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The optimality of post-Doppler STAP

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Abstract

Herein find a proof of the theorem of asymptotic optimality of post-Doppler Space-Time Adaptive Processing (STAP). The proof uses well known properties of block Toeplitz and block circulant matrices.

Résumé

Nous démontrons le théorème de l'efficacité asymptotique du traitement adaptatif espace-temps (STAP) post-Doppler au moyen des propriétés bien connues des matrices de Toeplitz par blocs et des matrices circulantes par blocs.

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Executive summary

The optimality of post-Doppler STAP

Ishuwa Sikaneta, Joachim Ender; DRDC Ottawa TM 2013-014; Defence R&D Canada – Ottawa; January 2014.

Although this memo contains a mathematical proof, it has practical importance with regards to the computational complexity of STAP. In order to apply optimum STAP processing to a measured signal containing many samples, a large matrix must be computed and inverted. This matrix is referred to as the covariance matrix. Rather than computing this matrix in the original domain in which the signal is measured (the time domain) it has been common practice to transform the data into a different domain (the frequency or Doppler domain) because, in this domain, the covariance matrix is “smaller” and easier to invert, thus the entire operation can be executed more quickly. The idea that the matrix is “smaller” and has a more ideal structure, however, is not quite right. Most practitioners don’t see this as a problem, as they quote a theorem that states that as the number of samples under analysis becomes large, the matrix becomes more and more “well-behaved” and the difference between not-quite-right and right becomes increasingly negligible. We refer to this as the asymptotic optimality theorem.

It isn’t just the computation of the covariance matrix that is at issue, however, as post-Doppler STAP performs matrix inversion in the frequency domain. Although proofs exist concerning the covariance matrix in the frequency domain, the author is not aware of a proof demonstrating that the entire STAP operation in the Doppler domain is asymptotically optimal. In particular, it is not clear that the inverse of the covariance matrix has the appropriate asymptotic behavior required for the post-Doppler STAP prescription. Further, it is of practical importance to quantitatively determine the error as a function of the size of data under analysis. This memo thus provides a proof of the asymptotic optimality of post-Doppler STAP.

Sommaire

The optimality of post-Doppler STAP

Ishuwa Sikaneta, Joachim Ender ; DRDC Ottawa TM 2013-014 ; R & D pour la défense Canada – Ottawa ; janvier 2014.

Bien que la présente note contienne une preuve mathématique, elle a une importance pratique en ce qui concerne la complexité algorithmique du STAP. En vue d'appliquer un STAP optimal à un signal mesuré contenant de nombreux échantillons, une vaste matrice doit être calculée et inversée. Il s'agit de la matrice de covariance. Plutt que de calculer cette matrice dans le domaine d'origine dans lequel le signal est mesuré (le domaine temporel), il est de pratique courante de transformer les données dans un domaine différent (la fréquence ou le domaine Doppler) parce que, dans ce domaine, la matrice de covariance est "plus petite" et plus facile à inverser, de sorte que l'opération complète peut être exécutée en moins de temps. L'idée selon laquelle la matrice est "plus petite" et a une structure qui convient mieux n'est cependant pas tout à fait juste. La plupart des praticiens ne considèrent pas qu'il s'agisse d'un problème et citent à cet égard un théor 'eme énonant qu'à mesure que le nombre d'échantillons analysés s'accroît, la matrice se comporte de mieux en mieux, et la différence entre ce qui n'est pas très juste et ce qui est juste devient de plus en plus négligeable. C'est le théorème de l'efficacité asymptotique.

Ce n'est pas seulement le calcul de la matrice de covariance qui est en cause, toutefois, étant donné que le STAP post-Doppler exécute l'inversion de la matrice dans le domaine de la fréquence. Bien qu'il existe des preuves à propos de la matrice de covariance dans le domaine de la fréquence, l'auteur n'est au courant d'aucune preuve démontrant que l'opération complète du STAP dans le domaine Doppler soit efficace asymptotiquement. En particulier, on ne sait pas avec certitude si l'inverse de la matrice de covariance se comporte de la façon asymptotique requise aux fins du STAP post-Doppler. De plus, il est important du point de vue pratique de déterminer quantitativement l'erreur en tant que fonction de la taille des données analysées. Cette note présente donc une preuve de l'efficacité asymptotique du STAP post-Doppler.

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1 Notation

Table 1: Notation

N	Number of channels
f_p	Pulse repetition frequency for each channel
ς_i	Sampling delay of antenna i
v_a	Radar platform velocity
κ_x	Spatial sample spacing v_a/f_p
$h_i(x)$	Measurement function of SAR antenna i
$(\cdot)^T$	Transpose operator
$(\cdot)^\dagger$	Conjugate transpose operator
$g_M(\cdot;\cdot)$	Dirichlet function
$\delta(x-y)$	Delta function
$\delta(n,m)$	Kronecker-delta function
$\mathcal{E}\{\cdot\}$	Expected value
$\sum_l$	$\sum_{l=-\infty}^{\infty}$

2 Introduction

This memo proves the following theorem:

Theorem 2.1. *Post-Doppler STAP using an M-point DFT is asymptotically optimal for increasing M.*

Since the memo deals with data in the DFT domain, it also proves the following:

Theorem 2.2. *At each frequency bin in the Doppler domain, the clutter covariance matrix of an adequately sampled multi-channel SAR GMTI system is asymptotically rank 1 for increasing M.*

Both theorems are commonly quoted and used. A proof of the first theorem for a configuration where the covariance matrix for the collected data assumes a banded block Toeplitz structure is provided in [1]; however, a more general proof has proven elusive. The second theorem has been proved in the literature and is included here only because it is somewhat related.

The proof presented here is designed for the applied mathematician.

A model of the system and the geometry are first used to describe the measured signal. The optimal STAP approach and the commonly used post-Doppler STAP approach are then described. Section 6 steps back briefly to review/introduce the tools needed to prove optimality. Much of the material in section 6 is based on [2], which, in the author's opinion, is an excellent article. Some basic proofs of pertinent lemmas and theorems are repeated from [2]; more often, however, this memorandum generalizes the mathematics from regular Toeplitz matrices to block-Toeplitz matrices. With the tools in hand, the last part of section 6 constructs the proof of the asymptotic optimality of post-Doppler STAP. In the final section, a proof is provided of the asymptotic rank 1 characteristic of the clutter covariance matrix.

3 The measured signal

Assume that the SAR measures homogeneous terrain. The terrain is such that each scatterer is statistically independent from its neighbour. Consult Figure 3 to visualize the measure-

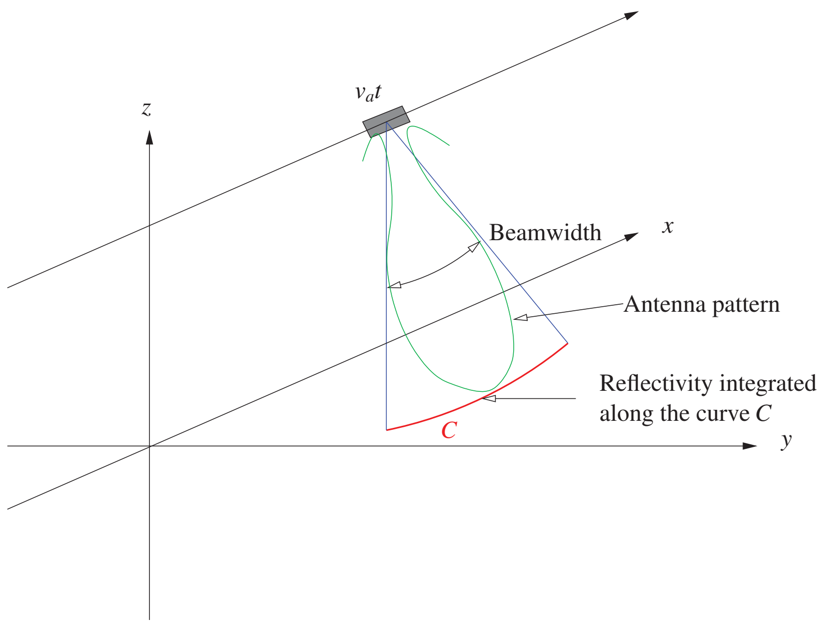

Figure 1: *Illustration measurement system.*

ment at time t. The measured, baseband, signal is a $\mathbb{C}^N$ vector given by

$$\mathbf{z}(t) = \begin{bmatrix} \int_C h_1(x) g(v_a(t-\varsigma_1) - x, t-\varsigma_1)\mathrm{d}x + \nu_1(t-\varsigma_1) \\ \int_C h_2(x) g(v_a(t-\varsigma_2) - x, t-\varsigma_2)\mathrm{d}x + \nu_2(t-\varsigma_2) \\ \vdots \\ \int_C h_N(x) g(v_a(t-\varsigma_N) - x, t-\varsigma_N)\mathrm{d}x + \nu_N(t-\varsigma_N) \end{bmatrix} \tag{1}$$

where $g(x,t)$ represents the stochastic process of scatterers on the ground and $h_l(x)$ represents the l^{th} SAR channel function which depends on the range, the beamwidth and the geometry, and $\nu_l(t)$ is the additive Gaussian noise (thermal) that accompanies each measurement. The SAR platform takes a noisy measurement of $g(x,t)$ from position $v_a(t+\varsigma_l$ using the l^{th} antenna represented by $h_l(x)$. Being a pulsed system, the m^{th} measurement is given by

$$\mathbf{z}(m) = \begin{bmatrix} \int_C h_1(x) g(v_a(\frac{m}{f_p}-\varsigma_1)-x, \frac{m}{f_p}-\varsigma_1)\mathrm{d}x + \nu_1(\frac{m}{f_p}-\varsigma_1) \\ \int_C h_2(x) g(v_a(\frac{m}{f_p}-\varsigma_2)-x, \frac{m}{f_p}-\varsigma_2)\mathrm{d}x + \nu_2(\frac{m}{f_p}-\varsigma_2) \\ \vdots \\ \int_C h_N(x) g(v_a(\frac{m}{f_p}-\varsigma_N)-x, \frac{m}{f_p}-\varsigma_N)\mathrm{d}x + \nu_N(\frac{m}{f_p}-\varsigma_N) \end{bmatrix} \tag{2}$$

4 The time and frequency domain covariance matrices

A good introduction and overview to the material in this section is provided in [1].

The covariance matrix elements are computed as

$$\begin{aligned} &\mathcal{E}\{z_i(m)z_j^*(m')\} = \mathcal{E}\Bigg\{\left[\int_C h_i(x) g\left(v_a\frac{m}{f_p} - v_a\varsigma_i - x, \frac{m}{f_p}-\varsigma_i\right)\mathrm{d}x + \nu_i(m/f_p - \varsigma_i)\right] \\ &\cdot \left[\int_C h_j(y) g\left(v_a\frac{m'}{f_p} - v_a\varsigma_j - y, \frac{m'}{f_p}-\varsigma_j\right)\mathrm{d}y + \nu_j(m'/f_p - \varsigma_j)\right]^\dagger\Bigg\} \\ &= \int_C\int_C h_i(x)h_j^*(y)\mathcal{E}\left\{g\left(v_a\frac{m}{f_p} - v_a\varsigma_i - x, \frac{m}{f_p}-\varsigma_i\right) g^*\left(v_a\frac{m'}{f_p} - v_a\varsigma_j - y, \frac{m'}{f_p}-\varsigma_j\right)\right\}\mathrm{d}x\mathrm{d}y \\ &+ \sigma_n^2\delta(m,m')\delta(i,j) \end{aligned} \tag{3}$$

Under the assumption that $\mathcal{E}\{g(x,t)g^*(x',t')\} = \delta(x-x')$ (spatially white with no temporal decorrelation)

$$\begin{aligned} &= \int_C\int_C h_i(x)h_j^*(y)\sigma_c^2\delta\left(\frac{v_a(m-m')}{f_p} - v_a(\varsigma_i-\varsigma_j) - x + y\right)\mathrm{d}x\mathrm{d}y + \sigma_n^2\delta(m,m')\delta(i,j) \\ &= \sigma_c^2\int_C h_i\left(y + \frac{v_a(m-m')}{f_p} - v_a(\varsigma_i-\varsigma_j)\right)h_j^*(y)\mathrm{d}y + \sigma_n^2\delta(m,m')\delta(i,j) \\ &= \sigma_c^2\int_C h_i\left(y - v_a\varsigma_i + \frac{v_a(m-m')}{f_p}\right)h_j^*(y - v_a\varsigma_j)\mathrm{d}y + \sigma_n^2\delta(m,m')\delta(i,j). \end{aligned} \tag{4}$$

Let us write that

$$H_j(\xi) = \int h_j(x - v_a\varsigma_j)\exp(-2\pi\imath\xi x)\mathrm{d}x. \tag{5}$$

These scalars can be aligned to form the vector $\mathbf{H}(\xi)$. One can then write that

$$\mathcal{E}\{\mathbf{z}(m)\mathbf{z}^\dagger(m')\} = \mathbf{C}_{m,m'} = \sigma_c^2 \int \mathbf{H}(\xi)\mathbf{H}^\dagger(\xi) e^{2\pi\imath\xi(m-m')\kappa_x} \mathrm{d}\xi + \sigma_n^2 \mathbf{I}_N \delta(m-m'), \tag{6}$$

where $\kappa_x = v_a/f_p$. Since the correlation matrix only depends on the difference $k = m - m'$, the above can be written as

$$\mathbf{C}_k = \sigma_c^2 \int \mathbf{C}_{\mathbf{HH}}(\xi) e^{2\pi\imath\xi k\kappa_x} \mathrm{d}\xi + \sigma_n^2 \mathbf{I}_N \delta(k), \tag{7}$$

where $\mathbf{C}_{\mathbf{HH}}(\xi) = \mathbf{H}(\xi)\mathbf{H}^\dagger(\xi)$. The covariance matrix in the time domain has a block Toeplitz matrix structure and is given by

$$\mathbf{C}^M = \begin{bmatrix} \mathbf{C}_0 & \mathbf{C}_{-1} & \dots & \mathbf{C}_{-M} \\ \mathbf{C}_1 & \mathbf{C}_0 & \dots & \mathbf{C}_{1-M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{C}_M & \mathbf{C}_{M-1} & \dots & \mathbf{C}_0 \end{bmatrix} \tag{8}$$

After applying the DFT, the covariance matrix in the Doppler domain has a block structure with $N \times N$ blocks given by

$$\begin{aligned} \mathbf{R}_{k_f,k'_f} &= \mathcal{E}\left\{ \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \mathbf{z}(m) \exp\left(\frac{-\imath 2\pi k_f m}{M}\right) \frac{1}{\sqrt{M}} \sum_{m'=0}^{M-1} \mathbf{z}^\dagger(m') \exp\left(\frac{\imath 2\pi k'_f m'}{M}\right) \right\} \\ &= \frac{1}{M} \sum_{m,m'=0}^{M-1} \mathcal{E}\{\mathbf{z}(m)\mathbf{z}^\dagger(m')\} \exp\left(\frac{-\imath 2\pi [k_f m - k'_f m']}{M}\right), \end{aligned} \tag{9}$$

By using (6), the above evaluates to

$$\begin{aligned} \mathbf{R}_{k_f,k'_f} = & \frac{\sigma_c^2}{M} \int \mathbf{C}_{\mathbf{HH}}(\xi) g_M\left(M; k_f - M\xi\kappa_x\right) g_M^*\left(M; k'_f - M\xi\kappa_x\right) \mathrm{d}\xi \\ & + \frac{\sigma_n^2}{M} g_M(M; k_f - k'_f)\mathbf{I}, \end{aligned} \tag{10}$$

where the Dirichlet function, $g_M(\cdot;\cdot)$, is defined in section B.

5 Processing to detect an added deterministic signal

Let the collection of M measurements (vectors) of the clutter and noise be concatenated into the $NM \times 1$ vector

$$\mathbf{z}^M = \begin{bmatrix} \mathbf{z}(0) \\ \mathbf{z}(1) \\ \vdots \\ \mathbf{z}(M-1) \end{bmatrix} \tag{11}$$

Now assume that at each pulse, the radar also measures a target signal $\mathbf{s}_i$. The target is assumed to be deterministic and can also be concatenated into a $NM \times 1$ vector

$$\mathbf{s}^M = \begin{bmatrix} \mathbf{s}(0) \\ \mathbf{s}(1) \\ \vdots \\ \mathbf{s}(M-1) \end{bmatrix} \tag{12}$$

Before any processing, the signal-to clutter ratio is defined as

$$\frac{\mathbf{s}^{M\dagger}\mathbf{s}^M}{\mathcal{E}\{\mathbf{z}^{M\dagger}\mathbf{z}^M\}} \tag{13}$$

5.1 Optimum processing

The measured clutter plus target signal $\mathbf{y}^M = \mathbf{z}^M + \mathbf{s}^M$ is processed by examining the norm-squared value of a linear combination of the elements of $\mathbf{y}^M$ via

$$|\mathbf{b}^\dagger \mathbf{y}^M|^2. \tag{14}$$

It is known that the vector $\mathbf{b}$ that optimizes the signal to clutter ratio

$$\frac{|\mathbf{b}^\dagger \mathbf{s}^M|^2}{\mathcal{E}\{|\mathbf{b}^\dagger \mathbf{z}^M|^2\}} \tag{15}$$

is given by

$$\mathbf{b} = \frac{[\mathbf{C}^M]^{-1}\mathbf{w}^M}{\sqrt{\mathbf{w}^{M\dagger}[\mathbf{C}^M]^{-1}\mathbf{w}^M}}. \tag{16}$$

The maximum occurs when $\mathbf{w} = \mathbf{s}^M$ and is given by $\mathbf{s}^{M\dagger}[\mathbf{C}^M]^{-1}\mathbf{s}^M$.

5.2 post-Doppler STAP

Because of the computational expense of computing $[\mathbf{C}^M]^{-1}$, post-Doppler STAP transforms the measured data into the Doppler Domain by using an M-point DFT

$$\mathbf{y}^M = \mathbf{z}^M + \mathbf{s}^M \rightarrow \mathbf{Y}^M = \mathbf{Z}^M + \mathbf{S}^M \tag{17}$$

and then examines $|\mathbf{b}_{pd}^\dagger \mathbf{Y}^M|^2$, where

$$\mathbf{b}_{pd} = \frac{[\mathbf{R}^M]^{-1}\mathbf{w}^M}{\sqrt{\mathbf{w}^{M\dagger}[\mathbf{R}^M]^{-1}\mathbf{w}^M}}, \tag{18}$$

and

$$\mathbf{R}^M = \begin{bmatrix} \mathbf{R}_{M-1,M-1} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{R}_{M-2,M-2} & \mathbf{0} & \dots \\ \vdots & \vdots & \ddots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R}_{0,0} \end{bmatrix}. \tag{19}$$

The matrices $\mathbf{R}_{m,n}$ are as defined in equation (9). The challenge of this document is to show that the prescription for post-Doppler STAP approaches the signal processing gain achieved by optimal processing as $M \to \infty$.

6 Proof of asymptotic optimality

The objective of showing asymptotic optimality can be approached by using the theory of Toeplitz and Circulant matrices. A engineers perspective on the topic is well presented in [2] which provides the basis for this proof. In contrast to the material in [2], our application deals with block Toeplitz and block circulant matrices. The approach is as follows:

1. Review the following material from [2]:
   (a) Matrix norms (strong and weak).
   (b) Asymptotic equivalence of two sequences of matrices $\mathbf{A}_n$ and $\mathbf{B}_n$, and the requirements for the asymptotic equivalance of $\mathbf{A}_n^{-1}$ and $\mathbf{B}_n^{-1}$.

   These concepts form the building blocks of the proof.
2. Show that a suitably chosen sequence of block circulant matrices is asymptotically equivalent to the sequence of block Toeplitz matrices.
3. Show that the inverse of the sequence of block circulant matrix is asymptotically equivalent to the inverse of the sequence of block Toeplitz matrices.
4. Show that the block circulant matrix implies the DFT.
5. Show that the prescription for post-Doppler STAP implies a sequence of matrices that are asymptotically equivalent to the sequence of inverse block Toeplitz matrices.

6.1 Prerequisites

6.1.1 Matrix Norms

This material is contained of [2] where two norms are considered. Both norms obey the usual rules

1. $\text{Norm}(\mathbf{A}) = 0$ iff $\mathbf{A} = 0$.
2. $\text{Norm}(\mathbf{A}+\mathbf{B}) \leq \text{Norm}(\mathbf{A}) + \text{Norm}(\mathbf{B})$
3. For scalar c, $\text{Norm}(c\mathbf{A}) = |c|\text{Norm}(\mathbf{A})$.

Definition 6.1. *The strong norm of a $N \times N$ matrix $\mathbf{A}$ is defined as*

$$\|\mathbf{A}\| = \max_k \sqrt{\lambda_k} \tag{20}$$

where λ_k are the eigenvalues of $\mathbf{A}^\dagger \mathbf{A}$.

Definition 6.2. *The weak norm is defined as*

$$|\mathbf{A}| = \left(\frac{1}{N} \sum_{j,k=0}^{N-1} |a_{jk}|^2 \right)^{\frac{1}{2}} = \left(\frac{1}{N} Tr \left[\mathbf{A}^\dagger \mathbf{A} \right] \right)^{\frac{1}{2}} = \left(\frac{1}{N} \sum_{k=0}^{N-1} \lambda_k \right)^{\frac{1}{2}}. \tag{21}$$

The adjectives strong and weak stem from the fact that $\max_k\{\lambda_k\} \geq \text{average}\{\lambda_k\}$. Lemma 2.3 in [2] states and proves that

$$|\mathbf{GH}| \leq \|\mathbf{G}\| \; |\mathbf{H}|. \tag{22}$$

6.1.2 Asymptotic equivalence of sequences of matrices

Again, from [2].

Definition 6.3. *Two sequences of matrices, $\mathbf{A}_m$ and $\mathbf{B}_m$ are said to be asymptotically equivalent, denoted by $\mathbf{A}_m \sim \mathbf{B}_m$, if*

1. *$\mathbf{A}_m$ and $\mathbf{B}_m$ are uniformly bounded in strong norm*

$$\|\mathbf{A}_m\|, \|\mathbf{B}_m\| \leq K < \infty, m = 0, 1, 2, \ldots \tag{23}$$

2. *$\mathbf{A}_m - \mathbf{B}_m = \mathbf{D}_m$ goes to zero in weak norm as $m \to \infty$*

$$\lim_{m\to\infty} |\mathbf{A}_m - \mathbf{B}_m| = \lim_{m\to\infty} |\mathbf{D}_m| = 0 \tag{24}$$

Most salient for this discussion is the following theorem and proof from [2]:

Theorem 6.1. *Properties of asymptotic equivalence*

1. *If $\mathbf{A}_m \sim \mathbf{B}_m$ and $\mathbf{B}_m \sim \mathbf{C}_m$, then $\mathbf{A}_m \sim \mathbf{C}_m$*
2. *if $\mathbf{A}_m \sim \mathbf{B}_m$, and $\|\mathbf{A}_m^{-1}\|, \|\mathbf{B}_m^{-1}\| \leq Q < \infty$ for all m, then $\mathbf{A}_m^{-1} \sim \mathbf{B}_m^{-1}$.*

Proof. Proofs of equivalence properties

1. $|\mathbf{A}_m - \mathbf{C}_m| = |\mathbf{A}_m - \mathbf{B}_m + \mathbf{B}_m - \mathbf{C}_m| \leq |\mathbf{A}_m - \mathbf{B}_m| + |\mathbf{B}_m - \mathbf{C}_m|$. The expression on the right goes to zero as $m \to \infty$.

2.

$$\begin{aligned}\lim_{m\to\infty}|\mathbf{A}_m^{-1}-\mathbf{B}_m^{-1}| &= \lim_{m\to\infty}|\mathbf{B}_m^{-1}\mathbf{B}_m\mathbf{A}_m^{-1}-\mathbf{B}_m^{-1}\mathbf{A}_m\mathbf{A}_m^{-1}| \\ &\leq \lim_{m\to\infty}\|\mathbf{A}_m^{-1}\|\;\|\mathbf{B}_m^{-1}\|\;|\mathbf{A}_m-\mathbf{B}_m| \\ &= 0.\end{aligned} \tag{25}$$

□

We will also use the following lemmas concerning the asymptotic equivalence of Hermitian matrices to construct the proof

Lemma 6.1. *Let $\mathbf{A}_n, \mathbf{B}_n$ be Hermitian with ordered eigenvalues $\{\alpha_{n,k}\}_{k=0\ldots n-1}, \{\beta_{n,k}\}_{k=0\ldots n-1}$ such that $\mathbf{A}_n = \mathbf{U}_n \mathrm{diag}(\alpha_{n,k})\mathbf{U}_n^\dagger$, and $\mathbf{B}_n = \mathbf{V}_n \mathrm{diag}(\beta_{n,k})\mathbf{V}_n^\dagger$. Then,*

1. $\mathrm{diag}(\alpha_{n,k}) \sim \mathrm{diag}(\beta_{n,k})$,
2. *and* $\mathbf{A}_n \sim \mathbf{V}_n \mathrm{diag}(\alpha_{n,k})\mathbf{V}_n^\dagger$.

The second point says that, asymptotically, the diagonalizing matrix of $\mathbf{B}_n$ also diagonalizes the matrix $\mathbf{A}_n$.

Proof. By the Wielandt-Hoffman theorem (see, for example [2], Theorem 2.5),

$$|\mathrm{diag}(\alpha_{n,k}) - \mathrm{diag}(\beta_{n,k})| \leq |\mathbf{A}_n - \mathbf{B}_n|, \tag{26}$$

hence, if $\mathbf{A}_n \sim \mathbf{B}_n$, then $\mathrm{diag}(\alpha_{n,k}) \sim \mathrm{diag}(\beta_{n,k})$. Since multiplication by a unitary matrix does not change the weak norm,

$$\begin{aligned}|\mathrm{diag}(\alpha_{n,k}) - \mathrm{diag}(\beta_{n,k})| &= |\mathbf{V}_n\mathrm{diag}(\alpha_{n,k})\mathbf{V}_n^\dagger - \mathbf{V}_n\mathrm{diag}(\beta_{n,k})\mathbf{V}_n^\dagger| \\ &= |\mathbf{V}_n\mathrm{diag}(\alpha_{n,k})\mathbf{V}_n^\dagger - \mathbf{B}_n| \\ &\leq |\mathbf{A}_n - \mathbf{B}_n|.\end{aligned} \tag{27}$$

Therefore, $\mathbf{V}_n\mathrm{diag}(\alpha_{n,k})\mathbf{V}_n^\dagger \sim \mathbf{B}_n$ and $\mathbf{A}_n \sim \mathbf{B}_n$ so by Theorem 6.1, $\mathbf{A}_n \sim \mathbf{V}_n\mathrm{diag}(\alpha_{n,k})\mathbf{V}_n^\dagger$

□

6.1.3 The block Toeplitz matrix for multi-channel SAR-GMTI

By using (7), we define the matrix function

$$\begin{aligned}\Phi(\alpha) &= \sum_{k=-\infty}^{\infty} \mathbf{C}_k \exp(\imath\alpha k)\\ &= \sigma_c^2 \int \mathbf{C_{HH}}(\xi) \sum_{k=-\infty}^{\infty} e^{\imath k(\alpha+2\pi\rho_x\xi)} \mathrm{d}\xi + \sigma_n^2 \mathbf{I}_N\\ &= 2\pi\sigma_c^2 \int \mathbf{C_{HH}}(\xi) \sum_l \delta(\alpha + 2\pi\rho_x\xi + 2\pi l) \mathrm{d}\xi + \sigma_n^2 \mathbf{I}_N\\ &= \frac{\sigma_c^2}{\rho_x} \sum_l \mathbf{C_{HH}}\left(-\frac{1}{\rho_x}\left[\frac{\alpha}{2\pi} + l\right]\right) + \sigma_n^2 \mathbf{I}_N.\end{aligned} \tag{28}$$

By definition

$$\mathbf{C}_k = \frac{1}{2\pi} \int_0^{2\pi} \Phi(\alpha) e^{-\imath k\alpha} \mathrm{d}\alpha. \tag{29}$$

If $\Phi(\alpha) = \Phi(\alpha)^\dagger$ for all α, then $\mathbf{C}_k = \mathbf{C}_{-k}^\dagger$ and $\mathbf{C}^M$ is Hermitian and, therefore, has real eigenvalues.

The block Toeplitz matrix defined in (8), $\mathbf{C}^M$, is completely defined by the matrix of functions $\Phi(\alpha)$. We thus denote the block Toeplitz matrix as $\mathbf{C}^M(\Phi[\alpha])$.

6.1.4 Bounds on the eigenvalues of the block Toeplitz matrix

In order to apply definition 6.3, one must first analyze the strong norm of the sequence of block Toeplitz matrices. Following in the same vein as [2], let us examine the nature of the minimum and maximum eigenvalues of $\mathbf{C}^M(\Phi[\alpha])$. The minimum and maximum eigenvalues are given by

$$\lambda_m = \min_{\mathbf{x}} \frac{\mathbf{x}^\dagger \mathbf{C}^M(\Phi[\alpha])\mathbf{x}}{\mathbf{x}^\dagger\mathbf{x}}$$
$$\lambda_M = \max_{\mathbf{x}} \frac{\mathbf{x}^\dagger \mathbf{C}^M(\Phi[\alpha])\mathbf{x}}{\mathbf{x}^\dagger\mathbf{x}}$$

Let the vector $\mathbf{x}$ be given by

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{M-1} \end{bmatrix} \tag{30}$$

where each $\mathbf{x}_i$ is a $N \times 1$ vector [1]. Then

$$\begin{aligned}
\mathbf{x}^\dagger \mathbf{C}^M(\Phi[\alpha])\mathbf{x} &= \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} \mathbf{x}_m^\dagger \mathbf{C}_{m-n}\mathbf{x}_n \\
&= \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} \mathbf{x}_m^\dagger \frac{1}{2\pi}\int_0^{2\pi} \Phi(\alpha) e^{-\imath(m-n)\alpha} \mathrm{d}\alpha \mathbf{x}_n \\
&= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{M-1}\sum_{m=0}^{M-1} \mathbf{x}_m^\dagger e^{-\imath m\alpha}\Phi(\alpha)\mathbf{x}_n e^{\imath n\alpha}\mathrm{d}\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{m=0}^{M-1} \mathbf{x}_m e^{\imath m\alpha}\right]^\dagger \Phi(\alpha) \left[\sum_{n=0}^{M-1} \mathbf{x}_n e^{\imath n\alpha}\right] \mathrm{d}\alpha.
\end{aligned} \tag{31}$$

Also,

$$\begin{aligned}
\mathbf{x}^\dagger \mathbf{x} &= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{m=0}^{M-1} \mathbf{x}_m e^{\imath m\alpha}\right]^\dagger \mathbf{I}_N \left[\sum_{n=0}^{M-1} \mathbf{x}_n e^{\imath n\alpha}\right] \mathrm{d}\alpha \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{n=0}^{M-1} \mathbf{x}_n e^{\imath n\alpha}\right|^2 \mathrm{d}\alpha.
\end{aligned} \tag{32}$$

Let us write the eigen-decomposition of $\Phi(\alpha)$ as

$$\Phi(\alpha) = \sum_{n=1}^{N} \lambda_n(\alpha)\mathbf{u}_n(\alpha)\mathbf{u}_n^\dagger(\alpha). \tag{33}$$

Then, since the orthonormal eigenvectors at each point α span $\mathbb{C}^N$, one can write

$$\sum_{m=0}^{M-1} \mathbf{x}_m e^{\imath m\alpha} = \sum_{n=1}^{N} a_n(\alpha)\mathbf{u}_n(\alpha), \tag{34}$$

hence

$$\min_{n,\alpha} \lambda_n(\alpha) \leq \frac{\mathbf{x}^\dagger \mathbf{C}^M(\Phi[\alpha])\mathbf{x}}{\mathbf{x}^\dagger \mathbf{x}} = \frac{\int_0^{2\pi} \sum_{n=1}^{N} |a_n(\alpha)|^2 \lambda_n(\alpha)\mathrm{d}\alpha}{\int_0^{2\pi} \sum_{n=1}^{N} |a_n(\alpha)|^2 \mathrm{d}\alpha} \leq \max_{n,\alpha} \lambda_n(\alpha). \tag{35}$$

and, therefore,

$$\lambda_m \geq \min_{n,\alpha} \lambda_n(\alpha) = e_\Phi \geq \sigma_n^2 \tag{36}$$

$$\lambda_M \leq \max_{n,\alpha} \lambda_n(\alpha) = E_\Phi < \infty \tag{37}$$

1. This is a slight change in notation - previously we have used the superscript M for such vectors, and we had used $\mathbf{x}(0)$ rather than $\mathbf{x}_0$. It is thought that the new notation is more aesthetically pleasing.

By (37), one sees that

$$\left\|\mathbf{C}^{M}(\Phi[\alpha])\right\| < \infty, \tag{38}$$

and by (36)

$$\left\|[\mathbf{C}^{M}(\Phi[\alpha])]^{-1}\right\| < \infty. \tag{39}$$

6.2 A candidate block circulant matrix for asymptotic equivalence

Define the $NM \times NM$ block circulant matrix

$$\mathbf{D}^{M} = \begin{bmatrix} \mathbf{D}_0^M & \mathbf{D}_1^M & \dots & \mathbf{D}_{M-1}^M \\ \mathbf{D}_{M-1}^M & \mathbf{D}_0^M & \dots & \mathbf{D}_{M-2}^M \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}_1^M & \mathbf{D}_2^M & \dots & \mathbf{D}_0^M \end{bmatrix}, \tag{40}$$

as $\mathbf{D}^{M}(\Phi[\alpha])$ with a top row of blocks given by

$$\mathbf{D}_k^M = \frac{1}{M}\sum_{n=0}^{M-1}\Phi\left(\frac{2\pi n}{M}\right)e^{2\pi\iota\frac{nk}{M}}. \tag{41}$$

One notes that

$$\lim_{M\to\infty}\mathbf{D}_k^M = \mathbf{C}_{-k}, \tag{42}$$

if the integral in (29) can be written as a Riemann sum.

Also, if $\Phi(\alpha)$ is Hermitian for all α, then one sees that

$$\mathbf{D}_k = \mathbf{D}_{M-k}^{\dagger}. \tag{43}$$

Finally, from (28), one notes that putting $\alpha = \frac{2\pi n}{M}$ yields

$$\Phi\left(\frac{2\pi n}{M}\right) = \sum_{l=-\infty}^{\infty}\mathbf{C}_l e^{2\pi\iota\frac{nl}{M}} \tag{44}$$

which, when substituted into (41) gives

$$\begin{aligned} \mathbf{D}_k^M &= \frac{1}{M}\sum_{n=0}^{M-1}\sum_{l=-\infty}^{\infty}\mathbf{C}_l e^{2\pi\imath\frac{nl}{M}} e^{2\pi\imath\frac{nk}{M}} \\ &= \sum_{l=-\infty}^{\infty}\mathbf{C}_l \frac{1}{M}\sum_{n=0}^{M-1} e^{2\pi\imath\frac{n(l+k)}{M}} \\ &= \sum_{l=-\infty}^{\infty}\mathbf{C}_l \delta[\mathrm{mod}_M(l+k)] \\ &= \sum_{l=-\infty}^{\infty}\mathbf{C}_{-k+lM} \end{aligned} \tag{45}$$

Theorem 6.2. *Suppose that*

$$\Phi\left(\frac{2\pi l}{M}\right)\mathbf{v} = \lambda\mathbf{v},$$

i.e. $\mathbf{v}$ *is an eigenvector of* $\Phi\left(\frac{2\pi l}{M}\right)$ *with eigenvalue* λ*. Then* λ *is also an eigenvalue of the block circulant matrix* $\mathbf{D}^M(\Phi[\alpha])$ *with eigenvector*

$$\mathbf{u}(\mathbf{v},l) = \frac{1}{\sqrt{M}}\begin{bmatrix} \mathbf{v}e^{-2\pi\imath\frac{l\cdot 0}{M}} \\ \mathbf{v}e^{-2\pi\imath\frac{l\cdot 1}{M}} \\ \vdots \\ \mathbf{v}e^{-2\pi\imath\frac{l\cdot(M-1)}{M}} \end{bmatrix}, \tag{46}$$

Proof.

$$\mathbf{D}^M(\Phi[\alpha])\mathbf{u}(\mathbf{v},l) = \begin{bmatrix} \mathbf{D}_0^M & \mathbf{D}_1^M & \cdots & \mathbf{D}_{M-1}^M \\ \mathbf{D}_{M-1}^M & \mathbf{D}_0^M & \cdots & \mathbf{D}_{M-2}^M \\ \vdots & \vdots & \ddots & \vdots \end{bmatrix}\mathbf{u}(\mathbf{v},l). \tag{47}$$

The $N \times 1$ block (times $\sqrt{M}$) in the $(p-1)^{\text{th}}$ row of the above is given by

$$\begin{aligned}
\sum_{k=0}^{M-1} \mathbf{D}_k^M \mathbf{v} e^{-2\pi\imath \frac{l(k+p)}{M}} &= \sum_{k=0}^{M-1} \frac{1}{M} \sum_{n=0}^{M-1} \Phi\left(\frac{2\pi n}{M}\right) e^{2\pi\imath \frac{nk}{M}} \mathbf{v} e^{-2\pi\imath \frac{l(k+p)}{M}} \\
&= \sum_{n=0}^{M-1} \Phi\left(\frac{2\pi n}{M}\right) \mathbf{v} e^{-2\pi\imath \frac{lp}{M}} \frac{1}{M} \sum_{k=0}^{M-1} e^{-2\pi\imath \frac{k(l-n)}{M}} \\
&= \sum_{n=0}^{M-1} \Phi\left(\frac{2\pi n}{M}\right) \mathbf{v} e^{-2\pi\imath \frac{lp}{M}} \delta(l,n) \\
&= \Phi\left(\frac{2\pi l}{M}\right) \mathbf{v} e^{-2\pi\imath \frac{lp}{M}} \\
&= \lambda \mathbf{v} e^{-2\pi\imath \frac{lp}{M}}
\end{aligned} \tag{48}$$

One thus sees that $\mathbf{u}(\mathbf{v}, l)$ is an eigenvector of $\mathbf{D}^M(\Phi[\alpha])$ with eigenvalue λ. □

Note that since $E_\Phi < \infty$

$$\left\| [\mathbf{D}^M(\Phi[\alpha])] \right\| \leq E_\Phi < \infty, \tag{49}$$

and, since $\lambda \geq e_\Phi > 0$

$$\left\| [\mathbf{D}^M(\Phi[\alpha])]^{-1} \right\| \leq e_\Phi^{-1} < \infty. \tag{50}$$

Also, $[\mathbf{D}^M(\Phi[\alpha])]^{-1}$ has eigenvalues given by $1/\lambda$, and has the same eigenvectors as $[\mathbf{D}^M(\Phi[\alpha])]$.

6.3 Asymptotic equivalence of $\mathbf{D}^M(\Phi[\alpha])$ and $\mathbf{C}^M(\Phi[\alpha])$

Denote $\Delta^M = \mathbf{D}^M(\Phi[\alpha]) - \mathbf{C}^M(\Phi[\alpha])$, then

$$\begin{aligned}
|\Delta^M|^2 = \frac{1}{M} \Big[& M|\mathbf{D}_0 - \mathbf{C}_0|^2 \\
& + (M-1)|\mathbf{D}_1 - \mathbf{C}_{-1}|^2 + (M-2)|\mathbf{D}_2 - \mathbf{C}_{-2}|^2 + \ldots + |\mathbf{D}_{M-1} - \mathbf{C}_{-(M-1)}|^2 \\
& + (M-1)|\mathbf{D}_{M-1} - \mathbf{C}_1|^2 + (M-2)|\mathbf{D}_{M-2} - \mathbf{C}_2|^2 + \ldots + |\mathbf{D}_1 - \mathbf{C}_{(M-1)}|^2 \Big] \\
= \frac{1}{M} \Big[& M|\mathbf{D}_0 - \mathbf{C}_0|^2 \\
& + (M-1)|\mathbf{D}_1 - \mathbf{C}_{-1}|^2 + (M-2)|\mathbf{D}_2 - \mathbf{C}_{-2}|^2 + \ldots + |\mathbf{D}_{M-1} - \mathbf{C}_{-(M-1)}|^2 \\
& + (M-1)|\mathbf{D}_1^\dagger - \mathbf{C}_{-1}^\dagger|^2 + (M-2)|\mathbf{D}_2^\dagger - \mathbf{C}_{-2}^\dagger|^2 + \ldots + |\mathbf{D}_{M-1}^\dagger - \mathbf{C}_{-(M-1)}^\dagger|^2 \Big] \\
= \frac{1}{M} \Big[& M|\mathbf{D}_0 - \mathbf{C}_0|^2 + 2\sum_{k=1}^{M-1} (M-k)|\mathbf{D}_k - \mathbf{C}_{-k}|^2 \Big].
\end{aligned} \tag{51}$$

By (45), this can be written as

$$
\begin{aligned}
|\Delta^M|^2 &= \left|\sum_{l\neq 0} \mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \left|\sum_{l\neq 0} \mathbf{C}_{-k+lM}\right|^2 \\
&\leq \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l\neq 0} \left|\mathbf{C}_{-k+lM}\right|^2 \\
&\leq \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{-k+lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{-k-lM}\right|^2 \\
&= \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+(l-1)M}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{-k-lM}\right|^2 \\
&= \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \left|\mathbf{C}_{k}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{-k-lM}\right|^2
\end{aligned}
\tag{52}
$$

In the case where $\mathbf{C}_k = \mathbf{C}^{\dagger}_{-k}$, the above evaluates to

$$
\begin{aligned}
|\Delta^M|^2 &\leq \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \left|\mathbf{C}_{k}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{M-k}{M} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+lM}\right|^2 \\
&= \sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \left|\mathbf{C}_{k}\right|^2 + 2\sum_{k=1}^{M-1} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+lM}\right|^2 \\
&\leq 2\sum_{l\neq 0} \left|\mathbf{C}_{lM}\right|^2 + 2\sum_{k=1}^{M-1} \frac{k}{M} \left|\mathbf{C}_{k}\right|^2 + 2\sum_{k=1}^{M-1} \sum_{l=1}^{\infty} \left|\mathbf{C}_{k+lM}\right|^2 \\
&= 2\sum_{k=0}^{M-1} \frac{k}{M} \left|\mathbf{C}_{k}\right|^2 + 2\sum_{k=M}^{\infty} \left|\mathbf{C}_{k}\right|^2
\end{aligned}
\tag{53}
$$

Now apply Kronecker's lemma 7.1 to see that the first term above goes to zero. Since the second term also goes to zero as $M \to \infty$ since, by assumption, $\mathbf{C}_k$ represents a Fourier series, one finds that

$$\lim_{M\to\infty} |\Delta^M|^2 = 0. \tag{54}$$

6.4 Putting it all together

Equations (38), (49) and (54) satisfy definition 6.3, therefore, $\mathbf{D}^M(\Phi[\alpha]) \sim \mathbf{C}^M(\Phi[\alpha])$. The equivalence, $\mathbf{D}^M(\Phi[\alpha]) \sim \mathbf{C}^M(\Phi[\alpha])$, along with equations (39) and (50) satisfy the conditions of theorem 6.1; hence, $[\mathbf{D}^M(\Phi[\alpha])]^{-1} \sim [\mathbf{C}^M(\Phi[\alpha])]^{-1}$. The asymptotic equivalence leads to the following theorem

Theorem 6.3. *For asymptotically increasing M, the covariance matrix of the DFT of the vector*

$$\mathbf{z}^M = \begin{bmatrix} \mathbf{z}(0) \\ \mathbf{z}(1) \\ \vdots \\ \mathbf{z}(M-1) \end{bmatrix} \tag{55}$$

is asymptotically block-diagonal.

Proof. The DFT of $\mathbf{z}^M$ is given by

$$\begin{bmatrix} \mathbf{Z}(0) \\ \mathbf{Z}(1) \\ \vdots \\ \mathbf{Z}(M-1) \end{bmatrix} = \frac{1}{\sqrt{M}} \begin{bmatrix} \left[\mathbf{I}_N e^{-2\pi\imath\frac{0\cdot 0}{M}}\right] & \left[\mathbf{I}_N e^{-2\pi\imath\frac{0\cdot 1}{M}}\right] & \dots & \left[\mathbf{I}_N e^{-2\pi\imath\frac{0\cdot(M-1)}{M}}\right] \\ \left[\mathbf{I}_N e^{-2\pi\imath\frac{1\cdot 0}{M}}\right] & \left[\mathbf{I}_N e^{-2\pi\imath\frac{1\cdot 1}{M}}\right] & \dots & \left[\mathbf{I}_N e^{-2\pi\imath\frac{1\cdot(M-1)}{M}}\right] \\ \vdots & \vdots & \vdots & \vdots \\ \left[\mathbf{I}_N e^{-2\pi\imath\frac{(M-1)\cdot 0}{M}}\right] & \left[\mathbf{I}_N e^{-2\pi\imath\frac{(M-1)\cdot 1}{M}}\right] & \dots & \left[\mathbf{I}_N e^{-2\pi\imath\frac{(M-1)\cdot(M-1)}{M}}\right] \end{bmatrix} \begin{bmatrix} \mathbf{z}(0) \\ \mathbf{z}(1) \\ \vdots \\ \mathbf{z}(M-1) \end{bmatrix}$$
$$= \mathbf{E}^M \begin{bmatrix} \mathbf{z}(0) \\ \mathbf{z}(1) \\ \vdots \\ \mathbf{z}(M-1) \end{bmatrix} \tag{56}$$

so

$$\begin{bmatrix} \mathbf{Z}(0) \\ \mathbf{Z}(-1) \\ \vdots \\ \mathbf{Z}(-[M-1]) \end{bmatrix} = [\mathbf{E}^M]^* \begin{bmatrix} \mathbf{z}(0) \\ \mathbf{z}(1) \\ \vdots \\ \mathbf{z}(M-1) \end{bmatrix} \tag{57}$$

Pre-multiply $[\mathbf{E}^M]^*$ by the block-diagonal matrix

$$\mathbf{V}^\dagger = \begin{bmatrix} \mathbf{V}_0^\dagger & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_1^\dagger & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{V}_{M-1}^\dagger \end{bmatrix} \tag{58}$$

to see that

$$\mathbf{V}^\dagger[\mathbf{E}^M]^* = \frac{1}{\sqrt{M}}\begin{bmatrix} \begin{bmatrix}\mathbf{V}_0^\dagger e^{2\pi\iota\frac{0\cdot 0}{M}}\\ \mathbf{V}_1^\dagger e^{2\pi\iota\frac{1\cdot 0}{M}}\end{bmatrix} & \begin{bmatrix}\mathbf{V}_0^\dagger e^{2\pi\iota\frac{0\cdot 1}{M}}\\ \mathbf{V}_1^\dagger e^{2\pi\iota\frac{1\cdot 1}{M}}\end{bmatrix} & \cdots & \begin{bmatrix}\mathbf{V}_0^\dagger e^{2\pi\iota\frac{0\cdot(M-1)}{M}}\\ \mathbf{V}_1^\dagger e^{2\pi\iota\frac{1\cdot(M-1)}{M}}\end{bmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix}\mathbf{V}_{M-1}^\dagger e^{2\pi\iota\frac{(M-1)\cdot 0}{M}}\end{bmatrix} & \begin{bmatrix}\mathbf{V}_{M-1}^\dagger e^{2\pi\iota\frac{(M-1)\cdot 1}{M}}\end{bmatrix} & \cdots & \begin{bmatrix}\mathbf{V}_{M-1}^\dagger e^{2\pi\iota\frac{(M-1)\cdot(M-1)}{M}}\end{bmatrix}\end{bmatrix}, \tag{59}$$

and, hence

$$\mathbf{E}^M\mathbf{V} = \frac{1}{\sqrt{M}}\begin{bmatrix} \begin{bmatrix}\mathbf{V}_0 e^{-2\pi\iota\frac{0\cdot 0}{M}}\\ \mathbf{V}_0 e^{-2\pi\iota\frac{0\cdot 1}{M}}\end{bmatrix} & \begin{bmatrix}\mathbf{V}_1 e^{-2\pi\iota\frac{1\cdot 0}{M}}\\ \mathbf{V}_1 e^{-2\pi\iota\frac{1\cdot 1}{M}}\end{bmatrix} & \cdots & \begin{bmatrix}\mathbf{V}_{M-1} e^{-2\pi\iota\frac{(M-1)\cdot 0}{M}}\\ \mathbf{V}_{M-1} e^{-2\pi\iota\frac{(M-1)\cdot 1}{M}}\end{bmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ \begin{bmatrix}\mathbf{V}_0 e^{-2\pi\iota\frac{0\cdot(M-1)}{M}}\end{bmatrix} & \begin{bmatrix}\mathbf{V}_1 e^{-2\pi\iota\frac{1\cdot(M-1)}{M}}\end{bmatrix} & \cdots & \begin{bmatrix}\mathbf{V}_{M-1} e^{-2\pi\iota\frac{(M-1)\cdot(M-1)}{M}}\end{bmatrix}\end{bmatrix}, \tag{60}$$

Now recognize that

$$\begin{aligned}\mathbf{V}^\dagger\mathbf{C}_{ZZ}\mathbf{V} &= \mathbf{V}^\dagger\mathcal{E}\left\{\begin{bmatrix}\mathbf{Z}(0)\\ \mathbf{Z}(-1)\\ \vdots\\ \mathbf{Z}(-[M-1])\end{bmatrix}\begin{bmatrix}\mathbf{Z}^\dagger(0) & \mathbf{Z}^\dagger(-1) & \dots & \mathbf{Z}^\dagger(-[M-1])\end{bmatrix}\right\}\mathbf{V} \\ &= \mathbf{V}^\dagger[\mathbf{E}^M]^*\mathcal{E}\left\{\begin{bmatrix}\mathbf{z}(0)\\ \mathbf{z}(1)\\ \vdots\\ \mathbf{z}(M-1)\end{bmatrix}\begin{bmatrix}\mathbf{z}^\dagger(0) & \mathbf{z}^\dagger(1) & \vdots & \mathbf{z}^\dagger(M-1)\end{bmatrix}\right\}\mathbf{E}^M\mathbf{V} \\ &= \mathbf{V}^\dagger[\mathbf{E}^M]^*\mathbf{C}^M(\Phi[\alpha])\mathbf{E}^M\mathbf{V}\end{aligned} \tag{61}$$

In the limit $M \to \infty$, the eigenvectors of $\mathbf{D}^M(\Phi[\alpha])$ are those of $\mathbf{C}^M(\Phi[\alpha])$, (by Lemma 6.1), therefore, since one recognizes, for appropriately chosen $\mathbf{V}$, that the columns of $\mathbf{E}^M\mathbf{V}$ are the eigenvectors in theorem 6.2 the entire right becomes diagonal in the limit $M \to \infty$. Since $\mathbf{V}$ is block diagonal with $N \times N$ blocks, $C_{ZZ} = [\mathbf{E}^M]^*\mathbf{C}^M(\Phi[\alpha])\mathbf{E}^M$ is also block diagonal with $N \times N$ blocks by lemma 7.2. □

The prescription for post-Doppler STAP is to invert the covariance matrix at each Doppler bin. We use the following theorem and the corollaries to prove that post Doppler STAP is asymtotically optimal

Theorem 6.4.

$$\lim_{M\to\infty}\mathbf{R}_{M-k_f,M-k_f} = \Phi\left(\frac{2\pi k_f}{M}\right) \tag{62}$$

Proof. By equation (9), the block-diagonal elements of covariance matrix in the Doppler domain are given by

$$\begin{aligned}\mathbf{R}_{k_f k_f} &= \frac{1}{M}\sum_{m,m'=0}^{M-1} \mathcal{E}\{\mathbf{z}(m)\mathbf{z}^\dagger(m')\}\exp\left(\frac{-\imath 2\pi[k_f(m-m')]}{M}\right) \\ &= \frac{1}{M}\sum_{m,m'=0}^{M-1} \mathbf{C}_{m-m'}\exp\left(\frac{-\imath 2\pi[k_f(m-m')]}{M}\right).\end{aligned} \tag{63}$$

Re-index the sum (sum along the diagonals) to see that

$$\begin{aligned}\mathbf{R}_{k_f k_f} &= \frac{1}{M}\sum_{k=-(M-1)}^{M-1} (M-|k|)\mathbf{C}_k \exp\left(\frac{-\imath 2\pi k k_f}{M}\right) \\ &= \sum_{k=-(M-1)}^{M-1} \mathbf{C}_k \exp\left(\frac{-\imath 2\pi k k_f}{M}\right) - \sum_{k=-(M-1)}^{M-1} \frac{|k|}{M}\mathbf{C}_k \exp\left(\frac{-\imath 2\pi k k_f}{M}\right).\end{aligned} \tag{64}$$

By lemma 7.1, the second term goes to zero as $M \to \infty$. Hence, by the definition in equation (28)

$$\begin{aligned}\lim_{M\to\infty} \mathbf{R}_{M-k_f, M-k_f} &= \lim_{M\to\infty} \sum_{k=-(M-1)}^{M-1} \mathbf{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right) \\ &= \Phi\left(\frac{2\pi k_f}{M}\right)\end{aligned} \tag{65}$$

□

Corollary 6.1. *The matrix*

$$\mathbf{R}^M = \begin{bmatrix} \mathbf{R}_{M-1,M-1} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{R}_{M-2,M-2} & \mathbf{0} & \dots \\ \vdots & \vdots & \ddots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{R}_{0,0} \end{bmatrix}. \tag{66}$$

is such that $\mathbf{R}^M \sim [\mathbf{E}^M]^* \mathbf{D}^M(\Phi[\alpha])\mathbf{E}^M$

Proof. The matrix $\mathbf{R}^M$ has a finite maximum eigenvalue since its eigenvalues are those of $\mathbf{R}_{k_f,k_f}$. Also is has a minimum lowest eigenvalue since all eigenvalues of $\mathbf{R}_{k_f,k_f}$ are greater than or equal to σ_n^2. Now, since $[\mathbf{E}^M]^*\mathbf{D}^M(\Phi[\alpha])\mathbf{E}^M$ is a block diagonal matrix with block elements given by $\Phi\left(\frac{2\pi k_f}{M}\right)$, then

$$\begin{aligned}
\left|\mathbf{R}^M - [\mathbf{E}^M]^*\mathbf{D}^M(\Phi[\alpha])\mathbf{E}^M\right|^2 &= \frac{1}{M}\left|\sum_{k_f=0}^{M-1}\left\{\Phi\left(\frac{2\pi k_f}{M}\right) - \sum_{k=-(M-1)}^{M-1}\mathrm{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right)\right\}\right|^2 \\
&= \frac{1}{M}\left|\sum_{k_f=0}^{M-1}\left\{\sum_{k=-\infty}^{\infty}\mathrm{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right) - \sum_{k=-(M-1)}^{M-1}\mathrm{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right)\right\}\right|^2 \\
&= \frac{1}{M}\left|\sum_{k_f=0}^{M-1}\left\{\sum_{k=-\infty}^{-M}\mathrm{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right) + \sum_{k=M}^{\infty}\mathrm{C}_k \exp\left(\frac{\imath 2\pi k k_f}{M}\right)\right\}\right|^2 \\
&\leq \frac{1}{M}\sum_{k_f=0}^{M-1}\sum_{k=-\infty}^{-M}|\mathrm{C}_k|^2 + \frac{1}{M}\sum_{k_f=0}^{M-1}\sum_{k=M}^{\infty}|\mathrm{C}_k|^2 \\
&= \sum_{k=-\infty}^{-M}|\mathrm{C}_k|^2 + \sum_{k=M}^{\infty}|\mathrm{C}_k|^2
\end{aligned} \tag{67}$$

By assumption of the absolute summability of C_k the above is seen to go to zero as $M \to \infty$. □

Corollary 6.2. $\mathrm{D}^M(\Phi[\alpha])] \sim \mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*$

Proof. Since $\mathbf{E}^M$ is unitary, the eigenvalues of $\mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*$ are the eigenvalues of $\mathbf{R}^M$, and the bounds on the eigenvalues of $\mathbf{R}^M$ satisfy definition 6.3. Also,

$$\begin{aligned}
\left|\mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^* - \mathbf{D}^M(\Phi[\alpha])\right| &= \left|\mathbf{E}^M\left(\mathbf{R}^M - [\mathbf{E}^M]^*\mathbf{D}^M(\Phi[\alpha])]\mathbf{E}^M\right)[\mathbf{E}^M]^*\right| \\
&\leq \|\mathbf{E}^M\| \cdot \left|\mathbf{R}^M - [\mathbf{E}^M]^*\mathbf{D}^M(\Phi[\alpha])]\mathbf{E}^M\right| \cdot \|[\mathbf{E}^M]^*\|.
\end{aligned} \tag{68}$$

Since $\|\mathbf{E}^M\| = \|[\mathbf{E}^M]^*\| = 1$, and since the right side goes to zero as $M \to \infty$ by corollary 6.4, the left side also goes to zero as $M \to \infty$. □

Corollary 6.3. $\mathrm{C}^M \sim \mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*$

Proof. This follows from the law of associativity of asymptotic equivalence: $\mathbf{C}^M \sim \mathbf{D}^M(\Phi[\alpha])]$ and $\mathbf{D}^M(\Phi[\alpha])] \sim \mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*$, hence $\mathbf{C}^M \sim \mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*$. □

Corollary 6.4. $[\mathrm{C}^M]^{-1} \sim \mathbf{E}^M[\mathbf{R}^M]^{-1}[\mathbf{E}^M]^*$

Proof. This follows from theorem 6.1 and the facts that

- $[\mathbf{E}^M\mathbf{R}^M[\mathbf{E}^M]^*]^{-1} = \mathbf{E}^M[\mathbf{R}^M]^{-1}[\mathbf{E}^M]^*$
- $\|[\mathbf{C}^M]^{-1}\| < \infty$ by 39, and
- $\|[\mathbf{R}^M]^{-1}\| \leq \frac{1}{\sigma_n^2} < \infty$.

□

This last corollary proves that the prescription of post-Doppler STAP is asymptotically optimal.

6.5 How good is post-Doppler STAP compared to the optimum?

It is practical to consider how post-Doppler STAP approaches the optimum so that one can choose an appropriate number of samples to process for the application in mind. In the following, it is assumed the processing gain is used as a measure.

$$[\mathbf{C}^M - \Delta^M]^{-1} \approx [\mathbf{C}^M]^{-1} + [\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1} + O([[\mathbf{C}^M]^{-1}\Delta^M]^2)[\mathbf{C}^M]^{-1} \tag{69}$$

We'll drop the second and higher order terms in the following. Now the optimum processing gain for a target s is achieved by the filter $\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}$. What is applied in post-Doppler STAP, however, is

$$\begin{aligned} \mathbf{s}^\dagger[\mathbf{D}^M(\Phi[\alpha])]^{-1} &= \mathbf{s}^\dagger[\mathbf{C}^M - \Delta^M]^{-1} \\ &\approx \mathbf{s}^\dagger[\mathbf{C}^M]^{-1} + \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1} \end{aligned} \tag{70}$$

The resulting processed signal to clutter ratio is, approximately, given by

$$\begin{aligned} SCNR &= \frac{|\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\mathbf{s} + \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\mathbf{s}|^2}{\left(\mathbf{s}^\dagger[\mathbf{C}^M]^{-1} + \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\right)\mathbf{C}^M\left([\mathbf{C}^M]^{-1}\mathbf{s} + [\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\mathbf{s}\right)} \\ &\approx \frac{|\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\mathbf{s} + \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\mathbf{s}|^2}{\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\mathbf{s}} \\ &\cdot\left(1 - \frac{\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}[2\Delta^M - \Delta^M[\mathbf{C}^M]^{-1}\Delta^M][\mathbf{C}^M]^{-1}\mathbf{s}}{\mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\mathbf{s}}\right) \\ &\approx \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\mathbf{s} - \mathbf{s}^\dagger[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\Delta^M[\mathbf{C}^M]^{-1}\mathbf{s} \end{aligned} \tag{71}$$

The first term is the optimum processing gain while the second is the first order approximation to the error introduced by using post-Doppler STAP.

7 The correlation matrix at a given frequency bin

Consider the case where $k_f = k'_f$. One then finds that

$$\mathbf{R}_{k_f,k_f} = \frac{\sigma_c^2}{M} \int \mathbf{H}(\xi)\mathbf{H}^\dagger(\xi) \left| g_M \left(M; k_f - M\xi\kappa_x \right) \right|^2 \mathrm{d}\xi + \sigma_n^2 \mathbf{I}. \tag{72}$$

The first term in the above matrix can have rank 1 since as $M \to \infty$ since

$$\lim_{M\to\infty} \frac{g_M(M;x)}{M} = \sum_l \delta(x - lM), \quad l \in \mathcal{Z}. \tag{73}$$

If the matrix $\mathbf{H}(\xi)\mathbf{H}^\dagger(\xi)$ is such that its support only intersects with one value in the set $\xi = \kappa_x^{-1}(\frac{k_f}{M} - l)$, $\forall l$ (the data are adequately sampled) then the matrix will have rank 1. If the data are not adequately sampled, but have spectral regions where the clutter is unambiguous, then values of k_f corresponding to this region will still, asymptotically, have rank 1.

For finite M, it is instructive to examine how the matrix approaches rank 1 with increasing M. The Doppler domain clutter-only covariance matrix has a term given by

$$\mathbf{R}^c_{k_f,k_f} = \sum_l \mathbf{R}^c(k_f, l), \tag{74}$$

where

$$\mathbf{R}^c(k_f, l) = \frac{\sigma_c^2}{M} \int_{-\kappa}^{\kappa} \mathbf{H}(\xi + \xi_0(l))\mathbf{H}^\dagger(\xi + \xi_0(l)) |g_M(M; M\kappa_x\xi)|^2 \mathrm{d}\xi, \tag{75}$$

where

$$\xi_0(l) = \kappa_x^{-1} \left(\frac{k_f}{M} - l \right), \tag{76}$$

and

$$\kappa = \frac{\kappa_x^{-1}}{2} \tag{77}$$

Let us assume that the $(i, j)^{\text{th}}$ element of $\mathbf{H}(\xi + \xi_0(l))\mathbf{H}^\dagger(\xi + \xi_0(l))$ has a phase given by $\exp(\imath(\xi + \xi_0(l))\Delta_{i,j})$ ($\Delta_{i,i} = 0$) and an amplitude, $d[\xi_0(l)]$, that remains constant over the small integration region picked out by the function $|g_M(M; M\kappa_x\xi)|^2$. We thus make the approximation

$$R^c_{i,j}(k_f, l) \propto \frac{d[\xi_0(l)] \exp(\imath\xi_0(l)\Delta_{i,j})}{M} \int_{\kappa}^{\kappa} \exp(\imath\xi\Delta_{i,j}) |g_M(M; M\kappa_x\xi)|^2 \mathrm{d}\xi. \tag{78}$$

Next, let us make the approximation

$$|g_M(M;x)|^2 = M^2 \left| \frac{\sin \pi x}{\pi x} \right|^2. \tag{79}$$

While the function on the left has equal multiple peaks, the sinc function on the right only has one dominant peak at zero. we can thus pass to infinite limits for the integration. Equation 78 becomes

$$R^c_{i,j}(k_f, l) \propto \exp(\imath \xi_0 \Delta_{i,j}) M \int \exp(\imath \xi \Delta_{i,j}) \left| \frac{\sin \pi M \kappa_x \xi}{\pi M \kappa_x \xi} \right|^2 \mathrm{d}\xi. \tag{80}$$

The above inverse Fourier transform is represented in the time domain as the autocorrelation of the function

$$\mathcal{F}^{-1} \left\{ \frac{\sin \pi M \kappa_x \xi}{\pi M \kappa_x \xi} \right\} = \frac{1}{M \kappa_x} \mathrm{rect}_{M\kappa_x} x, \tag{81}$$

evaluated at $\Delta_{i,j}$, where

$$\mathrm{rect}_X(x) = \begin{cases} 1 & |x| < X/2 \\ 0 & \text{otherwise.} \end{cases} \tag{82}$$

The convolution of the rectangular function is a triangular function. Thus we arrive at the conclusion that

$$R_{i,j}(k_f, l) \propto \frac{1}{\kappa_x} - \frac{2|\Delta_{i,j}|}{M \kappa_x^2}. \tag{83}$$

The correlation coefficient of the above matrix element is then, to good approximation,

$$\rho_{i,j}(k_f, l) = 1 - \frac{2|\Delta_{i,j}|}{M \kappa_x}. \tag{84}$$

The rank 1 property will depend on two factors. In the first place, there should only be one value for l that contributes to the covariance matrix at $\mathbf{R}(k_f)$. Additionally, correlation coefficients that are significantly different from 1 will destroy the rank 1 property of the matrix. For example, in the 2 channel case, the ratio of the small to the large eigenvalue becomes, approximately,

$$\frac{\lambda_2}{\lambda_1} \approx \frac{|\Delta_{i,j}|}{M \kappa_x}. \tag{85}$$

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Annex A: Various lemmas

The following text is repeated from [3], lemma 3.1.4

Lemma 7.1 (Kroneckers lemma)**.** *If the sequence* $\{a_k\}$ *is such that*

$$\lim_{n\to\infty}\sum_{k=0}^{n}|a_k| = A < \infty$$

then

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{k}{n}|a_k| = 0.$$

Proof. Because A is finite, there exists N_a such that

$$\sum_{k=N_a+1}^{\infty} |a_k| < \varepsilon \tag{A.1}$$

for any ε. Thus, for $n > N_a$

$$\sum_{k=0}^{n}\frac{k}{n}|a_k| < \frac{1}{n}\sum_{k=0}^{N_a}|a_k| + \varepsilon \tag{A.2}$$

In the limit $n \to \infty$, the first term goes to zero. The fact that ε can be made arbitrarily small proves the lemma. □

Lemma 7.2. *The product of two block-diagonal matrices with* $N \times N$ *sized blocks is block-diagonal with* $N \times N$ *blocks*

Proof. $\mathbf{C} = \mathbf{AB}$, where $\mathbf{A}$ and $\mathbf{B}$ are block-diagonal with $N \times N$ blocks has $N \times N$ block element in position α, β given by

$$\begin{aligned}\mathbf{C}_{\alpha\beta} &= \sum_{\gamma}\mathbf{A}_{\alpha\gamma}\mathbf{B}_{\gamma\beta}\\ &= \mathbf{A}_{\alpha\alpha}\mathbf{B}_{\alpha\beta}\\ &= \mathbf{0} \text{ unless } \alpha = \beta.\end{aligned} \tag{A.3}$$

□

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Annex B: Dirichlet function

Occasions will arise that make use of the function defined by

$$g_M(l;k) = \sum_{n=0}^{l-1} \exp\left(-2\pi\imath \frac{nk}{N}\right), \quad l \leq N. \tag{B.1}$$

By reorganizing the summation in $g_M(l;k)$, one finds that

$$\begin{aligned} g_M(N-M;k) &= g_M(N;k) - \exp\left(-2\pi\imath k \frac{N-M}{N}\right) g_M(M;k) \\ &= g_M(N;k) - \exp\left(2\pi\imath k \frac{M}{N}\right) g_M(M;k), \quad \text{if } k \in \mathcal{Z}. \end{aligned} \tag{B.2}$$

This function has the property that,

$$g_M(N;k) = \begin{cases} N & \text{If } k \text{ is some multiple of } N \text{ including } k = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{B.3}$$

The sum can be explicitly computed as

$$g_M(l;k) = \exp\left(-\pi\imath k \frac{l-1}{N}\right) \frac{\sin\left(\frac{\pi k l}{N}\right)}{\sin\left(\frac{\pi k}{N}\right)}. \tag{B.4}$$

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Herein find a proof of the theorem of asymptotic optimality of post-Doppler Space-Time Adaptive Processing (STAP). The proof uses well known properties of block Toeplitz and block circulant matrices.

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