the Heterogeneous tactic provides a similar but better PRA than the one of the SS-L-SS tactic. However, when $SSPH_2$ exceeds $SSPH_1$, there is a substantial improvement in the PRA. With two engagement opportunities, the Heterogeneous tactic provides a significant improvement to the PRA when compared to the SS-L-SS tactic. For example, when $SSPH_1 = SSPH_2 = 0.4$, the Heterogeneous tactic with two engagement opportunities yields a $PRA = 0.81$ while the Salvo tactic or the SS-L-SS tactic yields a $PRA = 0.5$ which is the same PRA as the one of the Heterogeneous tactic with one engagement opportunity. This is an improvement of more than 30 percent. This means that in a hundred battles, the defence will succeed 81 times if it employs the Heterogeneous tactic (with two engagement opportunities) and only 50 times if it uses the Salvo tactic or the SS-L-SS tactic.

There are two reasons why the Heterogeneous tactic provides a superior PRA. First, the number of interceptors allocated to each threat is chosen such that they maximize the PRA at each engagement opportunity. Second, the hit assessment sensors allow the defence to know if a threat is hit at the previous engagement opportunities in which case the defence can allocate more interceptors (normally allocated to the threats that were hit) to the threats that were missed previously.

As shown above, we can achieve substantial improvement in the PRA when we employ the Heterogeneous tactic. It would be natural then to ask ourselves how many interceptors would be needed in order to carry out the Heterogeneous tactic. Figure 4 displays the expected number of interceptors expended $(ENIE)$ as a function of $SSPH_2$. Once the allocations of interceptors are known e.g. through the optimization of the PRA, we can use them to determine the $ENIE$. However, the length of this paper does not allow us to go through the details of the derivations for the $ENIE$. Unlike the PRA, the Salvo tactic and the SS-L-SS tactic do not yield the same $ENIE$. This must be true as in the Salvo tactic, the defence launches all the interceptors in one engagement opportunity while the SS-L-SS tactic launches a salvo of two interceptors against each threat at the first engagement opportunity and re–launches another salvo of two interceptors against each threat only if it was missed at the first engagement opportunity. Since there is a chance that threats are hit at the first engagement opportunity and hence there is no need to re–engage. This implies that the $ENIE$ must be lower for the SS-L-SS tactic than the one for the Salvo tactic.

In general, we see that the Salvo tactic and the Heterogeneous tactic with one engagement opportunity requires twenty interceptors. The Heterogeneous tactic with two engagement opportunities requires a lower $ENIE$ than the one for the Salvo tactic and for the Heterogeneous tactic with one engagement opportunity. The SS-L-SS tactic requires the lowest $ENIE$.

Also, the $ENIE$ decreases with $SSPH$. This must hold since when the $SSPH$ increases, the chances of neutralizing the threats at the first engagement opportunity increases and hence the chances to re-engage decreases implying a lower $ENIE$.

VII. Conclusion

In this paper, we examined maritime air defence scenarios in which threats may be of different types (different sizes, different cross sections, different lethalities etc.). However, we assume that they have the same number of engagement opportunities. The case of different engagement opportunities is considered in [13].

When threats are of different types, the corresponding $SSPH$s may be different. We show that by balancing the same inventory of interceptors, we can significantly improve the probability of raid annihilation compared to the Salvo tactic and the SS-L-SS tactic. At the same type we can decrease the expected number of interceptors expended, relative to the Salvo tactic.

The simulation has shown that the improvement in PRA can be more than thirty percent which means that in a hundred battles, the additional number of successful missions is thirty, a substantial military achievement. The improvement will increase with the number of engagement opportunities. Of course, this improvement comes with several costs. The first cost is the cost for the hit assessment sensor. The hit assessment sensor allows the defence to determine if a threat was neutralized or not. The second aspect is the speed of interceptors combined with command & control parameters such as time delays that would allow more than one engagement opportunity.

In essence, given hit assessment and more than one engagement opportunity, the defence can allocate the interceptors where they are most needed. Generally, a threat with a low $SSPH$ should be assigned more interceptors than a threat with a high $SSPH$. The Heterogeneous tactic determines exactly how many interceptors should be assigned to each threat depending on the number of threats, their $SSPH$s, the inventory of interceptors, and the number of engagement opportunities in a way that the is globally maximized.

From a scientific view, to be able to globally maximize the PRA is a step forward in optimization; in general, it is non-trivial to achieve the global optimum in multiple variables. In

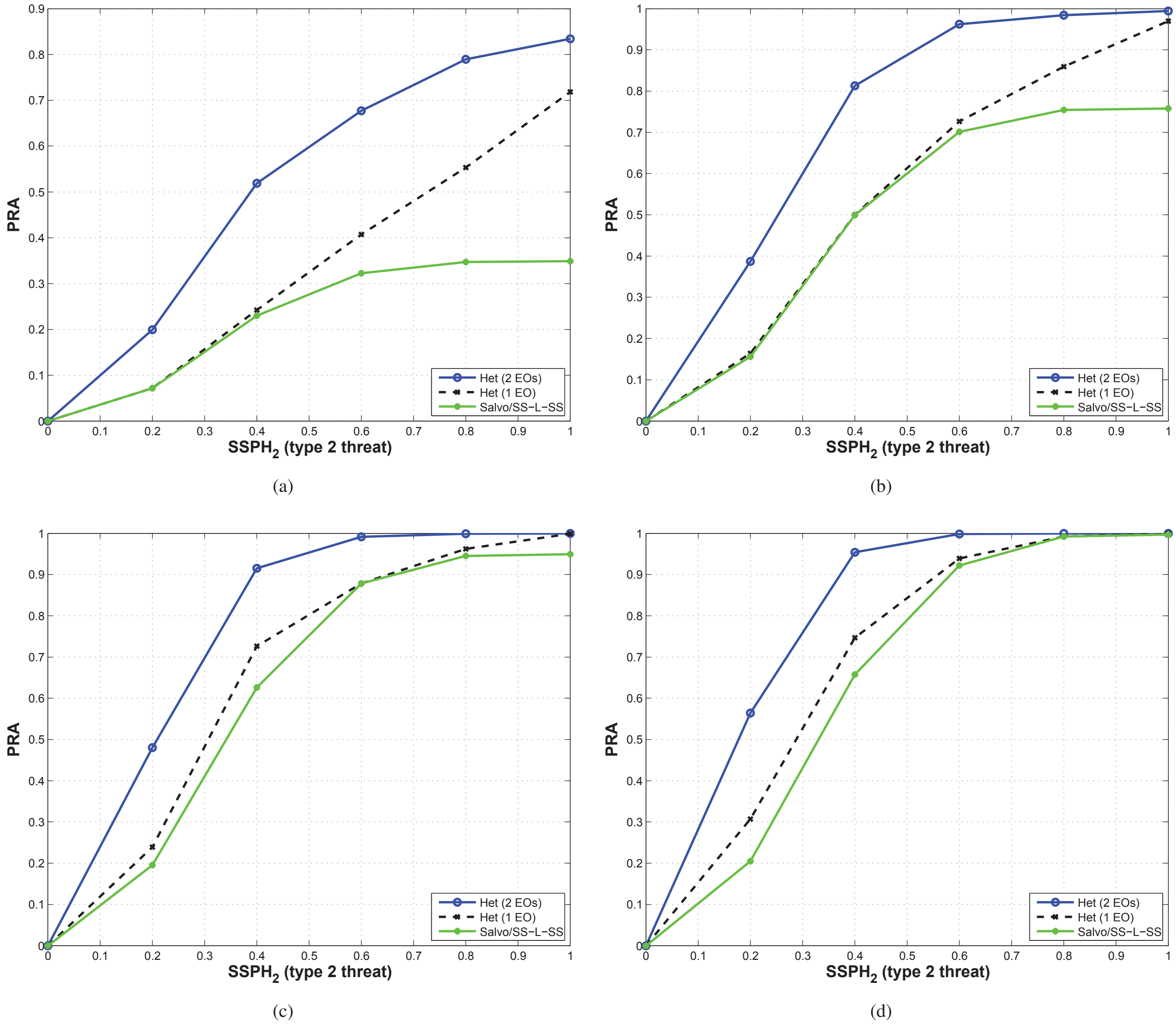

Fig. 3: PRA as a function of $SSPH_2$ with $SSPH_1 =$ (a) 0.2, (b) 0.4, (c) 0.6, and (d) 0.8.

this case, it was done by observing that the PRA is a concave function. The analytical expressions for the performances of the defence system such as the probability of detection are often similar to the PRA. Therefore, we could use the same approach to optimize other defence metrics.

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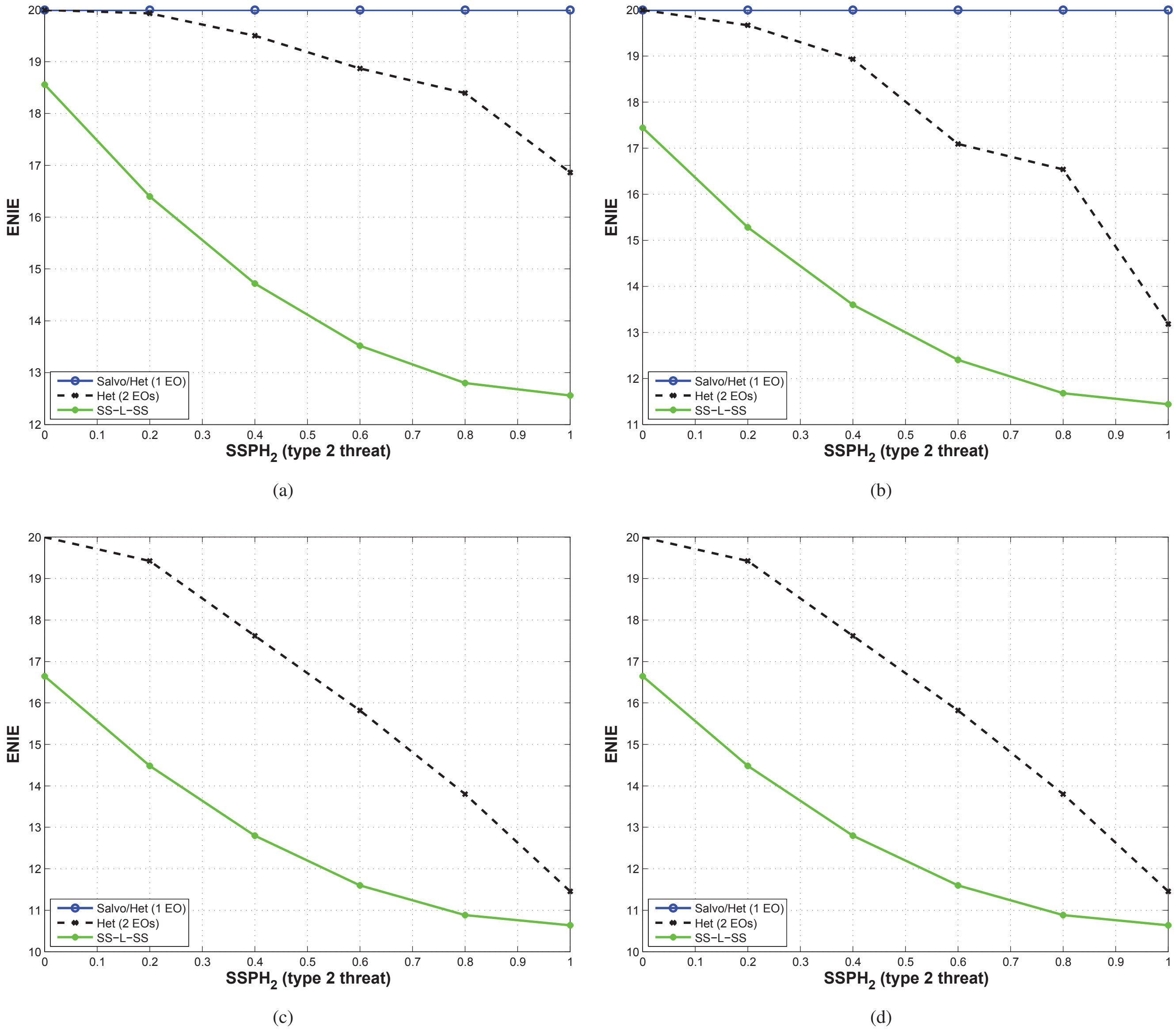

Fig. 4: $ENIE$ as a function of $SSPH_2$ with $SSPH_1 =$ (a) 0.2, (b) 0.4, (c) 0.6, and (d) 0.8.

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