DRDC-RDDC-2015-P040

Joint Power Optimization and Connectivity Control Problem over Underwater Random Sensor Networks

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Abstract— **In this paper, the problem of power optimization and connectivity control for an underwater random sensor network is investigated. The weighted edge connectivity is proposed as a metric to evaluate the connectivity of the weighted expected graph of a random sensor network where the elements of the weight matrix represent the operational probability of their corresponding communication links. The introduced connectivity measure is described as an explicit function of the transmission power of the sensors and the statistical parameters of the communication channels. An optimization problem is then presented to minimize the total power consumption of the network while a lower bound on the global weighted edge connectivity of the network is satisfied. An analytical solution to the optimization problem is obtained by considering a directed cycle topology for the expected communication graph. The efficacy of the obtained results is confirmed for log-normal distribution of the power gain of acoustic communication channels.**

I. Introduction

Ad-hoc wireless networks consist of a number of fixed or mobile sensors that are capable of data exchange through wireless channels without the support of a pre-existing infrastructure [1], [2]. The convergence time of numerous distributed algorithms for applications such as consensus, swarming, target localization, data aggregation and diffusion over graphs is highly dependent on the connectivity degree of the network [3]. For the case of random networks where the communication channels are described by random variables, the convergence time of the distributed algorithms running over these networks is strictly dependent on the connectivity of the underlying expected communication graphs [4], [5].

The problem of connectivity control has been investigated extensively in the literature. In [6], a distributed algorithm for the estimation and control of the connectivity of ad-hoc networks in the presence of a random topology is developed which is applicable to undirected networks with symmetric communication channels. A distributed topology control algorithm is proposed in [7] which provides a solution to the topology control problem in a network of heterogeneous wireless sensors with different maximum transmission ranges. However, the path loss model utilized in [7] cannot be extended to more complex communication channels and the transmission medium is assumed to be symmetric. The second smallest eigenvalue of the Laplacian matrix, known as the *algebraic connectivity*, is used in [6], [8] as a global measure to describe the connectivity of a symmetric network. Since the acoustic communication channels used in underwater sensor networks are mostly directed and their underlying communication graph is asymmetric [9], the algebraic connectivity is not able to describe the connectivity of such networks.

The connectivity control problem over an underwater acoustic sensor network using an optimal assignment of the transmission power of the sensors is investigated in this paper. Unlike the communication channels used in terrestrial sensor networks, there are several sources of uncertainty which influence the acoustic communication between underwater nodes such as multi-path propagation, variation of sound speed profile and underwater currents [10]. Since these sources of uncertainty vary over time and space in an unpredictable manner, a random directed graph (digraph) is employed to describe the communication links between sensors in the underwater environment [11], [12], [13]. A binary random variable is used to describe the communication channels of the network, and the expected communication graph is subsequently obtained as a weighted deterministic digraph. To address the shortcoming of the algebraic connectivity and edge connectivity in describing the connectivity of weighted digraphs, the *weighted edge connectivity* is proposed as a new metric which reflects the connectivity of the expected communication graph by considering the joint effects of the path reliability and the network robustness to link failure. By adopting a probability density function on the power gain of the acoustic communication channels, the existence probability of every link is explicitly described in terms of the transmission power of the sensors and the statistical characteristics of the channels. An optimization problem is then introduced to minimize the total power consumption of the network in such a way that a lower bound on the value of the weighted edge connectivity of the network is satisfied. Moreover, the optimization problem is solved analytically by considering a directed cycle topology for the expected communication graph of the network. The effectiveness of the optimal transmission power vector is demonstrated for a scenario in which the power gain of all acoustic channels is represented by log-normal distributions.

The remainder of the paper is organized as follows. In Section II, some relevant preliminaries and definitions are given. A new measure to assess the connectivity of the expected communication graph of a random network is introduced in

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This work has been supported by PWGSC contract no. W7707-145674/001/HAL funded by Defence Research and Development Canada.

Section III, and the relation between the network probability matrix and the transmission power vector of the network is addressed in Section IV. The joint power optimization and connectivity control problem along with an analytical solution for directed cycle topologies is presented in Section V, and the conclusions are summarized in Section VI.

II. Preliminaries

Throughout the paper, the set of positive and nonnegative real numbers are denoted by $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$, respectively. Moreover, $\mathbb{N}_n := \{1,2,\ldots,n\}$, $|\Phi|$ is the cardinality of a finite set Φ, and ϕ^i represents the i-th element of the set Φ. The power set of a finite set Φ, denoted by $\mathscr{P}(\Phi)$, includes all subsets of Φ. Also, $\mathbf{I}_n$ is the $n \times n$ identity matrix, and 1_n is the all-one column vector of length n.

Definition 1: Let $G = (V,E)$ denote a random digraph composed of a set of nodes V and a set of edges E. Let also the probability matrix $\mathbf{P} = [p_{ij}]$ represent the existence probability of all directed edges in G, where $p_{ij} \in [0,1]$ is the probability of the existence of edge $(j,i) \in E$. Define $\mathbf{A} = [a_{ij}]$ as the adjacency matrix of G, where a_{ij} is a binary random variable such that:

$$a_{ij} = \begin{cases} 1, & \text{with probability } p_{ij}, \\ 0, & \text{with probability } 1 - p_{ij}. \end{cases} \tag{1}$$

Definition 2: Define $\hat{G} = (\hat{V},\hat{E})$ as the expected graph of the random digraph $G = (V,E)$. Let $\hat{\mathbf{A}} = [\hat{a}_{ij}]$ represent the weighted adjacency matrix of $\hat{G}$, where $\hat{a}_{ij} = p_{ij}$ for every pair of distinct nodes $i, j \in \hat{V}$.

Let the *communication graph* of a network composed of n sensors be specified by a random digraph $G = (V,E)$ with the probability matrix $\mathbf{P} = [p_{ij}]$, where its node and edge sets are given by:

$$V = \{1,2,\ldots,n\}, \tag{2a}$$

$$E = \{(i,j) \in V \times V \mid a_{ji} = 1\}. \tag{2b}$$

Also, consider $\hat{G} = (\hat{V},\hat{E})$ as the *expected communication graph* of the network, where $\hat{V} = V$ and:

$$\hat{E} = \{(i,j) \in \hat{V} \times \hat{V} \mid p_{ji} \neq 0\}. \tag{3}$$

III. Weighted Edge Connectivity Metric

The main objective of this section is to propose a global metric for evaluating the connectivity of the expected communication graph $\hat{G}$ associated with a random sensor network. Consider a random digraph $G = (V,E)$ whose node set V corresponds to a group of sensors and the existence of every directed link in its edge set E is characterized by a binary random variable, which is assumed to be independent of all other binary random variables. The concept of *edge connectivity* (EC) is introduced in [14] to evaluate the global connectivity of a graph. This notion is defined as the minimum number of edges that should be removed such that the digraph is no longer strongly connected. However, this measure does not account for the probability matrix of random networks, and merely demonstrates the robustness of the network to link failure. This calls for a more accurate measure of connectivity to capture the probabilistic nature of the communication links. The *weighted edge connectivity* (WEC) measure is introduced here to extend the notion of the EC degree to the more general cases of weighted digraphs, where the elements of the weight matrix represent the operational probability of their corresponding communication links. This nonnegative measure is strictly positive for a strongly connected digraph, and a larger value of this measure represents "stronger" connectivity by considering the combined effects of the operational probability of the paths and the network robustness to link failure. To clarify this new concept, the multiplicative weight of a path is defined next, based on the mutual independence of the binary random variables used for describing the probabilistic nature of the edges of the random network.

Definition 3: Let $\Psi_{i,j}$ denote the set of all directed paths from node i to node j in the expected communication graph $\hat{G} = (\hat{V},\hat{E})$ with probability matrix $\mathbf{P} = [p_{ij}]$. Let also $\pi^k_{i,j}$ represent the k-th element of $\Psi_{i,j}$ defined as $\pi^k_{i,j} = \{(v^k_0, v^k_1), \ldots, (v^k_{m_k-1}, v^k_{m_k})\}$, which denotes a directed path of length m_k from node i to node j such that $v^k_0 = i$, $v^k_{m_k} = j$, and $(v^k_{l-1}, v^k_l) \in \hat{E}$ for all $l \in \mathbb{N}_{m_k}$. Then, the multiplicative weight of path $\pi^k_{i,j}$, denoted by $W(\pi^k_{i,j})$, is defined as follows:

$$W(\pi^k_{i,j}) = \prod_{l=1}^{m_k} p_{v^k_l v^k_{l-1}}. \tag{4}$$

Since each element of $\mathbf{P}$ represents the probability of the existence of its corresponding edge in $\hat{G}$, the multiplicative weight can be interpreted as the operational probability of a given path.

Definition 4: Consider $\pi^s_{i,j}$ and $\pi^t_{i,j}$ as two distinct directed paths from node i to node j in $\hat{G}$ which are described by the edge sets $\pi^s_{i,j} = \{(v^s_0, v^s_1), \ldots, (v^s_{m_s-1}, v^s_{m_s})\}$ and $\pi^t_{i,j} = \{(v^t_0, v^t_1), \ldots, (v^t_{m_t-1}, v^t_{m_t})\}$, respectively. Let also $v^s_0 = v^t_0 = i$ and $v^s_{m_s} = v^t_{m_t} = j$, where m_s and m_t denote the lengths of two directed paths $\pi^s_{i,j}$ and $\pi^t_{i,j}$, respectively. Then, $\pi^s_{i,j}$ and $\pi^t_{i,j}$ are edge-disjoint paths if $\pi^s_{i,j} \cap \pi^t_{i,j} = \emptyset$.

The notion of local WEC measure for any pair of distinct nodes $i, j \in \hat{V}$ in the expected communication graph $\hat{G} = (\hat{V},\hat{E})$ is now introduced. This measure, denoted by $\hat{\kappa}_{i,j}(\hat{G})$, is defined as the maximum of the summation of the operational probability of edge-disjoint paths from node i to node j in $\hat{G}$. Consider $\mathscr{P}(\Psi_{i,j})$ as the power set of $\Psi_{i,j}$, and let $\hat{\mathscr{P}}(\Psi_{i,j}) \subseteq \mathscr{P}(\Psi_{i,j})$ contain all nonempty subsets of $\Psi_{i,j}$ which are composed of a set of mutually edge-disjoint paths from i to j in $\hat{G}$. Then:

$$\hat{\kappa}_{i,j}(\hat{G}) = \sum_{k=1}^{|\hat{\Psi}_{i,j}|} W(\hat{\pi}^k_{i,j}), \tag{5}$$

where,

$$\hat{\Psi}_{i,j} = \underset{\Psi \in \hat{\mathscr{P}}(\Pi_{i,j})}{\operatorname{argmax}} \sum_{k=1}^{|\Psi|} W(\pi^k), \tag{6}$$

and $\Psi = \{\pi^k \mid k \in \mathbb{N}_{|\Psi|}\}$ denotes a path set composed of $|\Psi|$ elements. The WEC metric of $\hat{G}$, denoted by $\hat{\kappa}(\hat{G})$, is defined as the global connectivity measure relating to the previous local WEC measure as follows:

$$\hat{\kappa}(\hat{G}) = \min_{i,j\in\hat{V},\, i\neq j} \hat{\kappa}_{i,j}(\hat{G}). \tag{7}$$

Example 1: An illustrative example is given here to demonstrate the required steps for finding the proposed WEC measure for a random network composed of six nodes. Let Fig. 1 depict the expected graph $\hat{G}$ of the network. Note that the existence probability of each link appears as a weight on its corresponding edge in $\hat{G}$, which yields the following probability matrix $\mathbf{P}$:

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0.9 & 0.8 \\ 0.9 & 0 & 0 & 0 & 0 & 0.9 \\ 0.6 & 0.8 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.9 & 0 \end{bmatrix}. \tag{8}$$

Since at least two edge-disjoint paths exist between any pair

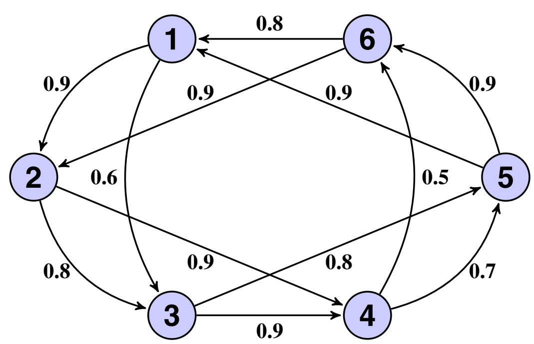

Fig. 1. The expected graph $\hat{G}$ of Example 1.

of distinct nodes in $\hat{G}$, it follows that the EC degree of $\hat{G}$ is two. For this example, the minimum local WEC measure corresponds to the directed paths from node 4 to node 3, *i.e.*, $\hat{\kappa}(\hat{G}) = \hat{\kappa}_{4,3}(\hat{G})$. In order to find $\hat{\kappa}_{4,3}(\hat{G})$, it is first required to find the set $\Psi_{4,3}$ containing all distinct paths from node 4 to node 3 in $\hat{G}$. This results in eight different paths $\pi^k_{4,3}$, $k \in \mathbb{N}_8$, as follows:

$\pi^1_{4,3} = \{(4,5),(5,1),(1,3)\}$, $\pi^2_{4,3} = \{(4,6),(6,1),(1,3)\}$,
$\pi^3_{4,3} = \{(4,6),(6,2),(2,3)\}$, $\pi^4_{4,3} = \{(4,5),(5,1),(1,2),(2,3)\}$,
$\pi^5_{4,3} = \{(4,5),(5,6),(6,1),(1,3)\}$, $\pi^6_{4,3} = \{(4,5),(5,6),(6,2),(2,3)\}$,
$\pi^7_{4,3} = \{(4,6),(6,1),(1,2),(2,3)\}$, $\pi^8_{4,3} = \{(4,5),(5,6),(6,1),(1,2),(2,3)\}$.

The path set $\hat{\mathscr{P}}(\Psi_{4,3})$ is subsequently given by:

$$\hat{\mathscr{P}}(\Psi_{4,3}) = \Big\{\pi^1_{4,3}, \pi^2_{4,3}, \ldots, \pi^8_{4,3}, \{\pi^1_{4,3}, \pi^3_{4,3}\}, \{\pi^1_{4,3}, \pi^7_{4,3}\}, \{\pi^2_{4,3}, \pi^4_{4,3}\}, \{\pi^2_{4,3}, \pi^6_{4,3}\}, \{\pi^3_{4,3}, \pi^5_{4,3}\}\Big\}, \tag{9}$$

every element of which is composed of a set of edge-disjoint paths belonging to $\Psi_{4,3}$. By solving the combinatorial optimization problem (6), one arrives at $\hat{\Psi}_{4,3} = \{\pi^1_{4,3}, \pi^3_{4,3}\}$. In other words, the path set $\hat{\Psi}_{4,3}$ belongs to $\hat{\mathscr{P}}(\Psi_{4,3})$ and the summation of the multiplicative weight of its paths is maximum. It can then be concluded that $\hat{\kappa}_{4,3}(\hat{G}) = W(\pi^1_{4,3}) + W(\pi^3_{4,3}) = 0.738$ or $\hat{\kappa}(\hat{G}) = 0.738$ according to (5) and (7).

IV. Existence Probability of Communication Links as a Function of the Transmission Power

Consider a network composed of n sensors and let $\hat{G}$ denote its expected communication graph. Let also P^i_T, $i \in \hat{V}$, denote the transmission power of the i-th sensor and define ξ_{ij} as a set of statistical parameters characterizing the probabilistic nature of the communication channel from sensor j to sensor i. Define $h(\cdot)$ as a function which represents the existence probability of the communication links in terms of the power of transmitting sensors and the statistical characteristics of the channels. Thus, the existence probability of a directed link from sensor j to sensor i, denoted by p_{ij}, is given by:

$$p_{ij} = h(P^j_T; \xi_{ij}), \tag{10}$$

for any $(j,i) \in \hat{E}$. The function $h(\cdot)$ can be explicitly characterized by adopting an appropriate probability density function for the power gain of the communication channels, such as log-normal, Nakagami, and Rician distributions [15], [16].

A. Log-Normal Power Gain Distribution

For simplicity of analysis, a log-normal distribution for the power gain of the communication channel from sensor j to sensor i is considered in this subsection, which is described as:

$$f_{ij}(x; \mu_{ij}, \sigma_{ij}) = \frac{1}{x\sigma_{ij}\sqrt{2\pi}} e^{-\frac{(\log(x)-\mu_{ij})^2}{2\sigma^2_{ij}}}, \tag{11}$$

for all $x > 0$, where μ_{ij} and σ_{ij} represent the mean value and standard deviation of the normal distribution of $\log(f_{ij})$, respectively. The log-normal distribution has been considered in [16], [17] to characterize the large-scale phenomena that affect the locally-averaged power gain of underwater acoustic communication channels. It should be noted that the large-scale effects describe the slowly varying phenomena such as location uncertainty and changing environmental conditions, while the small-scale effects with fast variations such as scattering and motion-induced Doppler shifting are not captured by the proposed distribution [17]. It is now desirable to analytically characterize the function $h(\cdot)$ for any communication link $(j,i) \in \hat{E}$ in terms of the transmission power of the j-th sensor, which acts as a transmitter, and the set of statistical parameters $\xi_{ij} = \{\mu_{ij}, \sigma_{ij}\}$. Since the underwater communication is constrained to a limited bandwidth, all channels are assumed to operate in a finite frequency bandwidth B with a center frequency of f_c. Let $H_{ij}(f,t)$ represent the transfer function of the underwater acoustic channel at frequency f and time instant t. According to [16], the instantaneous power gain of the received signal over the considered bandwidth is given by:

$$K_{ij}(t) = \frac{1}{B}\int_{f_c-B/2}^{f_c+B/2} |H_{ij}(f,t)|^2 \mathrm{d}f. \tag{12}$$

Consider a quasi-stationary channel with slow time variations. The locally-averaged power gain $\overline{K}_{ij}$ over a time window of length T_w is obtained as follows [16]:

$$\overline{K}_{ij} = \frac{1}{T_w}\int_{t-T_w}^{t} K_{ij}(s)\mathrm{d}s. \tag{13}$$

Using a *time division multiple access* (TDMA) protocol in the MAC layer of the network, it can be ensured that the channel interference is negligible. Consider P_N as the noise power and let γ_{ij} denote the signal-to-interference-plus-noise ratio (SINR) of a signal transmitted from sensor j to sensor i with transmission power P_T^j through a channel with locally-averaged power gain $\overline{K}_{ij}$. Then, γ_{ij} is defined as [16]:

$$\gamma_{ij} = \frac{P_T^j}{P_N}\overline{K}_{ij}, \tag{14}$$

where no interference exists due to the utilized MAC protocol. Also, let p_{ij}^{out} denote the outage probability of the transmitted signal, which is defined as:

$$p_{ij}^{out} = \mathbb{P}\{\gamma_{ij} < \gamma_0\} = \mathbb{P}\{\overline{K}_{ij} < \overline{K}_{ij}^0\}, \tag{15}$$

where γ_0 is a threshold on the SINR for successfully receiving a message, and $\overline{K}_{ij}^0 = \frac{P_N}{P_T^j}\gamma_0$. Then, the existence probability of a directed link from j to i is defined as the probability that the inequality $\gamma_{ij} \geq \gamma_0$ holds, which is given by:

$$\begin{aligned} p_{ij} =& \mathbb{P}\{\gamma_{ij} \geq \gamma_0\} = 1 - \mathbb{P}\{\gamma_{ij} < \gamma_0\} = 1 - p_{ij}^{out} \\ =& \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\log(\frac{P_T^j}{P_N\gamma_0}) + \mu_{ij}}{\sqrt{2}\sigma_{ij}}\right)\right] = h(P_T^j;\mu_{ij},\sigma_{ij}), \end{aligned} \tag{16}$$

where $\mathrm{erf}(\cdot)$ denotes the error function. Thus, the (i,j)-th element of the probability matrix $\mathbf{P}$ can be explicitly described as a function of the transmission power of the j-th sensor and the statistical characteristics of the directed channel from j to i specified by the parameter set $\xi_{ij} = \{\mu_{ij},\sigma_{ij}\}$, for every $(j,i) \in \hat{E}$. As an example, the existence probability of a communication channel with log-normal power gain distribution versus the normalized transmission power with a constant mean value and different standard deviations is depicted in Fig. 2.

V. Transmission Power Optimization and Connectivity Control Problem

Consider a network composed of n sensors and let $\hat{G} = (\hat{V},\hat{E})$ denote its underlying expected communication graph with a directed cycle topology, such that:

$$\hat{V} = \{0,1,\ldots,n-1\}, \tag{17a}$$
$$\hat{E} = \{(i,j) \in \hat{V}\times\hat{V} \mid j \equiv i+1 \ (\mathrm{mod}\, n)\}. \tag{17b}$$

Let also $\mathrm{P}_T = [\, P_T^0\ P_T^1\ \ldots\ P_T^{n-1}\,]^T$ denote the transmission power vector of the network and consider P_T^i, $i \in \hat{V}$, as the transmission power used by the i-th sensor to broadcast its data throughout the network. The main objective is to determine the optimal transmission power vector of the network by minimizing the total power consumption of all sensors such that the WEC measure $\hat{\kappa}(\hat{G})$ is lower-bounded

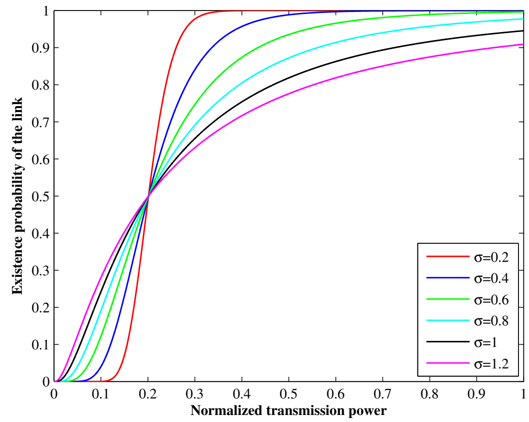

Fig. 2. Existence probability of a channel with log-normal power gain distribution versus the normalized power of the transmitter.

by $\hat{\kappa}_0$, *i.e.*:

$$\begin{aligned} \underset{\mathrm{P}_T}{\text{minimize}} \quad & \sum_{i=0}^{n-1} P_T^i, \\ \text{s.t.} \quad & \hat{\kappa}(\hat{G}) \geq \hat{\kappa}_0. \end{aligned} \tag{18}$$

Based on the definition of the global WEC measure for a digraph $\hat{G}$ given by (7), $\hat{\kappa}(\hat{G})$ is lower-bounded by $\hat{\kappa}_0$ if the local metric $\hat{\kappa}_{i,j}(\hat{G})$ is lower-bounded by $\hat{\kappa}_0$ for all ordered pairs of distinct nodes $i,j \in \hat{V}$. In other words, the inequality $\hat{\kappa}(\hat{G}) \geq \hat{\kappa}_0$ which represents the constraint of the optimization problem (18) holds if all the inequalities belonging to the *constraints set* $C(\hat{G})$ defined as:

$$C(\hat{G}) = \{\hat{\kappa}_{i,j}(\hat{G}) \geq \hat{\kappa}_0 \mid i,j \in \hat{V},\ i \neq j\}, \tag{19}$$

are satisfied. However, some of the inequality constraints belonging to $C(\hat{G})$ are redundant and it is not necessary to examine all of them to make sure that $\hat{\kappa}(\hat{G}) \geq \hat{\kappa}_0$ holds. This leads to the definition of the *dominant constraints set* $C_D(\hat{G})$, as a subset of $C(\hat{G})$, which includes the minimum number of the inequality constraints whose satisfaction ensures that $\hat{\kappa}(\hat{G})$ is lower-bounded by $\hat{\kappa}_0$. The dominant constraints set for the power optimization problem (18) over a network with directed cycle topology is addressed in the next lemma.

Lemma 1: Consider a network composed of n sensors and let its expected communication graph $\hat{G} = (\hat{V},\hat{E})$ have a directed cycle topology. Then, the set of dominant constraints $C_D(\hat{G})$ for the power optimization problem (18) defined over $\hat{G}$ with $\hat{\kappa}_0$ as a lower bound on $\hat{\kappa}(\hat{G})$ contains n constraints, and is described as follows:

$$\begin{aligned} C_D(\hat{G}) =& \left\{\hat{\kappa}_{j,i}(\hat{G}) \geq \hat{\kappa}_0 \mid i \in \hat{V},\ j \equiv i+1\ (\mathrm{mod}\, n)\right\} \\ =& \left\{\prod_{k=0,\,k\neq i}^{n-1} p_{jk} \geq \hat{\kappa}_0 \mid i \in \hat{V},\ j \equiv k+1\ (\mathrm{mod}\, n)\right\}. \end{aligned} \tag{20}$$

Proof: The proof is omitted due to space limitations. ■

For the simplicity of analysis and without loss of generality, assume that all communication channels have similar statistical characteristics, *i.e.*, $\xi = \xi_{ij}$ for all $(j,i) \in \hat{E}$. From this assumption and according to (10), the existence probability of any communication link $(j,i) \in \hat{E}$ is only a function of the power of transmitting sensor j, *i.e.*:

$$p_{ij} = h(P_T^j). \tag{21}$$

According to (21), the set of dominant constraints (20) given by Lemma 1 can be represented in terms of the transmission power of the sensors as follows:

$$C_D(\hat{G}) = \Big\{ \prod_{k=0,\, k\neq i}^{n-1} h(P_T^k) \geq \hat{\kappa}_0 \mid i \in \hat{V} \Big\}. \tag{22}$$

Furthermore, the power optimization problem (18) can be reformulated as follows:

$$\begin{aligned} &\underset{\mathrm{P}_T}{\text{minimize}} \quad \sum_{i=0}^{n-1} P_T^i, \\ &\text{s.t.} \qquad \hat{\kappa}_0 - \prod_{j=1,\, j\neq i}^{n-1} h(P_T^j) \leq 0, \quad i \in \hat{V}. \end{aligned} \tag{23}$$

Before solving the constrained optimization problem (23), two important assumptions are presented in the sequel.

Assumption 1: Assume that $g(\cdot): \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ represents a continuously differentiable and strictly increasing function of the transmission power of the sensors, which is defined as $g(P_T^k) = \frac{h(P_T^k)}{h'(P_T^k)}$ for any $P_T^k \in \mathbb{R}_{>0}$ and $k \in \hat{V}$.

Assumption 2: Assume that the set of feasible solutions of the constrained optimization problem (23) is nonempty.

Then, the Lagrangian of (23) is formed as follows:

$$L = \sum_{i=0}^{n-1} P_T^i + \sum_{i=0}^{n-1} \lambda_i \big(\hat{\kappa}_0 - \prod_{j=0,\, j\neq i}^{n-1} h(P_T^j)\big), \tag{24}$$

where λ_i, $i \in \hat{V}$, denotes the Lagrange multiplier associated with the $(i+1)$-th constraint. Note that $\lambda_i \geq 0$ if the $(i+1)$-th constraint is active, *i.e.*, $\hat{\kappa}_0 - \prod_{j=0,\, j\neq i}^{n-1} h(P_T^j) = 0$. However, $\lambda_i = 0$ when the $(i+1)$-th constraint is inactive, *i.e.*, $\hat{\kappa}_0 - \prod_{j=0,\, j\neq i}^{n-1} h(P_T^j) < 0$. In order to proceed, apply the first-order Karush-Kuhn-Tucker optimality condition to L and note that the cost function and the set of all inequality constraints in (23) are continuously differentiable [18]. Then, the partial derivative of L w.r.t. P_T^k, $k \in \hat{V}$, is obtained as:

$$\frac{\partial L}{\partial P_T^k} = 1 - \sum_{i=0,\, i\neq k}^{n-1} \lambda_i \frac{h'(P_T^k)}{h(P_T^k)} \prod_{j=0,\, j\neq i}^{n-1} h(P_T^j). \tag{25}$$

Define $\eta = \prod_{j=0}^{n-1} h(P_T^j)$ and $\rho_k = \frac{h'(P_T^k)}{h(P_T^k)}$ for any $k \in \hat{V}$. Then, (25) can be simplified as:

$$\frac{\partial L}{\partial P_T^k} = 1 - \eta \rho_k \sum_{i=0,\, i\neq k}^{n-1} \frac{\lambda_i}{h(P_T^i)}. \tag{26}$$

Hence, the solution to the following set of $2n$ nonlinear equations gives a candidate optimal solution for (23):

$$1 - \eta \rho_k \sum_{i=0,\, i\neq k}^{n-1} \frac{\lambda_i}{h(P_T^i)} = 0, \quad k \in \hat{V}, \tag{27a}$$

$$\lambda_k \big(\hat{\kappa}_0 - \prod_{i=0,\, i\neq k}^{n-1} h(P_T^i)\big) = 0, \quad k \in \hat{V}. \tag{27b}$$

The solution of the above set of nonlinear equations is addressed by the next lemma.

Lemma 2: Consider the set of $2n$ nonlinear equations given by (27) and let Assumptions 1 and 2 hold. Without loss of generality, assume that the first m constraints in (23) are active while the remaining $n-m$ are inactive. Then, the solution of (27) is denoted by the transmission power vector $\hat{\mathrm{P}}_T$ whose elements are given by:

$$\hat{P}_T^0 = \hat{P}_T^1 = \cdots = \hat{P}_T^{m-1} = \hat{P}_1, \tag{28}$$

and,

$$\hat{P}_T^m = \hat{P}_T^{m+1} = \cdots = \hat{P}_T^{n-1} = \hat{P}_2, \tag{29}$$

where $\hat{P}_i$, $i \in \{1,2\}$, are the solution of the following set of equations for a known $m \in \mathbb{N}_n$:

$$(h(\hat{P}_1))^{m-1} (h(\hat{P}_2))^{n-m} - \hat{\kappa}_0 = 0, \tag{30a}$$

$$m g(\hat{P}_1) - (m-1) g(\hat{P}_2) = 0. \tag{30b}$$

Proof: The proof is omitted due to space limitations. ■

Remark 1: Since the constrained optimization problem (23) is trivial for the case that $n = 1$, it is assumed that $n \geq 2$. Consider a case in which all constraints of (23) are inactive represented by $m = 0$. It then follows from (30) that the candidate optimal vector for this scenario is given by $\hat{P}_T^0 = \hat{P}_T^1 = \cdots = \hat{P}_T^{n-1} = \hat{P}_2 = 0$, which is not an acceptable solution. Also, $m = 1$ represents a case in which all constraints are inactive except for one. However, it is implied from (30) that $\hat{P}_T^0 = \hat{P}_1 = 0$ for this scenario, which is unacceptable. Therefore, it can be concluded from the above two scenarios that the number of active constraints, denoted by $m \in \mathbb{N}$, should satisfy $2 \leq m \leq n$.

Theorem 1: Consider the constrained optimization problem (23) and let Assumptions 1 and 2 hold. Then, the candidate optimal transmission power vector $\hat{\mathrm{P}}_T = \hat{P} 1_n$ is the minimizer of (23), where:

$$\hat{P} = h^{-1}(\hat{\kappa}_0^{\frac{1}{n-1}}). \tag{31}$$

Proof: According to Lemma 2 and Remark 1, the candidate solutions for (23) are specified by an integer m where $2 \leq m \leq n$. Therefore, the main optimization problem can be transformed into a simpler one to find m in such a way that the total power consumption of the network is minimized subject to two equations in (30) as the equality constraints which characterize the candidate solution. Furthermore, the optimization problem is reformulated as follows:

$$\begin{aligned} &\underset{m\in\mathbb{N},\, 2\leq m\leq n}{\text{minimize}} \quad m\hat{P}_1 + (n-m)\hat{P}_2, \\ &\text{s.t.} \qquad (h(\hat{P}_1))^{m-1}(h(\hat{P}_2))^{n-m} - \hat{\kappa}_0 = 0, \\ &\qquad\qquad m g(\hat{P}_1) - (m-1) g(\hat{P}_2) = 0. \end{aligned} \tag{32}$$

The above problem has $n-1$ different possible solutions, given by $m \in \mathbb{N}_n \setminus \{1\}$, which correspond to all possible values that m can accept. To proceed with the proof, consider a positive real $\alpha \in (\frac{m}{n}, 1]$ for two integers $m, n \in \mathbb{N}$, where $2 \leq m \leq n$. Then:

$$0 < n\alpha - m. \tag{33}$$

Since $g(\cdot)$ is a strictly increasing function over its domain from Assumption 1, it can be concluded from (30b) that $\hat{P}_2 > \hat{P}_1$ or $\hat{P}_2 - \hat{P}_1 > 0$. Then it yields from (33) that:

$$0 < (n\alpha - m)(\hat{P}_2 - \hat{P}_1). \tag{34}$$

After manipulation, (34) can be rewritten as follows:

$$n\alpha\hat{P}_1 + n(1-\alpha)\hat{P}_2 < m\hat{P}_1 + (n-m)\hat{P}_2. \tag{35}$$

According to (35), the total transmission power of the network given in the right-hand side of (35) is lower-bounded by the total power consumption of the network in a scenario that all n sensors have a common transmission power $\hat{P}$ defined as $\hat{P} = \alpha\hat{P}_1 + (1-\alpha)\hat{P}_2$. Moreover, since $\alpha \in (\frac{m}{n}, 1]$, hance $\hat{P}_1 \leq \hat{P} < \hat{P}_2$, which confirms that all the inequality constraints of (23) hold at the new optimal transmission power $\hat{P}$. Therefore, one can conclude that the minimizer of (32) corresponds to a scenario in which all constraints are active, *i.e.*, $m = n$ and $\hat{P}_T^i = \hat{P}_1 = \hat{P}$ for all $i \in \hat{V}$. Moreover, the transmission power vector $\hat{\mathrm{P}}_T = \hat{P}1_n$ is a minimizer of the constrained optimization problem (23), where $\hat{P}$ given in (31) is obtained from (30a) by considering $m = n$. This completes the proof. ■

According to the above results, the total power consumption of a random sensor network with directed cycle topology whose communication links have the same statistical characteristics is minimized when all sensors have a similar transmission power, while a lower bound on the WEC measure of its expected communication graph is maintained. Moreover, the optimal transmission power of each sensor can be obtained explicitly as a function of the network size, the WEC lower bound, and the statistical parameters of the communication channels.

Theorem 2: Conditions of Assumption 1 hold for the case when the power gain of every communication channel in the network is given by a log-normal distribution.

Proof: The proof is omitted due to space limitations. ■

VI. CONCLUSIONS

Optimal transmission power assignment is proposed in this work for connectivity control in an underwater acoustic sensor network. The information flow throughout the network is described by a random communication graph. The weighted edge connectivity is proposed as a novel measure to evaluate the connectivity of the corresponding expected communication graph. By adopting a convenient probability density function for the power gain of the acoustic communication channels, an explicit relation is obtained describing the existence probability of the links in terms of the transmission power of the sensors and the statistical parameters of the channels. The connectivity control problem is then formulated as an optimization problem to minimize the total power consumption of the network while a lower bound on the weighted edge connectivity is satisfied. An analytical solution is subsequently obtained for the optimal transmission power vector by considering a directed cycle topology for the expected communication graph of the network. The effectiveness of the optimal transmission power vector is then verified for a log-normal power gain distribution of the communication channels.

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