Foreign Exchange Exposure Model: FOREX Optimization

Model and Simulation DLL Documentation

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Contract Project Manager: P.E. Desmier
Contract Number: EN537-8-4015/082/ZJ
Contract Scientific Authority: P.E. Desmier

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Defence R&D Canada
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Assistant Deputy Minister (Materiel)

National Defence   Défense nationale

Canada

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Contract Report

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Abstract

The Foreign Exchange Exposure Model (FOREX) was designed and built as two separate components, that when integrated provide the user with full functionality in determining Value-at-Risk (VaR) results for a variety of account transactions and foreign exchanges. This contract report describes the optimization module component.

Résumé

Le modèle FOREX, un modèle d'évaluation du risque de change, est formé de deux modules distincts qui, une fois combinés, permettent à l'utilisateur de déterminer la valeur à risque (VaR) associée à toute une gamme d'opérations en devises. Le présent rapport vise à décrire le module d'optimisation.

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Executive summary

Foreign Exchange Exposure Model: FOREX Optimization

Jay Adamsson,
Jay Adamsson Consulting Inc.; DRDC CORA CR 2010–232; Defence R&D Canada – CORA; November 2010.

Background: Value-at-Risk (VaR), as applied to foreign exchange exposure, is an estimation of the probability of losses that could arise from changes in exchange rates. It has become very popular in financial risk management because it is easily understood and condenses a vast amount of information into a single, summary statistical measure of market risk under normal market conditions.

With the aim of automating the process and creating a web-based departmental application, it became necessary to incorporate an automated process where the time series models for Financial and Managerial Accounting Systems (FMAS) expenditures and foreign currency exchange rates were developed at the outset, but had their coefficients adjusted quarterly as actual data became available. Once a year, it would be necessary to recalculate the coefficients as the model structure may have to be adjusted due to radical changes in spending or currency patterns.

The Foreign Exchange Exposure Model (FOREX) was designed and built as two separate components, that when integrated provide the user with full functionality in determining VaR results for a variety of account transactions and foreign exchanges. They are the user interface, as described in [1], and the optimization module described herein.

Aim: The aim of this report is to document the optimization calculations as a called VB.Net module (DLL [1]). This contract report documents the FOREX DLL module which will directly access the data sets; generate additional coefficient data as required, and produce the VaR output for the scenarios.

Results: The primary purpose of the FOREX application is to determine the Value-at-Risk, or VaR, of the major DND foreign currency expenditures. In order to calculate the VaR for a given foreign currency expenditure, several data sets are required. The data sets are:

1. The forecasted future (budget) currency exchange rate, which is entered through the application;
2. The historical currency exchange rate, which is obtained from the Bank of Canada;
3. FMAS fund data, which is downloaded from the FMAS web site, and loaded into the application; and,
4. Coefficients, which are generated separately from the web application.

1. A dynamic-link library (DLL) is an executable file that acts as a shared library of functions.

Once the data is collected, a scenario is built, using a specific currency, FMAS Fund, and forecast date. The calculation is complex, and is built as a called VB.Net module (DLL). The FOREX DLL will directly access the data sets; generate additional coefficient data as required, and produce the VaR output for the scenario. The data can then be viewed online.

Way Ahead: The VaR methodology and its resulting FOREX application should be adopted as part of the department's integrated risk management framework for managing the budgetary risk attributed to exposure to foreign currency fluctuations for all acquisitions. Currently there is no tool available to assess the in-year impact of foreign exchange fluctuations on Defence budget allocations. FOREX will offer this capability.

Sommaire

Foreign Exchange Exposure Model: FOREX Optimization

Jay Adamsson,
Jay Adamsson Consulting Inc. ; DRDC CORA CR 2010–232 ; R & D pour la défense Canada – CARO ; novembre 2010.

Contexte : La méthode de la valeur à risque (VAR) telle qu'elle est appliquée au risque de change consiste à estimer la probabilité que des pertes soient subies à cause de la fluctuation des taux de change. Les spécialistes de la gestion des risques financiers font très souvent appel à cette méthode en raison de sa simplicité et parce qu'elle permet de condenser une vaste quantité de renseignements en une seule mesure statistique sommaire du risque du marché dans des conditions normales.

Dans le but, d'une part, d'automatiser le processus et, d'autre part, de créer une application Web ministérielle, il fallut créer un processus automatisé selon lequel les modèles de séries chronologiques associés aux taux de change et aux dépenses consignées dans le Système de comptabilité financière et de gestion (SCFG) sont élaborés au début, mais où les coefficients sont rajustés chaque trimestre lorsque les données réelles sont accessibles. Une fois par année, il faut revoir les modèles afin d'évaluer si des changements importants dans les dépenses ou les taux de change commandent un rajustement.

Le modèle FOREX, un modèle d'évaluation du risque de change, est formé de deux modules distincts qui, une fois combinés, permettent à l'utilisateur de déterminer la VAR associée à toute une gamme d'opérations en devises. Le premier module, présenté en [1], est l'interface utilisateur ; la deuxième composante - sujet du présent document - est le module d'optimisation.

Objectif : Le présent rapport vise à décrire les calculs d'optimisation faits à partir d'un module VB.Net (DLL) et à expliquer le module DLL du modèle FOREX, lequel accède directement aux ensembles de données, génère les coefficients additionnels nécessaires et, enfin, produit la VAR associée aux différents scénarios.

Résultats : Le modèle FOREX a pour but premier d'établir la VAR associée aux grandes dépenses en devises réalisées par le MDN. Le calcul de la VAR pour une opération en devises déterminée nécessite plusieurs ensembles de données :

1. le taux de change prévu, qu'on consigne dans l'application ;
2. le taux de change historique, qu'on obtient auprès de la Banque du Canada ;
3. les données financières du SCFG, qu'on télécharge à partir du site Web du SCFG puis qu'on verse dans l'application ;
4. les coefficients, qui sont générés séparément à partir de l'application Web.

Une fois les données recueillies, on élabore un scénario pour une devise et une date précises, à l'aide des données financières du SCFG. Le calcul, qui est complexe, se fait à l'aide d'un module VB.Net (DLL). Le module DLL du modèle FOREX accède

directement aux ensembles de données pour ensuite générer les coefficients additionnels nécessaires et, enfin, produire la VAR associée au scénario en question. Les données peuvent ensuite être consultées en ligne.

Suite des choses : La méthode VAR et le modèle FOREX devraient être incorporés au Cadre de gestion intégrée du risque du MDN afin que l'on puisse gérer, pour toutes les acquisitions, le risque associé aux fluctuations des taux de change. à l'heure actuelle, il n'existe aucun outil permettant d'évaluer l'incidence, en cours d'exercice, des fluctuations des taux de change sur les affectations budgétaires du MDN. Le modèle FOREX comblera cette lacune.

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1 Introduction

1.1 Background

Value-at-Risk (VaR), as applied to foreign exchange exposure, is an estimation of the probability of losses that could arise from changes in exchange rates. It has become very popular in financial risk management because it is easily understood and condenses a vast amount of information into a single, summary statistical measure of market risk under normal market conditions.

In previous studies [2, 3, 4], financial expenditure models were developed through Box-Jenkins mechanisms, and the conditional variances of the financial return series through the basic Generalized Autoregressive Conditional Heteroskedasticity (GARCH)(1,1) model, where the GARCH weights were specified by maximizing the log-likelihood of the standardized *t(d)* distribution for Canadian/U.S. dollar (CAD/USD) and Canadian dollar/U.K. sterling (CAD/GBP) transactions, and the normal distribution for Canadian dollar/Euro (CAD/EUR) transactions.

With the aim of automating the process and creating a web-based departmental application, it became necessary to remove the manual methods of [2, 3] for developing expenditure and currency models, and incorporate an automated process where the time series models for Financial and Managerial Accounting Systems (FMAS) expenditures and foreign currency exchange rates were developed at the outset, but had their coefficients adjusted quarterly as actual data became available. Once a year, it would be necessary to recalculate the models as their structure may have to be adjusted due to radical changes in spending or currency patterns.

The Foreign Exchange Exposure Model (FOREX) was designed and built as two separate components, that when integrated provide the user with full functionality in determining VaR results for a variety of account transactions and foreign exchanges. They are the user interface, as described in [1], and the optimization module described herein.

1.2 Aim

The optimization calculations are complex, and are built as a called VB.Net module (DLL). This contract report documents the FOREX DLL module which will directly access the data sets; generate additional coefficient data as required, and produce the VaR output for the scenarios.

1.3 Scope

In this contract report, the mathematics behind the FOREX optimization module is described in detail. This report is divided into nine sections. Following the introduction, section 2 describes the overall program flow, from opening the connection to the database, running a Monte Carlo simulation to optimize the GARCH coefficients and forecast the VaR, and output results to the database; section 3 describes the procedure for determining the coefficients of the expenditure models based on an interior point optimization process; section 4 describes the forecasting procedure for both expenditures and exchange rates using filtered historical simulation (fully documented in references [2, 4]); section 5 discusses the limits of the simulation and the degree of accuracy of the results, which are based on 25,000 iterations of a Monte Carlo simulation; section 6 lists the complete list of classes found in the FOREX DLL; section 7 describes the database structure and the tables therein; section 8 describes the complex mathematics behind the optimization of the foreign exchange model using maximum likelihood estimation algorithms developed from scratch, i.e., without the use of third party software; and section 9 provides a short conclusion to the report.

2 Program Flow

The overall program flow for the FOREX DLL [2] is shown in Figure 3 on the next page. Most of the operations depicted by the various boxes will be further detailed below. The DLL exposes only one class to the calling program, named OptimizationMod. A sample of code used to call this method is given in Figure 1.

The first step is to open the database connection. The database connection is passed through the ConnectionString property. In Figure 1, it is represented as "****". In reality, the connection string for the actual database would be passed. The required database structure is given in section 7 of this report.

```
Dim opt as New
 OptimizationMod

opt.Currency="USD"
opt.Fund="L101"
opt.ScenarioID=3
opt.SetProjectioToDate(4,2009)
opt.ConnectionString="****"
```

Figure 1: OptimizationMod: Sample Code

Once the data has been read in, the VaR is simulated and the results are output to the database. The program ends by returning a string variable to the calling program. If an error was encountered, a string is returned describing the error.

2.1 Project Option

```
Dim opt as New Optimization-
Mod
 opt.Currency="USD"

opt.ContractAmount=1000000
opt.ScenarioID=3
```

Figure 2: Code for running contract option

The first decision for the flow chart in Figure 3 concerns a request for a single project, e.g., major crown contract (see blue dashed box in Figure 3). Normally, a request to the DLL would result in a simulation of both the future expenditures and the foreign exchange. The DLL provides the option to bypass the simulation of future expenditures, and provide a VaR for a set project contract amount. In this way, the risk of known future expenditures due to currency fluctuations can be quantified in an analogous manner to the full VaR calculations. The only difference is the use of a known quantity for one of the simulated variables. The code used to determine the VaR for a set project contract amount is given in Figure 2. In this example, a contract with a value of one million dollars is evaluated. Note that the fund is no longer required in this instance. In fact, if both a fund and a contract amount are given, the contract amount will be ignored and the full VaR for the fund will be calculated.

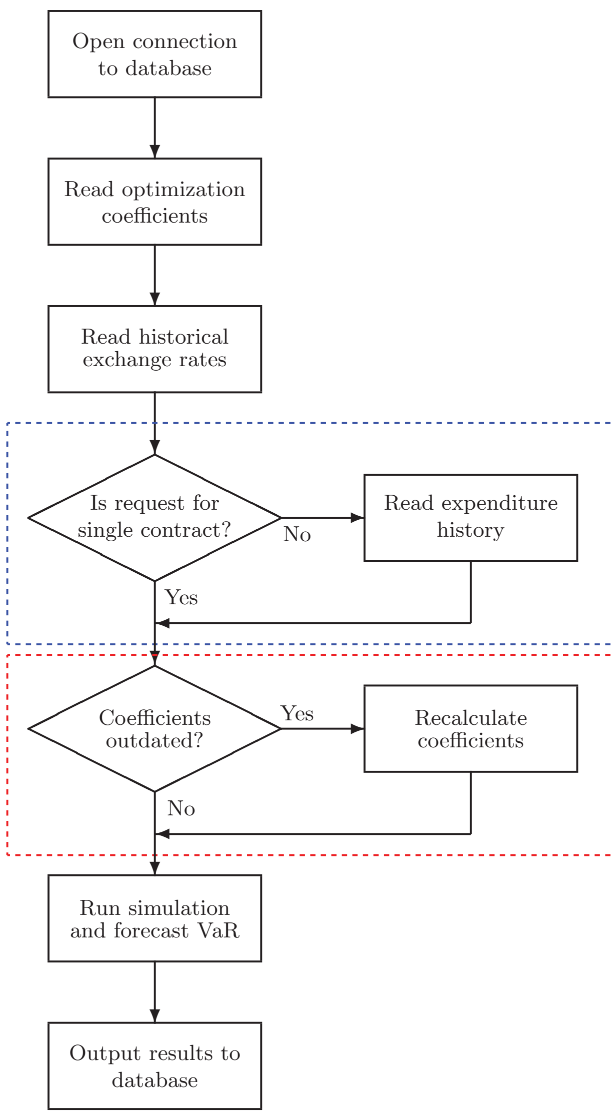
2.2 Coefficients Option

The second decision box determines whether or not the optimization coefficients are outdated (see the red dashed box in Figure 3). The coefficients for the GARCH analysis

2. A dynamic-link library (DLL) is an executable file that acts as a shared library of functions.

of the historical foreign exchange data are stored in the table tblMLECoefficients. Along with the coefficients, a date is also stored in this table. Whenever the coefficients are recalculated, the date of the last foreign exchange value is stored. The next time the procedure is called, this date is used to determine if the foreign exchange values have been updated. If the foreign exchange values have not been updated since the previous run, the stored coefficients are used and the optimization routine is bypassed. If the values have been updated, the previous coefficients are used to seed the optimization routine, and new coefficients are produced.

Figure 3: Program flow



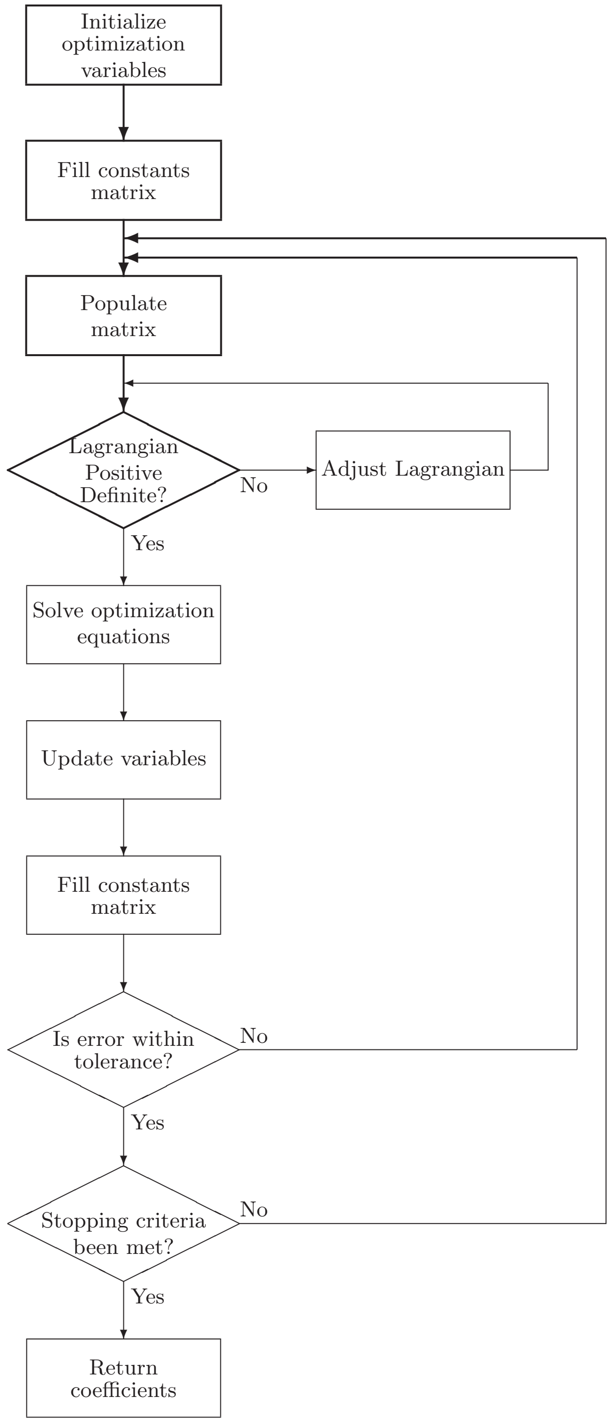
Results of the VaR analysis are saved in the tblScenarioOutput table. The simulated results for future expenditures, future exchange rates, and future gain/loss are saved for each month of the forecast. Percentiles are recorded from the 0th through to the 100th percentile, in increments of five.

3 Foreign Expenditure Model

By far, the most complex part of the FOREX DLL code lies in determining the Autoregressive coefficients for the foreign expenditure model. A broad overview of the process is shown in Figure 4. The optimization algorithm outlined by Nocedal and Wright [5] was used as a basis for this process. Essentially, the procedure solves iteratively a set of linear equations following the Newton method for constrained optimization. First, a collection of intermediate variables, which represent slack variables, dual variables, and other attributes of the optimization, are initialized. The next step is to fill the constant vector and to populate the matrix. For the optimization to work, the upper left corner of the matrix has to be positive definite. If not, it has to be adjusted until it meets this criterion. The equations are then solved and the variables updated.

This process is an interior point optimization process. Conceptually, it works by optimizing in an interior region. Once this has been accomplished, the region is expanded to closer approximate the entire domain. Therefore, there are two loops which surround the procedure. The inner loop is used to optimize within a region. The calculations are repeated until the solution is within a given error amount. Once this is reached, the region is expanded to the outer loop. The outer loop continues to expand until it is reasonable close to the entire domain of the function. Once the two loops have both been determined to be sufficiently close to the solution, the four GARCH coefficients (α, β, ω and d) are saved to the database, along with the final date of the foreign exchange rates that were used in their calculation.

Figure 4: Expenditure model



4 Forecasting: Exchange Rates and Expenditures

The forecasting portion of FOREX makes use of two models: the expenditure model and the foreign exchange model. Both of these models have already been documented in references [2, 4].

The exchange rate forecasting, as outlined in Figure 5, begins the day after the last day for which actual exchange data is available. The number of weekdays remaining in the actual month is determined and used to project foreign exchange rates to the end of the month. For the remaining months, a generalized value of 22 trading days per month is used.

In [2], it was determined that Filtered Historical Simulation (FHS) was the preferred method for representing actual market behaviour as it captures all possible values of the historical distribution of price returns, in particular the tail events critical to VaR calculations, with the least number of assumptions about the statistical properties of future price changes.

Consider the set of past returns $\{r_{t+1-\tau} : \tau = 1, 2, \ldots, T\}$. From the equation

$$r_t = \sigma_t z_t \quad \text{with } z_t \sim \tilde{t}(d) , \tag{1}$$

where z_t is the error term defined by the standardized $t(d)$ distribution, we can write the one-day ahead return as the product of the estimated standard deviation and the error term, i.e.,

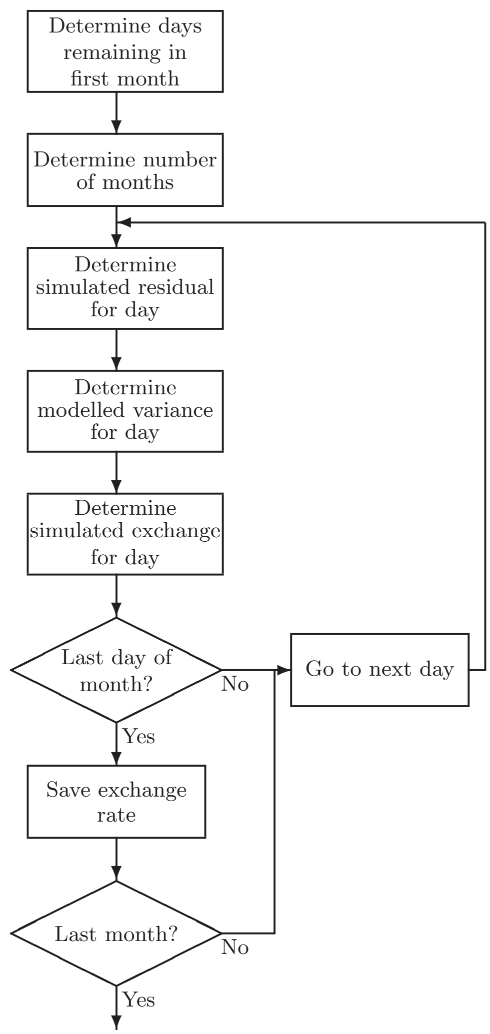
$$r_{t+1} = \sigma_{t+1} z_{\,t+1} , \tag{2}$$

where σ_{t+1} is defined through the GARCH variance, whose coefficients have already been calibrated using the historical data. Using the data set $\{r_{t+1-\tau} : \tau = 1, 2, \ldots, T\}$ we can now estimate the model parameters and calculate the set of realized standardized returns, $\{\hat{z}_{t+1-\tau} : \tau = 1, 2, \ldots, T\}$, defined by

$$\hat{z}_{\,t+1-\tau} = r_{t+1-\tau}/\sigma_{t+1-\tau} , \; \tau = 1, 2, \ldots, T \tag{3}$$

Therefore, given actual returns up to time t, we can immediately evaluate the

Figure 5: Exchange forecasting model



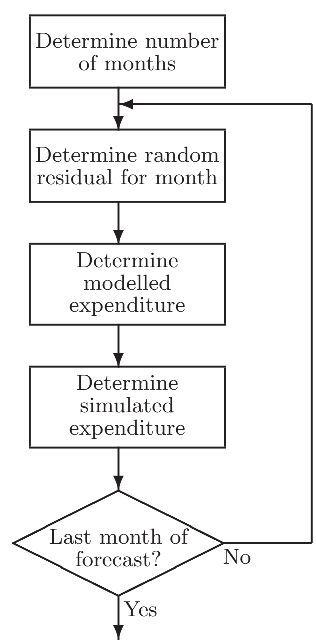
GARCH variance for time $t + 1$. To compute hypothetical returns for tomorrow, we draw with replacement from the set of past standardized residuals, $\{\hat{z}_{t+1-\tau} : \tau = 1, 2, \ldots, T\}$, through sampling a discrete uniform distribution of elements consisting of the $\tau = 1, 2, \ldots, T$ standardized returns defined by equation (3). The estimated exchange rate, P_{t+1}, on day $t + 1$ is then defined to be

$$P_{t+1} = e^{\,r_{t+1}} P_t \,, \tag{4}$$

where P_t is defined as the exchange rate on day t.

In the FOREX application, a random residual from the historical set of residuals is used as the simulated return. In this manner, the simulated distribution of residuals is identical to the historical distribution. In addition, further efforts were made to have the future distribution more closely match the historical distribution. Each given day is simulated 25,000 times. Looking over these 25,000 iterations, the residuals were not simply chosen purely randomly. They were chosen by random permutations of the historical values. So, for example, if there were 1,000 historical values, each historical residual would be chosen exactly 25 times in the simulation for each day. This approximation of a Latin Squares helps the simulated distribution better match the historical distribution.

Figure 6: Expenditure forecasting model



Once the daily residual has been calculated, the modelled variance for the day is determined, based on the model coefficients determined earlier. This, along with the simulated residual, is used to calculate the simulated return. This return is then used to calculate the simulated exchange rate. This continues for the forecast period, with the last exchange rate for each month being saved.

The procedure for forecasting expenditures, as outlined in Figure 6, is very similar, but only has to forecast a single value for each month. The expenditure model is derived from the values calculated outside the DLL and saved in a set of tables in the database.

As in the foreign exchange forecast (Figure 5), the residuals for each month are chosen in a pseudo-random fashion [3]. Over the 25,000 iterations for each month, the historical residuals are each chosen an equal number of times, again forcing the distribution of residuals to match the historical distribution.

This procedure results in 25,000 simulated expenditures and 25,000 simulated exchanges per month. The

3. Pseudorandom numbers are important in practice for simulations with the Monte Carlo method.

program then determines the difference between the simulated exchange and the forecasted exchange for each month, and multiplies this by the simulated expenditure. This gives a set of 25,000 gain/loss values for the month. These are then sorted, and the VaR is extracted from this list as the 95th percentile loss.

5 Accuracy of Results

The VaR is a statistical measure of the potential losses that could occur due to currency fluctuations. Foreign exchange values are notoriously difficult to predict and enormous amounts of research have gone into this question over the years. The VaR, as determined through the FOREX application, is dependent upon the difference between the Department of National Defence (DND) currency exchange forecast (which is used for budget projections) and the modelled exchange rates. This leads to four areas for consideration in evaluating the accuracy of the VaR:

- Expenditure model
- Foreign exchange model
- DND foreign exchange forecasts
- Simulation procedures

5.1 Expenditure Model

The expenditure model produces large variations from month to month. The model values are created through the Autobox model, outside of the FOREX DLL. The question of how accurately the model (with the random variation) depicts future expenditure has been addressed by [4]. In addition, this implementation of the FOREX model includes the option to override the Autobox model and enter specific numbers for future expenditures. Given that budgeting would require external estimates and forecasts of expenditure levels in any case, the VaR can be determined for the known (or estimated) levels of expenditure, giving added flexibility to the user and the ability to circumvent this potential variation.

5.2 Foreign Exchange Model

Predicting the fluctuations of foreign exchange rates is not an exact science. However, the model proposed by Christoffersen [6], and used in this project, makes use of some general numerical observations and trends to provide a forecast that has been shown to be better than random. In fact, this process is quite well suited to creating a reasonably reliable distribution, which in the case of estimating VaR, is exactly what is needed. Although we still do not possess the ability to predict exchange rates, for the determination of VaR, this is not necessary – all that is important is the potential distribution.

The main caution of this model is that only an attempt is made to model the day-to-day fluctuations of currency in a relatively steady-state situation. Foreign currencies are quite subjective to political, domestic, and international shocks. It is impossible to predict such events.

5.3 DND Foreign Exchange Forecasts

These forecasts are produced internally within DND and entered into the FOREX system. They take into account not only the current foreign exchange levels, but also known or anticipated geo-political events that may have an impact upon currency levels. This value is the baseline from which budget requirements are determined, and from which the VaR is calculated.

5.4 Number of Iterations

Since the VaR calculation depends on a Monte Carlo simulation, care must be taken to ensure that the VaR converges. In the Excel version of FOREX, the Monte Carlo simulation used 10,000 iterations to produce the VaR. Since the number of iterations is hard coded into the DLL, a quick review of the variability encountered with different numbers of iterations was conducted. As a result of this review, as well as the increased speed of this implementation of the simulation, the number of iterations was increased to 25,000.

6 List of Classes

The following is a complete alphabetical listing of the classes used in the FOREX DLL, along with a short description of its purpose.

EX
: Contains all historical foreign exchange information.

EXDaily
: Contains foreign exchange information for a single day.

Expend
: Contains all historical expenditure information.

ExpendMonthly
: Contains historical expenditure information for a single month.

Forecasts
: Contains a collection of procedures to create VaR forecasts.

ForecastingException
: Used in error trapping and to return a string to the calling program describing the error.

Functions
: Contains a number of utility functions used throughout the application.

FutureExchangeCollection
: Contains forecasted exchange data, as well as procedures to extract distribution of future exchanges.

FutureExchangeCollectionMember
: Contains a forecasted exchange rate for a single month.

FutureExpendituresCollection
: Contains forecasted expenditure data, as well as procedures to extract distribution of future expenditures.

FutureRisk
: Contains all simulated gain/loss values, and procedures to extract the VaR from the collection.

Intervention
: Contains data for a single intervention.

InterventionGroup
: Contains all interventions.

Level
: Contains data for a single level intervention.

LevelCollection
: Contains all level interventions.

Matrix
: Class representing a matrix, which holds the entries in the matrix, as well as a number of matrix processing functions.

MatrixInertia
Contains the inertia of a given matrix (used during the optimization).

Memory
Contains a single intervention.

MemoryCollection
Contains a collection of interventions.

OptimizationMod
The principal member of the DLL. Contains all other classes and functions to determine the VaR.

PermutationCollection
Contains all permutation classes and coordinates random number seeds to ensure random number sequences are not repeated.

Permutation
Creates a random permutation of integers.

Pulse
Contains data for a single pulse intervention.

PulseCollection
Contains all pulse interventions.

RateForecast
Contains all forecasted foreign exchange rates.

Seasonal
Contains data for a single seasonal intervention.

SeasonalCollection
Contains all seasonal interventions.

7 Database Structure

The following tables give the database requirements for the FOREX DLL to function. In each of the tables, there are additional fields which have been added for the use of the overall application. Although they are required for the FOREX web site, these additional fields are not required for the optimization.

Table 1: Database table for GARCH coefficients

tblMLECoefficients		
Field Name	**Type**	**Size**
FMASCurrencyCode	Text	10
a	Double	8
b	Double	8
w	Double	8
d	Double	8
Day	Integer	2
Month	Integer	2
Year	Integer	2

Table 2: Database table for Autoregressive coefficients

tblCoefficientsAR		
Field Name	**Type**	**Size**
FMASFundCode	Text	4
FMASCurrencyCode	Text	10
Shift	Integer	2
Value	Single	4

Table 3: Database table for constant coefficients

tblCoefficientsConstant		
Field Name	**Type**	**Size**
FMASFundCode	Text	4
FMASCurrencyCode	Text	10
Coefficient	Double	8
StartYear	Integer	2
StartMonth	Integer	2

Table 4: Database table for Level intervention coefficients

tblCoefficientsLevel		
Field Name	**Type**	**Size**
FMASFundCode	Text	4
FMASCurrencyCode	Text	3
Value	Single	4
BFY	Integer	2
Shift	Integer	2
Period	Integer	2
Month	Integer	2

Table 5: Database table for Pulse intervention coefficients

tblCoefficientsPulse		
Field Name	**Type**	**Size**
FMASFundCode	Text	4
FMASCurrencyCode	Text	3
Value	Single	4
BFY	Integer	2
Period	Integer	2
Shift	Integer	2
Month	Integer	2

Table 6: Database table for Seasonal Pulse intervention coefficients

tblCoefficientsSeasonal		
Field Name	**Type**	**Size**
FMASFundCode	Text	4
FMASCurrencyCode	Text	3
Value	Single	4
Recurrence	Integer	2
BFY	Integer	2
Period	Integer	2
Shift	Integer	2
Month	Integer	2

Table 7: Database table for exchange rates

tblCurrencyExchangeRate		
Field Name	**Type**	**Size**
ToFMASCurrencyCode	Text	10
ExchangeRateDate	Date/Time	8
ExchangeRate	Single	4
ExchangeRateNA	Yes/No	1

Table 8: Database table for exchange rate forecasts

tblCurrencyExchangeRateForecast		
Field Name	**Type**	**Size**
ToFMASCurrencyCode	Text	10
ExchangeRateDate	Date/Time	8
ExchangeRate	Single	4

Table 9: Database table for scenario output results

tblScenarioOutput		
Field Name	**Type**	**Size**
BFY	Integer	2
Period	Integer	2
Fund	Text	4
Currency	Text	3
Cdn_Amount	Currency	8

Table 10: Database table for FMAS summary results

tblFMASSummaryData		
Field Name	**Type**	**Size**
ScenarioOutputId	Long Integer	4
Year	Integer	2
Month	Integer	2
Percentile	Integer	2
GainOrLoss	Single	4
Exchange	Single	4
Expenditure	Single	4

8 Exchange Model Optimization

8.1 Overview

The most difficult aspect of creating the FOREX forecasting module was the creation of the Maximum Likelihood Estimate (MLE) optimization algorithms. Due to security restrictions on the use of third party software on DND computer systems, it was decided early that an investment in creating an optimization algorithm from scratch would be easier and faster than obtaining the required security clearances for using third party software.

The overall optimization scheme is based on an interior point algorithm for a constrained objective function [5]. However, before this algorithm could be implemented, the derivatives of the objective function had to be determined.

The objective function, or MLE estimator, as implemented in FOREX, is dependent on four parameters (ω, α, β, and d). If we consider the standardized return as a random variable defined by $z_t = r_t/\sigma_t$, then the log-likelihood of the sample of returns is given by

$$\begin{aligned} \ln L &= \sum_{t=1}^{T} \ln(f(r_t; d)) - \sum_{t=1}^{T} \ln(\sigma_t^2)/2 \\ &= T\{\ln(\Gamma((d+1)/2)) - \ln(\Gamma(d/2)) - \ln(\pi)/2 - \ln(d-2)/2\} \\ &\quad -\frac{1}{2}\sum_{t=1}^{T}(1+d)\ln(1+(r_t/\sigma_t)^2/(d-2)) - \sum_{t=1}^{T}\ln(\sigma_t^2)/2 \; , \end{aligned} \tag{5}$$

where the last term in equation (5) takes into account the variance, and the unknown parameters (ω, α, β, d) are estimated through maximizing equation (5). Once the values of (ω, α, β) are estimated by MLE, the conditional variances are estimated.

The derivatives in terms of the four input variables have to be determined for the optimization process. Initially, a quick glance at the equation only shows the variable d explicitly. The other three variables are embedded in the calculation of the standard deviations.

8.2 Derivatives of Gamma function

Fortunately for our purposes, the $\ln\Gamma$ function is commonly used in statistical analysis, and as such has been well studied. In particular, the derivatives of this function are known, and numerical estimates of both the $\ln\Gamma$ function and its first two derivatives (which are required for the MLE optimization) are readily available.

The first derivative of the $\ln\Gamma$ function is often called the digamma or psi function, denoted by ψ. The second derivative is called the trigamma function.

Very accurate and fast numerical implementations of the calculation of these three functions are readily available. The procedures given by reference [7] have been used as a basis for the implementation of these calculations.

8.3 First derivatives of MLE function

8.3.1 Observations

The first three variables (α, β, and ω) can actually be grouped together to a great extent. They are only embedded in the standard deviation calculation, and are not explicitly given in the MLE Equation.

Since GARCH models define the characteristics of financial return series such as time-varying volatilities, the conditional variances must be specified through MLE estimation. The initial estimator $\hat{\sigma}_1^2$ is the variance of the recorded returns. Hence, given a set of returns $\{R_1, \ldots, R_{T-1}\}$, this initial estimator is a constant. The conditional variances are then given by the recursive formula:

$$\nu_{t+1} = \sigma_{t+1}^2 = \omega + \alpha R_t^2 + \beta \sigma_t^2 \ . \tag{6}$$

Note that the variable d does not appear in this recursion. Hence, the calculations of the derivatives fall into two distinct categories. Using this recursion formula for the variance, a similar recursion formula can be derived for the first derivative of the variance in respect to α, β, and ω.

$$\begin{aligned} \frac{\partial}{\partial \alpha}\nu_{t+1} &= \frac{\partial}{\partial \alpha}(\omega + \alpha R_t^2 + \beta \nu_t) \\ &= \frac{\partial}{\partial \alpha}\omega + \frac{\partial}{\partial \alpha}\left(\alpha R_t^2\right) + \frac{\partial}{\partial \alpha}(\beta \nu_t) \\ &= R_t^2 + \beta \frac{\partial}{\partial \alpha}\nu_t \ . \end{aligned} \tag{7}$$

Combining this recursive formula with the fact that $\frac{\partial}{\partial \alpha}\nu_1 = 0$ gives a means to determine this derivative for all values. In a similar manner, the other two derivatives can be calculated as

$$\frac{\partial}{\partial \beta}\nu_{t+1} = \nu_t + \beta \frac{\partial}{\partial \beta}\nu_t \ , \tag{8}$$

$$\frac{\partial}{\partial \omega}\nu_{t+1} = 1 + \beta \frac{\partial}{\partial \omega}\nu_t \ . \tag{9}$$

Again, keeping in mind that $\frac{\partial}{\partial \beta}\nu_1 = 0$ and $\frac{\partial}{\partial \omega}\nu_1 = 0$ gives a recursive formula to calculate these derivatives.

8.3.2 Derivative with respect to α

The α term does not appear directly in the MLE function, only indirectly through the variance terms. Hence, the derivative calculation is equivalent to

$$\frac{\partial}{\partial\alpha}\ln L = \frac{\partial}{\partial\alpha}\left(-\frac{1}{2}\sum_{t=1}^{T}\left\{(1+d)\ln\left(1+\frac{R_t^2/\nu_t}{d-2}\right)-\ln\nu_t\right\}\right) , \tag{10}$$

which, if we define $z_t = R_t/\sqrt{\nu_t}$, then eq. (10) becomes

$$\begin{aligned}\frac{\partial}{\partial\alpha}\ln L &= -\frac{1}{2}\sum_{t=0}^{T}\left[(1+d)\left(\frac{d-2}{d-2+z_t^2}\right)\left(-\frac{R_t^2}{(d-2)\nu_t^2}\right)\left(\frac{\partial}{\partial\alpha}\nu_t\right)+\frac{1}{\nu_t}\left(\frac{\partial}{\partial\alpha}\nu_t\right)\right] \\ &= \frac{1}{2}\sum_{t=0}^{T}\left[\left(\frac{\partial}{\partial\alpha}\nu_t\right)\left[\frac{(1+d)R_t^2}{(d-2+z_t^2)\nu_t^2}-\frac{1}{\nu_t}\right]\right] . \end{aligned} \tag{11}$$

If we further define

$$K_t = \frac{(1+d)R_t^2}{(d-2+z_t^2)\nu_t^2}-\frac{1}{\nu_t} = \frac{dz_t^2-d+2}{(d-2+z_t^2)\nu_t} = \frac{dz_t^2-d+2}{(d-2)\nu_t+R_t^2} , \tag{12}$$

and

$$A_t = \frac{\partial}{\partial\alpha}\nu_t , \tag{13}$$

then

$$\frac{\partial}{\partial\alpha}\ln L = \frac{1}{2}\sum_{t=1}^{T}A_tK_t . \tag{14}$$

Of importance here is that this equation gives a closed form for the calculation of the derivative with respect to α. Also, this form is easily implemented and, through the above recursion, very quick to calculate in practice.

8.3.3 Derivative with respect to β

The derivative with respect to β is very similar to that of α. In fact a close look at the derivation above reveals that, since α is not even explicitly present in the original formula, the calculation is exactly the same for β.

As a result, if we define

$$B_t = \frac{\partial}{\partial\beta}\nu_t , \tag{15}$$

then we get

$$\frac{\partial}{\partial\beta}\ln L = \frac{1}{2}\sum_{t=1}^{T}B_tK_t . \tag{16}$$

8.3.4 Derivative with respect to ω

Similar to the above two derivations, using

$$W_t = \frac{\partial}{\partial \omega}\nu_t \ , \tag{17}$$

we get

$$\frac{\partial}{\partial \omega} \ln L = \frac{1}{2}\sum_{t=1}^{T} W_t K_t \ . \tag{18}$$

8.3.5 Derivative with respect to d

The variable d is not embedded in the calculations of the variance, hence this term is a constant when calculating the derivative with respect to d. Removing the constant terms, we get

$$\begin{aligned}
\frac{\partial}{\partial d} \ln L &= \frac{\partial}{\partial d} T \left[\ln\Gamma((d+1)/2) - \ln\Gamma(d/2) - \ln(d-2)/2)\right] \\
&- \frac{1}{2}\sum_{t=1}^{T}(1+d)\ln(1+z_t^2/(d-2)) \\
&= T\left\{\frac{1}{2}\psi\left(\frac{d+1}{2}\right) - \frac{1}{2}\psi\left(\frac{d}{2}\right) - \frac{1}{2(d-2)}\right\} \\
&- \frac{1}{2}\sum_{t=1}^{T}\left[\ln\left(1+\frac{z_t^2}{d-2}\right) - (1+d)\left(\frac{d-2}{d-2+z_t^2}\right)\left(\frac{z_t^2}{(d-2)^2}\right)\right] \\
&= \frac{T}{2}\left\{\psi\left(\frac{d+1}{2}\right) - \psi\left(\frac{d}{2}\right) - \frac{1}{(d-2)}\right\} \\
&- \frac{1}{2}\sum_{t=1}^{T}\left[\ln\left(1+\frac{z_t^2}{d-2}\right) - \left(\frac{(1+d)z_t^2}{(d-2+z_t^2)(d-2)}\right)\right] \ .
\end{aligned} \tag{19}$$

While this computation is not quite as clean and compact as the previous derivations, it is still a closed form for the derivative which is easily computable.

8.4 Second derivatives of MLE function

8.4.1 Second derivatives with respect to d

The first three derivatives derived above involve only two summation terms,

$$\frac{\partial}{\partial \alpha} \ln L = \frac{1}{2}\sum_{t=1}^{T} A_t K_t \ . \tag{20}$$

Noting that the first term is a constant with respect to d, we get that

$$\frac{\partial^2}{\partial \alpha \partial d} \ln L = \frac{1}{2}\sum_{t=1}^{T} A_t \frac{\partial}{\partial d} K_t \ . \tag{21}$$

This is similar for the next two derivatives as well, simply replacing A with B or W as appropriate. To determine these three derivatives, we need only compute $\frac{\partial}{\partial d}K_t$. i.e.,

$$\begin{aligned}\frac{\partial}{\partial d}K_t &= \frac{\partial}{\partial d}\left(\frac{dz_t^2-d+2}{(d-2)\nu_t+R_t^2}\right)\\ &= \frac{\left((d-2)\nu_t+R_t^2\right)\left(z_t^2-1\right)-\left(dz_t^2-d+2\right)\nu_t}{\left((d-2)\nu_t+R_t^2\right)^2}\\ &= \frac{R_t^2z_t^2-2\nu_tz_t^2-R_t^2}{\left((d-2)\nu_t+R_t^2\right)^2} = R_t^2\frac{z_t^2-3}{\left((d-2)\nu_t+R_t^2\right)^2}\ . \end{aligned}\tag{22}$$

If we define $K_t^d = \frac{\partial}{\partial d}K_t$, then

$$\frac{\partial^2}{\partial\alpha\partial d}\ln L = \frac{1}{2}\sum_{t=1}^{T}A_tK_t^d\ , \tag{23}$$

$$\frac{\partial^2}{\partial\beta\partial d}\ln L = \frac{1}{2}\sum_{t=1}^{T}B_tK_t^d\ , \tag{24}$$

$$\frac{\partial^2}{\partial\omega\partial d}\ln L = \frac{1}{2}\sum_{t=1}^{T}W_tK_t^d\ . \tag{25}$$

This still leaves the problem of determining $\frac{\partial^2}{\partial d^2}\ln L$, i.e.,

$$\begin{aligned}\frac{\partial^2}{\partial d^2}\ln L &= \frac{\partial}{\partial d}\left[\frac{T}{2}\left\{\psi\left(\frac{d+1}{2}\right)-\psi\left(\frac{d}{2}\right)-\frac{1}{d-2}\right\}\right.\\ &- \left.\frac{1}{2}\sum_{t=1}^{T}\left[\ln\left(1+\frac{z_t^2}{d-2}\right)-\left(\frac{(1+d)z_t^2}{(d-2+z_t^2)(d-2)}\right)\right]\right]\\ &= \frac{T}{2}\left\{\frac{1}{2}\psi'\left(\frac{d+1}{2}\right)-\frac{1}{2}\psi'\left(\frac{d}{2}\right)+\frac{1}{(d-2)^2}\right\}\\ &- \frac{1}{2}\sum_{t=1}^{T}\left[-\frac{(d-2)z_t^2}{(d-2+z_t^2)(d-2)^2}-\frac{(d-2+z_t^2)(d-2)z_t^2-(1+d)z_t^2(2d-4+z_t^2)}{(d-2+z_t^2)^2(d-2)^2}\right]\\ &= \frac{T}{2}\left\{\frac{1}{2}\psi'\left(\frac{d+1}{2}\right)-\frac{1}{2}\psi'\left(\frac{d}{2}\right)+\frac{1}{(d-2)^2}\right\}\\ &+ \frac{1}{2(d-2)}\sum_{t=1}^{T}z_t^2\left[\frac{1}{(d-2+z_t^2)}+\frac{8-3z_t^2-2d-d^2}{(d-2+z_t^2)^2(d-2)}\right]\ . \end{aligned}\tag{26}$$

Again, although it appears relatively complex, this is a closed function that can be easily computed. The function ψ', used twice in the above function, is the previously mentioned trigamma function, which is easily computed numerically with a high degree of accuracy.

8.4.2 Second derivatives with respect to α

One of these derivatives, $\partial^2/\partial\alpha\partial d$, has been calculated above. This leaves the second derivatives with respect to α, β, and ω.

First, to get the second derivative $\partial^2/\partial\alpha^2$, use as a starting point

$$\frac{\partial}{\partial\alpha}\ln L = \frac{1}{2}\sum_{t=1}^{T} A_t K_t \ . \tag{27}$$

Then

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\ln L &= \frac{1}{2}\frac{\partial}{\partial\alpha}\sum_{t=1}^{T} A_t K_t \\ &= \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\alpha}K_t + K_t\frac{\partial}{\partial\alpha}A_t\right) \ ,\end{aligned} \tag{28}$$

where

$$\begin{aligned}\frac{\partial}{\partial\alpha}K_t &= \frac{\partial}{\partial\alpha}\left(\frac{dz_t^2 - d + 2}{(d-2)\nu_t + R_t^2}\right) \\ &= -A_t\frac{(d-2+z_t^2)(dz_t^2) + (dz_t^2 - d + 2)(d-2)}{\left((d-2)\nu_t + R_t^2\right)^2} \\ &= -A_t\frac{2d^2z_t^2 - 4dz_t^2 + dz_t^4 - d^2 + 4d - 4}{\left((d-2)\nu_t + R_t^2\right)^2} \ .\end{aligned} \tag{29}$$

For the derivative of A_t, using the recursive formula

$$\begin{aligned}\frac{\partial}{\partial\alpha}A_t &= \frac{\partial}{\partial\alpha}\left(R_{t-1}^2 + \beta A_{t-1}\right) \\ &= \beta\frac{\partial}{\partial\alpha}A_{t-1} \ ,\end{aligned} \tag{30}$$

and recalling that $\frac{\partial}{\partial\alpha}A_1 = 0$, it follows that $\frac{\partial}{\partial\alpha}A_t = 0$ for all t, we define

$$M_t = \frac{2d^2z_t^2 - 4dz_t^2 + dz_t^4 - d^2 + 4d - 4}{\left((d-2)\nu_t + R_t^2\right)^2} \ , \tag{31}$$

and get

$$\begin{aligned}\frac{\partial^2}{\partial\alpha^2}\ln L &= \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\alpha}K_t\right) \\ &= -\frac{1}{2}\sum_{t=1}^{T}\left(A_t^2 M_t\right) \ .\end{aligned} \tag{32}$$

For the derivatives with respect to β and ω, first note

$$\begin{aligned}\frac{\partial^2}{\partial\alpha\partial\beta}\ln L &= \frac{1}{2}\frac{\partial}{\partial\beta}\sum_{t=1}^{T}A_tK_t \\ &= \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\beta}K_t + K_t\frac{\partial}{\partial\beta}A_t\right) \ , \end{aligned} \tag{33}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\alpha\partial\omega}\ln L &= \frac{1}{2}\frac{\partial}{\partial\omega}\sum_{t=1}^{T}A_tK_t \\ &= \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\omega}K_t + K_t\frac{\partial}{\partial\omega}A_t\right) \ . \end{aligned} \tag{34}$$

Again using the recursion of eq. 30, we get that

$$\begin{aligned}\frac{\partial}{\partial\beta}A_t &= \frac{\partial}{\partial\beta}\left(R_t^2 + \beta A_{t-1}\right) \\ &= A_{t-1} + \beta\frac{\partial}{\partial\beta}A_{t-1} \ , \end{aligned} \tag{35}$$

and

$$\begin{aligned}\frac{\partial}{\partial\omega}A_t &= \frac{\partial}{\partial\omega}\left(R_{t-1}^2 + \beta A_{t-1}\right) \\ &= \beta\frac{\partial}{\partial\omega}A_{t-1} = 0 \ . \end{aligned} \tag{36}$$

This leads to the solutions

$$\begin{aligned}\frac{\partial^2}{\partial\alpha\partial\beta}\ln L &= = \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\beta}K_t + K_t\frac{\partial}{\partial\beta}A_t\right) \\ &= \frac{1}{2}\sum_{t=1}^{T}\left(-A_tB_tM_t + K_t\frac{\partial}{\partial\beta}A_t\right) \ , \end{aligned} \tag{37}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial\alpha\partial\omega}\ln L &= = \frac{1}{2}\sum_{t=1}^{T}\left(A_t\frac{\partial}{\partial\omega}K_t\right) \\ &= -\frac{1}{2}\sum_{t=1}^{T}\left(A_tW_tM_t\right) \ . \end{aligned} \tag{38}$$

Both of these solutions include recursive elements, and are both easily computed.

8.4.3 Second derivatives with respect to β

The derivatives $\partial^2/\partial\beta^2$ and $\partial^2/\partial\beta\partial\omega$ have to be calculated (the other two were computed in the previous sections). For the former, we get

$$\begin{aligned}
\frac{\partial^2}{\partial\beta^2}\ln L &= \frac{\partial}{\partial\beta}\left(\frac{1}{2}\sum_{t=1}^{T} B_t K_t\right) \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(B_t\frac{\partial}{\partial\beta}K_t + K_t\frac{\partial}{\partial\beta}B_t\right) \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(-B_t^2 M_t + K_t\frac{\partial}{\partial\beta}B_t\right) \ . \qquad (39)
\end{aligned}$$

Also, we again get a recursive relationship to compute the remainder of this derivative

$$\begin{aligned}
\frac{\partial}{\partial\beta}B_t &= \frac{\partial}{\partial\beta}\left(\nu_{t-1} + \beta B_{t-1}\right) \\
&= 2B_{t-1} + \beta\frac{\partial}{\partial\beta}B_{t-1} \ . \qquad (40)
\end{aligned}$$

For $\partial^2/\partial\beta\partial\omega$, we get the recursion formula

$$\begin{aligned}
\frac{\partial}{\partial\omega}B_t &= \frac{\partial}{\partial\omega}\left(\nu_{t-1} + \beta B_{t-1}\right) \\
&= W_{t-1} + \beta\frac{\partial}{\partial\omega}B_{t-1} \ . \qquad (41)
\end{aligned}$$

This is used to compute

$$\begin{aligned}
\frac{\partial^2}{\partial\beta\partial\omega}\ln L &= \frac{\partial}{\partial\omega}\left(\frac{1}{2}\sum_{t=1}^{T} B_t K_t\right) \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(B_t\frac{\partial}{\partial\omega}K_t + K_t\frac{\partial}{\partial\beta}B_t\right) \\
&= \frac{1}{2}\sum_{t=1}^{T}\left(-B_t W_t M_t + K_t\frac{\partial}{\partial\omega}B_t\right) \ . \qquad (42)
\end{aligned}$$

8.4.4 Second derivatives with respect to ω

The only derivative remaining is $\partial^2/\partial\omega^2$. Analogous to the above derivations, we get

$$\begin{aligned}
\frac{\partial^2}{\partial\omega^2}\ln L &= \frac{1}{2}\sum_{t=1}^{T}\left(W_t\frac{\partial}{\partial\omega}K_t\right) \\
&= -\frac{1}{2}\sum_{t=1}^{T}\left(W_t^2 M_t\right) \ . \qquad (43)
\end{aligned}$$

8.5 Determining the optimization coefficients

Finding a model for foreign exchange is not an easy task, although there are a number of models that do exist. Given a historical set of exchange rates $\{S_t\}_{t=0}^{T}$, and the corresponding returns $R_t = S_t/S_{t-1}$ for $t \in \{1, \ldots, T\}$, the GARCH variance model proposed by Christoffersen [8] states

$$\begin{aligned} \sigma_1^2 &= Var\left(\{R_t\}_{t=1}^{T}\right) , & (44) \\ \sigma_{t+1}^2 &= \omega + \alpha R_t^2 + \beta \sigma_t^2 , & (45) \end{aligned}$$

where Var refers to the non-conditional variance of the dataset.

This is the model used for foreign exchange rates in the FOREX system. The values for the three variables α, β, and ω have to be determined. The values of these three variables, along with the value for the variable d, should be chosen such that the objective function is maximized. This leads to the non-linear system

$$\begin{aligned} \text{Maximize} \quad f(x) &= \ln L = T\left\{\Gamma((d+1)/2) - \ln\Gamma(d/2) - \ln(\pi)/2 - \ln(d-2)/d\right\} \\ &- \frac{1}{2}\sum_{t=1}^{T}(1+d)\ln(1+(R_t/\sigma_t)^2/(d-2)) \\ &- \sum_{t=1}^{T}\ln(\sigma_t^2)/2 , \qquad (46) \\ \text{subject to:} \quad \alpha+\beta &< 1 \\ \alpha &> 0 \\ \beta &> 0 \\ \omega &> 0 \\ d &> 2 . \qquad (47) \end{aligned}$$

With the derivatives calculated in the previous section, an interior-point primal-dual algorithm with Newtonian approximation for constrained optimization was used to approximate the solution for this system of equations [4].

Using the vector notation $x = (x_1, \ldots, x_4) = (\alpha, \beta, \omega, d)$ to represent the variables, the inequality constraints can be summarized as

$$c_I(x) = \begin{bmatrix} 1 - x_1 - x_2 \\ x_1 \\ x_2 \\ x_3 \\ x_4 - 2 \end{bmatrix} . \qquad (48)$$

4. The remainder of this section is a quick overview of the process. Full details may be found in reference [5]

Now letting $s = (s_1, \ldots, s_5)$ and $z = (z_1, \ldots, z_5)$ and further defining $S = diag(s_1, \ldots, s_5)$ and $e = diag(1, \ldots, 1)$ and letting A_I^T be the Jacobian of the vector $c_I(x)$, the Karush-Kuhn Tucker conditions for this problem are given as

$$\begin{aligned} \nabla f(x) - A_I^T(x)z &= 0 \\ -\mu S^{-1}e + z &= 0 \\ c_I(x) - s &= 0 \,. \end{aligned} \tag{49}$$

The algorithm proceeds by initially setting $\mu = 0.2$. In addition, the initial settings for the remaining vectors are

$$\begin{aligned} x &= (x_1, \ldots, x_4) = (0.4,\, 0.4,\, 0.01,\, 8)\,, \\ s &= (s_1, \ldots, s_5) = (1,\, 1,\, 1,\, 1,\, 1)\,, \\ z &= (z_1, \ldots, z_5) = (1,\, 1,\, 1,\, 1,\, 1)\,. \end{aligned} \tag{50}$$

Next the algorithms solves the system of equations

$$\begin{bmatrix} \nabla^2_{xx}L & 0 & A_I^T(x) \\ 0 & \sum & -I \\ A_I(x) & I & 0 \end{bmatrix} \begin{bmatrix} p_x \\ p_s \\ -p_z \end{bmatrix} = - \begin{bmatrix} \nabla f(x) - A_I^T(x)z \\ z - \mu S^{-1}e \\ c_I(x) - s \end{bmatrix} . \tag{51}$$

The two non-zero diagonal terms of the left hand matrix are defined as

$$\begin{aligned} \sum &= S^{-1}Z = [diag(s)]^{-1} \times [diag(z)] \\ L &= f(x) - z^T(c_I(x) - s)\,. \end{aligned} \tag{52}$$

Before proceeding any further, there are constraints on the left hand matrix related to positive definiteness. The 14×14 left hand matrix has to have exactly 9 positive eigenvalues, otherwise the process may lead away from a solution instead of towards a solution. Often, it will have less than this, so the Lagrangian portion (the upper left portion) of this matrix has to be adjusted by adding a constant repeatedly to the diagonal elements until the desired number of eigenvalues is reached. The equation can then be solved for $(p_x,\, p_s,\, p_z)$. Solving this equation provides the length-14 vector $(p_x,\, p_s,\, p_z)$. These results are used to determine an iterative solution to the objective function.

First determine

$$\begin{aligned} a_s^{\max} &= \max(\alpha \in (0,\, 1] : s + \alpha p_s \geq (1-\tau)s;\; x + \alpha p_x \geq (1-\tau)x) \\ a_z^{\max} &= \max(\alpha \in (0,\, 1] : s + \alpha p_z \geq (1-\tau)z)\,. \end{aligned} \tag{53}$$

The iterative variables are then updated by

$$\begin{aligned} x^{\text{new}} &= x^{\text{old}} + \alpha_s^{\max} p_x \\ s^{\text{new}} &= s^{\text{old}} + \alpha_s^{\max} p_s \\ z^{\text{new}} &= z^{\text{old}} + \alpha_z^{\max} p_z \ . \end{aligned} \tag{54}$$

The procedure is then run iteratively until the desired proximity to the actual maximum is reached.

9 Conclusions

This contract report documents the VB.Net module for foreign exchange optimization. Documentation includes flowcharts depicting overall program flow as well as the complex mathematics behind the optimization procedures. Together with the user interface documented in [1], the reports provide complete documentation for determining VaR amounts for departmental accounts and major project contracts.

The VaR methodology and its resulting FOREX application should be adopted as part of the department's integrated risk management framework for managing the budgetary risk attributed to exposure to foreign currency fluctuations for all acquisitions. Currently there is no tool available to assess the in-year impact of foreign exchange fluctuations on Defence budget allocations. FOREX will now offer this capability.

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List of Acronyms

Autobox	Automatic Box-Jenkins
CAD	Canadian Dollar
DLL	Dynamic-Link Library
DND	Department of National Defence
EUR	Euro
FMAS	Financial and Managerial Accounting Systems
FOREX	FOReign EXchange
GARCH	Generalized Autoregressive Conditional Heteroskedasticity
GBP	U.K. Pound Sterling
USD	U.S. Dollars
VaR	Value-at-Risk
VB.Net	Visual Basic.Net

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The Foreign Exchange Exposure Model (FOREX) was designed and built as two separate components, that when integrated provide the user with full functionality in determining Value-at-Risk (VaR) results for a variety of account transactions and foreign exchanges. This contract report describes the optimization module component.

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