Modelling the US Dollar Trading Range

Bounds from the risk neutral measure

David W. Maybury
Defence Economics Team

DRDC CORA TM 2013-086
June 2013

Defence R&D Canada
Centre for Operational Research and Analysis

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Abstract

ADM(Fin CS) and senior decision makers at the Department of National Defence (DND) require insight into financial risks stemming from foreign exchange obligations in procurements and program delivery. We implement three popular derivative based quantitative financial models which provide the conditional Canada-US exchange rate trading range, under the risk neutral distribution, within a 95% confidence region up to a one year horizon. ADM(Fin CS) can use the model inferred trading range to help decide on a hedging rule in connection with foreign exchange budget obligations. Our results give a useful thumbnail sketch of the underlying probability distribution and confidence regions but, to gain a better understanding of foreign exchange market conditions, we require access to over-the-counter derivative data. Finally, ADM(Fin CS) staff can use the derived trading range in DND's foreign exchange reporting documents and internal monitoring services.

Résumé

Le SMA (Fin SM) et les hauts fonctionnaires du ministère de la Défense nationale (DND) doivent être informés du risque associé aux obligations en devises dans le cadre des achats et de la prestation de programmes. Nous appliquons trois modèles quantitatifs financiers bien connus axés sur les produits dérivés pour obtenir la fourchette de négociation conditionnelle du dollar américain par rapport au dollar canadien sur un horizon d'un an, suivant une distribution risque neutre et à l'intérieur d'une région de confiance de 95%. Le SMA (Fin SM) peut se servir de la fourchette de négociation donnée par les modèles pour décider d'une règle de couverture du risque de change associé à ses obligations budgétaires. Nos résultats donnent un aperçu utile de la distribution des probabilités et des régions de confiance. Cela dit, il conviendrait d'examiner les données sur les produits dérivés de gré à gré pour avoir une meilleure vue d'ensemble des conditions du marché des devises. Enfin, le personnel du SMA (Fin SM) peut utiliser la fourchette de négociation dans les documents du MDN concernant les opérations sur devises et dans les services de surveillance interne du Ministère.

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Executive summary

Modelling the US Dollar Trading Range

David W. Maybury; DRDC CORA TM 2013-086; Defence R&D Canada – CORA; June 2013.

Each year, the Department of National Defence (DND) expends approximately 10% of its annual budget in foreign currencies thereby exposing DND to financial risks that can interfere with program delivery. In recent discussions with ADM(Mat), ADM(Fin CS) has contemplated entering a formal relationship with the Department of Finance to build a foreign currency hedging program. Senior decision makers at DND feel that program delivery and military procurements can run more smoothly if DND has the ability to control or eliminate unwanted financial risks. To help ADM(Fin CS) staff understand foreign currency risk, we implement three popular derivative based quantitative financial models which provide the conditional Canada-US exchange rate trading range, under the risk neutral distribution, within a 95% confidence region over a one year horizon. The risk neutral distribution implied by market prices is the pricing distribution that ensures the absence of arbitrage. By having information on the US dollar trading range, ADM(Fin CS) staff can better gauge the amount of foreign exchange risk residing in DND's annual budget while also gaining insight into the market's assessment of the volatility in the Canada-US dollar exchange rate. As an example of the method, in figure 1, we display the model averaged results from January 29, 2013 along with the model specific results in table 1.

With the models presented in this paper, ADM(Fin CS) has insight into the trading range of the US dollar over a one year horizon. By observing changes in the trading range under the risk neutral distribution, ADM(Fin CS) staff can better understand the evolution of market risks and properly weigh the benefits of using a foreign exchange hedge with budget obligations. ADM(Fin CS) can use the results from this paper to help ascertain when hedging for the DND becomes appropriate, given the market conditions. For example, if the 95% confidence region at the six month horizon grows too large for the foreign exchange obligations becoming due over a prescribed period of time, ADM(Fin CS) staff could use that information as a signal to enter into a zero cost structure or a forward contract.

We must emphasize that our estimate of the risk neutral distribution produces a crude picture. We do not have access to over-the-counter market data – we use the publicly available data on the Canadian Derivatives Exchange. Our data is contaminated by stale quotes and inexact deposit rate estimates. Furthermore, the Exchange has the US dollar as its only foreign currency for derivative trading. To overcome these shortcomings, we recommend that ADM(Fin CS) staff obtain access to more detailed financial data, which will give DND a much more accurate picture of the global foreign exchange market, which will include exchange rate data across all currencies. With over-the-counter data, we have access to the implied risk neutral distribution based on the most recent market data yielding a much

Table 1: USD trading range 95% confidence region, January 29, 2013

	Contract Months				
95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Implied volatility model					
Upper	102.3	103.6	105.3	107.2	109.9
Lower	98.0	96.7	95.9	94.5	92.4
Lognormal mixture model					
Upper	102.5	103.9	105.4	107.9	110.5
Lower	97.8	96.7	95.6	93.8	91.9
Heston stochastic volatility model					
Upper	102.8	104.4	106.0	108.3	111.3
Lower	97.9	96.6	95.5	93.9	92.0
Model Average					
Upper	102.5	104.0	105.5	107.8	110.6
Lower	97.9	96.7	95.7	94.1	92.1

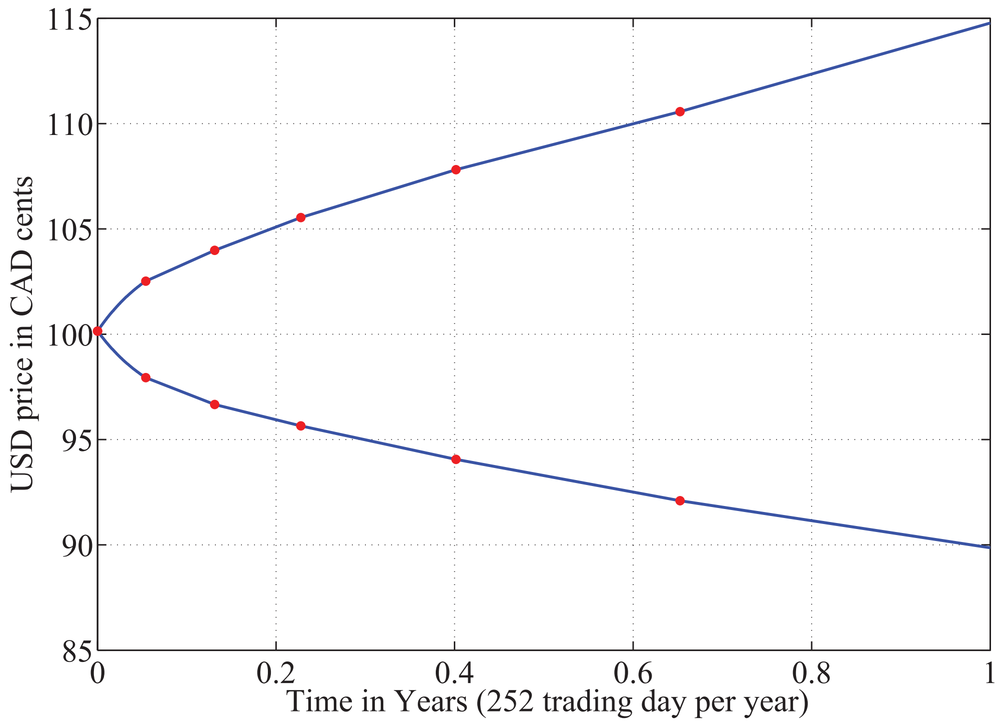

Figure 1: The model averaged 95% trading contour of the USD with one year horizon generated from data on January 29, 2013.

more accurate description of the trading ranges than the results of this study. More detailed financial data would also allow us to run counterfactual hedging scenarios (with mark-to-market performance) using actual over-the-counter market prices matched to DND's foreign exchange obligations.

Setting internal bounds on the US dollar trading range in conjunction with foreign currency payment dates can help ADM(Fin CS) recognize when the market senses more risk than usual and understand which type of hedging programs would best suit the DND's needs. Finally, ADM(Fin CS) staff can use the trading range in DND's foreign exchange reporting documents and internal monitoring services.

Sommaire

Modelling the US Dollar Trading Range

David W. Maybury ; DRDC CORA TM 2013-086 ; R & D pour la défense Canada – CARO ; juin 2013.

Chaque année, environ 10% des dépenses budgétées du ministère de la Défense nationale (MDN) se font en devises, ce qui expose le Ministère à des risques financiers susceptibles de perturber la prestation de programmes. Au cours de discussions récentes avec le SMA (Mat), le SMA (Fin SM) a envisagé la possibilité d'établir une relation officielle avec le ministère des Finances afin de mettre sur pied un programme de couverture du risque de change. Selon les hauts fonctionnaires du MDN, la prestation de programmes et l'approvisionnement en fournitures militaires pourraient être facilités si le Ministère arrivait à restreindre, voire à éliminer, les risques financiers indésirables. Pour aider le SMA (Fin SM) à cerner le risque associé aux devises, nous appliquons trois modèles quantitatifs financiers bien connus axés sur les produits dérivés pour obtenir la fourchette de négociation conditionnelle du dollar américain par rapport au dollar canadien sur un horizon d'un an, suivant une distribution risque neutre et à l'intérieur d'une région de confiance de 95%. Dans le contexte des prix sur le marché, une distribution risque neutre est une distribution des prix garantissant l'absence d'arbitrage. En ayant accès à des données sur la fourchette de négociation du dollar américain, le SMA (Fin SM) est mieux à même de mesurer le risque de change associé au budget annuel du MDN et il peut aussi mieux comprendre l'évaluation que fait le marché de la fluctuation du taux de change entre le dollar canadien et le dollar américain. À titre d'exemple de la méthode, la figure 2 donne les résultats moyens tirés des modèles pour la période commençant le 29 janvier 2013 ; les résultats détaillés sont présentés dans le tableau 2.

Tableau 2: USD trading range 95% confidence region, January 29, 2013

	Contract Months				
95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Implied volatility model					
Upper	102.3	103.6	105.3	107.2	109.9
Lower	98.0	96.7	95.9	94.5	92.4
Lognormal mixture model					
Upper	102.5	103.9	105.4	107.9	110.5
Lower	97.8	96.7	95.6	93.8	91.9
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Upper	102.8	104.4	106.0	108.3	111.3
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Model Average					
Upper	102.5	104.0	105.5	107.8	110.6
Lower	97.9	96.7	95.7	94.1	92.1

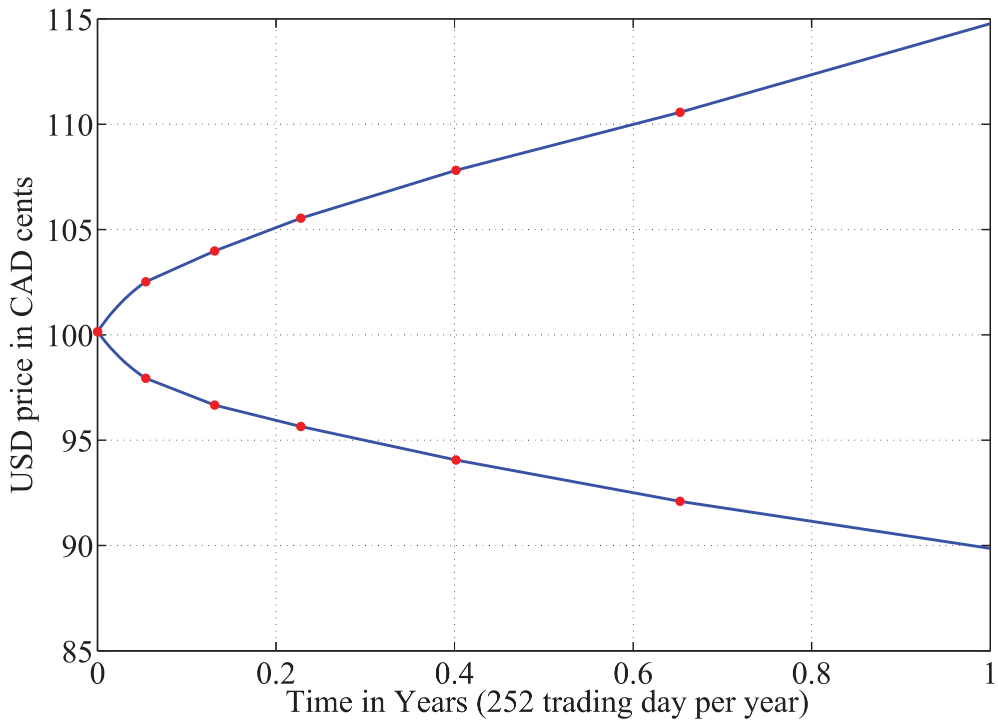

Figure 2: The model averaged 95% trading contour of the USD with one year horizon generated from data on January 29, 2013.

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1 Introduction

Prices are important not because money is considered paramount but because prices are a fast and effective conveyor of information through a vast society in which fragmented knowledge must be coordinated.

— Thomas Sowell

Each year, the Department of National Defence (DND) expends approximately 10% of its annual budget in foreign currencies thereby exposing DND to financial risks that can interfere with program delivery. In recent discussions with ADM(Mat), ADM(Fin CS) has contemplated entering a formal relationship with the Department of Finance to construct a foreign currency hedging program. Senior decision makers at DND feel that program delivery and military procurements can run more smoothly if DND has the ability to control or eliminate unwanted financial risks [1]. To help ADM(Fin CS) staff understand foreign currency risk, we implement three popular derivative based quantitative financial models which provide the conditional Canada-US exchange rate trading range, under the risk neutral distribution, within a 95% confidence region over a one year horizon. By having information on the US dollar trading range, ADM(Fin CS) staff can better gauge the amount of foreign exchange risk residing in DND's annual budget while also gaining insight into the market's assessment of the volatility in the Canada-US dollar exchange rate.

1.1 Background

Canada uses a floating exchange rate regime under which the value of the Canadian dollar adjusts automatically to demands in the foreign exchange market [2]. Over the long term, the market determined Canadian dollar exchange rates reflect Canada's economic fundamentals, fiscal and monetary policy, and trade balances with the rest of the world. Monetary and fiscal authorities in Canada do not use an exchange rate target and the Bank of Canada will directly intervene in markets to stabilize the Canadian dollar only under the most extreme of circumstances. The government of Canada feels that the market should set the value of the Canadian dollar.

Over the last five years, senior DND decision makers have sought a better understanding of the risks that foreign exchange transactions present to operational planning and military procurement [3]. At the request of ADM(Mat), in 2007, an analysis by the Centre for Operational Research and Analysis's (CORA) research team, DMGOR (Directorate Materiel Group Operational Research), examined value-at-risk (VaR) associated with foreign exchange transactions for both the National Procurement and Capital accounts [4]. DMGOR established departmental loss thresholds resulting from adverse currency movements over time horizons matched to spending. The DMGOR model uses generalized autoregressive conditional hetroskedasticity techniques (GARCH) applied to foreign exchange spot rates with a time series analysis of actual DND foreign currency payments [5]. In addition to VaR insight, senior decision makers sought counterfactual studies to understand possible risk mitigation strategies. Essaddam et al. [6], in 2005, examined hedging DND's foreign exchange risk using forward contracts[1] under two simple hedging rules with data from April 1990 to October 2002. They found that the forward contract hedging strategy surpassed DND's status quo no hedge policy while offering lower volatility. In 2011, DMGOR provided a complete counterfactual hedging study for ADM(Mat)'s US dollar obligations from November 2009 to July 2010 [3]. Using in-house derivative pricing expertise, the study showed the hedging performance of six derivative based hedging strategies with the corresponding reduction in variance of the budget.

Following the Essaddam et al. study, Director Strategic Finance and Costing (DSFC 7) examined foreign exchange risk in DND and concluded that DND should develop a well-defined foreign exchange risk mitigation strategy involving the Department of Finance [7]. Furthering the efforts in foreign exchange risk identification, the Defence Economics Team in CORA (DET CORA) completed a study [1] in 2010 on corporate risk prioritization using subject matter expert opinion with rank correlation. DET CORA found that "*increased financial pressure due to external factors represents one of the highest risk factors among all corporate activities at DND*". In follow up discussions with ADM(Fin CS) and ADM(Mat), DET CORA discovered that both assistant deputy ministers felt that foreign exchange risk represented one of the most important external risk factors facing DND.

1.2 Scope

ADM(Fin CS) requires an understanding of the trading range of the US dollar over the budget year. In this paper we provide:

- the 95% confidence trading envelope of the CAD/USD exchange rate over a one year horizon, based on call and put option data listed on the Canadian Derivatives Exchange.

We obtain all US dollar option data from the Canadian Derivatives Exchange (TMX) based in Montreal and we obtain Canadian and US LIBOR (London Interbank Offer Rate) interest rates from global-rates.com [8].

[1] Foreign currency forwards are derivative contracts that obligate the parties to exchange currencies at a predetermined fixed price at a predetermined maturity date.

2 Model application

Asset markets help participants share risk, and derivatives represent the plumbing-works in the risk exchange. Derivatives are financial contracts whose value depends on some set of underlying variables or assets (for more details on derivatives and derivative pricing, see [9], [17], [10]). The name *derivative* arises from the sense in which the contract's value is *derived* from its underlying features. In a foreign exchange context, some of the input underlying variables that determine a derivative contract price include the foreign and domestic interest rates, the spot exchange rate, and the exchange rate volatility. Derivatives allow participants to gain or eliminate risk exposure efficiently, and as a result they are widely traded around the globe. As of 2007 year end, the notional amount of all outstanding positions in derivative contracts worldwide stood at almost $600 trillion - approximately 10 times the world GDP (Gross Domestic Product). The derivative industry has witnessed tremendous growth over the last 30 years as corporate risk managers have become comfortable using derivative based strategies to limit risk exposure. Given the size of the foreign exchange market – with a nearly $4 trillion daily global currency turnover – and the volatility of exchange rates, derivatives feature centrally in controlling foreign exchange risk.

Market participants find derivatives useful because they behave like insurance contracts in a market where traditional pooled premium insurance does not exist. The insurance-like nature of derivatives increases economic efficiency of the market. Globally, derivative trading is divided between exchanges and the over-the-counter (OTC) market. In the exchange traded derivative market, the exchange (e.g., Chicago Board Options Exchange) lists derivatives, and uses a clearinghouse to ensure the performance of all parties. Furthermore, the exchange standardizes derivative contracts, and ensures that market participants make use of margin accounts to guard against default. In this fashion, the exchange creates and facilitates an ordered market. By contrast, the OTC market involves participants who deal directly with financial institutions to create highly flexible and tailored derivative contracts to address a specific need. The OTC market deals with sophisticated investors and traders, usually at a corporate level.

Like all traded assets, supply and demand ultimately determines derivative prices, but derivatives have an additional strong pricing constraint feature. Because derivatives depend on underlying variables, the price of the derivative must link to those variables in a precise fashion. If that link breaks down, the market will allow someone to make a riskless profit, call arbitrage[2], by directly trading the derivative against the underlying variables. Thus, derivative pricing becomes an intricate mathematical problem, the solution of which ensures arbitrage free prices.

[2] The absence of arbitrage implies that all portfolios with payoffs that are almost surely non-negative, which also have a positive probability of a positive payoff, must come with a positive price.

The trading efforts that keep derivative prices arbitrage free also help markets understand the probability of price movements. By reverse engineering derivative prices, market participants can learn the implied probabilities of price movements under the distribution that leads to arbitrage free prices. This distribution is called the risk neutral probability distribution. At the corporate treasurer level, changes in implied probabilities can help the firm decide when to hedge unwanted financial risks. For a corporate treasurer to make hedging decisions, she must understand the risks the firm faces with a realistic internal assessment of the chance that those risks materialize as damaging outcomes for the firm. In DND's foreign exchange context, senior decision makers require access to internal forecasting of trading ranges for the currency pairs that represent high risk elements in procurement and planning activities.

In this paper, we use three popular quantitative models[3] that deliver a trading range forecast for the Canada-US dollar exchange rate, based on the risk neutral distribution:

- An implied volatility model with Black-Scholes interpolation;
- A two dimensional lognormal mixture model; and
- The Heston stochastic volatility model.

We display the final trading envelope, enclosing the 95% confidence region, by averaging the results from the the three models. We give details of each model in Annex A.

2.1 Implied volatility with Black-Scholes Interpolation

Throughout this paper, we make use of two financial instruments: call options and put options. A Canadian call option on the US dollar gives the option holder the right but not the obligation to buy the US dollar for a predetermined Canadian dollar price on a predetermined date. Since the holder of the option has the privilege to exercise in favourable situations, the contract requires the holder to pay a premium to acquire the option. The price at which the option holder can buy the US dollar is called the strike price. Figure 3 illustrates the payoff of a call option, adjusted by the contract's premium. Notice that if the US dollar at the maturity date is less than the strike price, the option expires worthless. Similar to the call option, a Canadian put option on the US dollar gives the option holder the right but not the obligation to sell the US dollar for the stated strike price on a predetermined date. Figure 4 shows the payoff of a put option and we see that if the option expires with the US dollar above the strike price, the option expires worthless. Generally, derivatives can have an unlimited number of features and payoff structures, but call and put options have simple payouts with easy to understand inputs and thus they form the

[3]The three models we choose are relatively easy to implement with minimal computational overhead. We did not use the elegant Vanna-Volga method [11] – a fast and accurate interpolation scheme – as we do not have access to over-the-counter data.

backbone of the foreign exchange derivative market. Calls and puts with a predetermined exercise date are called European style options and the foreign exchange market heavily relies on this option style.

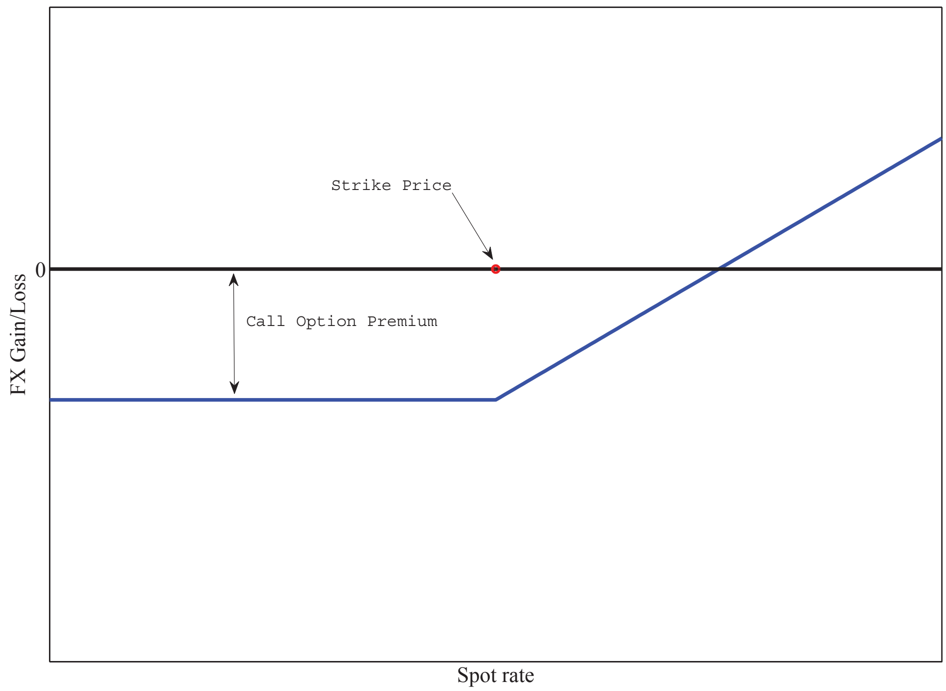

Figure 3: The payout of a call option.

The Canadian Derivative Exchange lists standardized European style call and put options on the US dollar (symbol USX) with premiums and strike prices quoted in Canadian cents – the US dollar is treated as the asset. Figure 5 shows the option chain at market close on January 29, 2013. Notice that the exchange lists options in half cent increments in strike prices along with bid and ask prices for the option premium and the last traded price. In our study, we focus on the midpoint between the bid and the ask price[4].

The option chain lists the contract months with each strike. In addition to standardizing contracts into half cent strike increments, the exchange offers contracts expiring only in the first three months plus the next two quarterly months in the March, June, September, December cycle. Contracts expire at noon on the third Friday of the expiration month. As an example, let's look at line five of the option chain in figure 5. Here we see February

[4]In asset markets, dealers quotes two prices – the price at which they are willing to buy (the bid price), and the price at which they are willing to sell (the ask price).

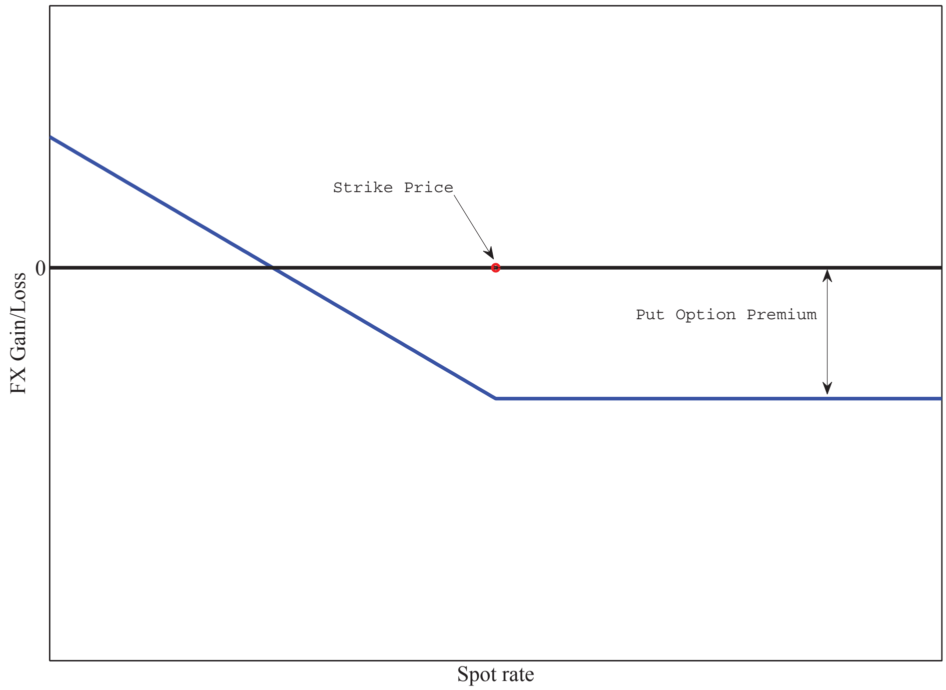

Figure 4: The payout of a put option.

contracts with strike of 98.500 cents with a bid price for the call of 1.810 cents and a bid price of 0.030 cents for the put. The February contracts will expire on Friday, February 15, 2013. On January 29, 2013 the spot price for the US dollar was 100.15 Canadian cents, which means that the call option strike is currently in-the-money (the strike price is lower than the current spot price) while the put option is out-of-the-money. With the option chain data, we can reverse engineer the option prices to give the market's assessment of price movements under the probability density that precludes arbitrage in the derivative market. To make use of this information, we need a market model of price movements to which we now turn – the Black-Scholes-Merton (BSM) equation.

We save technical details of the models for the annex, but we will briefly outline the important features. In 1973, Black and Scholes, and independently, Merton, showed [13], [14] that under the assumption that prices follow a continuous-time geometric random walk, it is possible to find an exact pricing solution to the European style call and put in closed form. A straightforward extension of this model by Garman and Kohlhagen [15] gives the solution to foreign exchange European style calls and puts. The BSM model is essentially

USX - Options on the US Dollar

Last update: January 29, 2013 at 5:30 p.m.

Calls

Month / Strike	Bid price	Ask price	Last price	Impl. vol.	Open int.	Vol.
13 FEB 96.500	3.760	3.820	3.820	--	0	0
13 FEB 97.000	3.260	3.330	3.330	--	0	0
13 FEB 97.500	2.760	2.830	2.830	--	0	0
13 FEB 98.000	2.280	2.350	2.350	--	0	0
13 FEB 98.500	1.810	1.880	1.880	--	0	0
13 FEB 99.000	1.380	1.450	1.450	--	20	0
13 FEB 99.500	1.000	1.060	1.060	--	5	0
13 FEB 100.000	0.680	0.740	0.740	--	10	0
13 FEB 100.500	0.440	0.500	0.500	--	0	0
13 FEB 101.000	0.270	0.330	0.330	--	0	0
13 FEB 101.500	0.150	0.210	0.210	--	0	0
13 FEB 102.000	0.090	0.140	0.140	--	0	0
13 FEB 102.500	0.050	0.100	0.100	--	0	0
13 MAR 95.000	5.320	5.380	5.380	--	0	0
13 MAR 95.500	4.830	4.890	4.890	--	0	0
13 MAR 96.000	4.340	4.400	4.400	--	0	0
13 MAR 96.500	3.850	3.920	3.920	--	0	0
13 MAR 97.000	3.380	3.440	3.440	--	10	0
13 MAR 97.500	2.910	2.980	2.980	--	0	0
13 MAR 98.000	2.470	2.530	2.530	--	0	0
13 MAR 98.500	2.050	2.100	2.100	--	40	0
13 MAR 99.000	1.670	1.730	1.730	--	10	0
13 MAR 99.500	1.330	1.390	1.390	--	10	0
13 MAR 100.000	1.040	1.100	1.100	--	10	0
13 MAR 100.500	0.810	0.850	0.850	--	0	0

Puts

Month / Strike	Bid price	Ask price	Last price	Impl. vol.	Open int.	Vol.
13 FEB 96.500	0	0.050	0.050	--	0	0
13 FEB 97.000	0	0.050	0.050	--	0	0
13 FEB 97.500	0	0.050	0.050	--	16	0
13 FEB 98.000	0	0.060	0.060	--	5	0
13 FEB 98.500	0.030	0.090	0.090	--	10	0
13 FEB 99.000	0.100	0.160	0.160	--	14	0
13 FEB 99.500	0.210	0.270	0.270	--	20	0
13 FEB 100.000	0.400	0.450	0.450	--	0	20
13 FEB 100.500	0.650	0.710	0.710	--	0	0
13 FEB 101.000	0.980	1.040	1.040	--	0	0
13 FEB 101.500	1.360	1.430	1.430	--	0	0
13 FEB 102.000	1.790	1.860	1.860	--	0	0
13 FEB 102.500	2.250	2.310	2.310	--	0	0
13 MAR 95.000	0	0.050	0.050	--	0	0
13 MAR 95.500	0	0.050	0.050	--	0	0
13 MAR 96.000	0	0.060	0.060	--	0	0
13 MAR 96.500	0.010	0.070	0.070	--	0	0
13 MAR 97.000	0.030	0.090	0.090	--	16	0
13 MAR 97.500	0.070	0.130	0.130	--	10	0
13 MAR 98.000	0.130	0.190	0.190	--	0	0
13 MAR 98.500	0.210	0.270	0.270	--	10	0
13 MAR 99.000	0.320	0.390	0.390	--	0	0
13 MAR 99.500	0.480	0.540	0.540	--	10	0
13 MAR 100.000	0.700	0.760	0.760	--	25	0
13 MAR 100.500	0.960	1.020	1.020	--	10	0

Figure 5: Part of the USX option chain for January 29, 2013 from the Canadian Derivatives Exchange.

the application of optimal control theory for hedging the short position[5] of an option. The authors show that in the absence of trading costs, a trading strategy exists that hedges the short position with probability one (almost surely). The optimal control problem becomes a replicating portfolio that reproduces the option's payoff at all times. If a financial institution can synthetically replicate the option with a trading strategy, the financial institution can sell an option while hedging the position through replication, thereby completely eliminate the risk of the option's payoff to its own balance sheet (under the market model assumption of a geometric random walk). The BSM equation tells us the amount of capital we require to start the replication process, which sets the option's premium. Any price different from

[5] To be short an option means that you are the counterparty of the option contract.

the replicating portfolio price will allow arbitrage as agents[6] can always go long or short the mispriced option and start the replication process to generate a riskless profit. Given that asset prices are often well approximated by a geometric random walk, the BSM model sheds tremendous insight into derivative pricing.

A key input required to price an option with the BSM model is the volatility of the asset. Roughly, the volatility measures how much the asset tends to fluctuate in price over time. For example, the S&P 500 (an index which approximately captures the returns of the US stock market) has an approximately normally distributed daily percent return centered on zero with a standard deviation of 1.3%. The daily standard deviation of 1.3% is the approximate long term daily volatility of the S&P 500. Notice that we do not directly observe the volatility, but we must infer it from the return data. Determining the volatility presents a problem for applying the BSM equation as different estimations of the volatility will yield different option prices.

Since the BSM equation assumes a geometric random walk, the model has a constant volatility parameter – the volatility parameter is independent of all market data or conditions. Rigorously applied, the BSM model demands that log-market returns follow a normal distribution, which even a cursory examination of asset returns allows us to reject.[7] We can see that something other than a pure geometric random walk describes market returns by taking market option price data as given and using the BSM equation in reverse to find the volatility consistent with the prices. This calculated volatility is called the *implied volatility*[8]. In a market described by a pure geometric random walk, the implied volatility will not depend on the option's strike price or other market inputs. Figure 6 shows the implied volatility as a function of strike prices using the call option data in the chain give figure 5 for February contracts. Notice that the implied volatility bends upward. Traders refer to this shape as the volatility smile. The presence of the smile in implied volatility indicates non-lognormal returns and thus the market corrects a naive application of the BSM equation with constant volatility. In essence, the option market recognizes that extreme market movements are more common than the BSM model implies and that deep out-of-the-money contracts are less liquid than near at-the-money counterparts. Instead of using the BSM equation to price options directly, we can use the equation to interpolate between observed option prices by applying a parameterization of the volatility smile. By using the BSM equation as an interpolation device, we can generate a continuum of option prices as a function of strike.

[6] Replication has a subtle implication. If it is always possible to perfectly replicate a derivative, then we would not see derivatives trading in the market – anyone could always synthetically create the derivative. We see derivatives traded precisely because replication is not perfect, meaning basis risk remains, and large complex financial institutions have a comparative advantage when implementing an approximate replication strategy.

[7] Again, in a perfect BSM world the market would contain no options as agents could always perfectly replicate the payoff of any option synthetically. That the BSM model with constant volatility fails to describe the real world should not surprise us – after all, we find options traded in the market.

[8] In option markets, by convention, traders often quote implied volatility instead of prices.

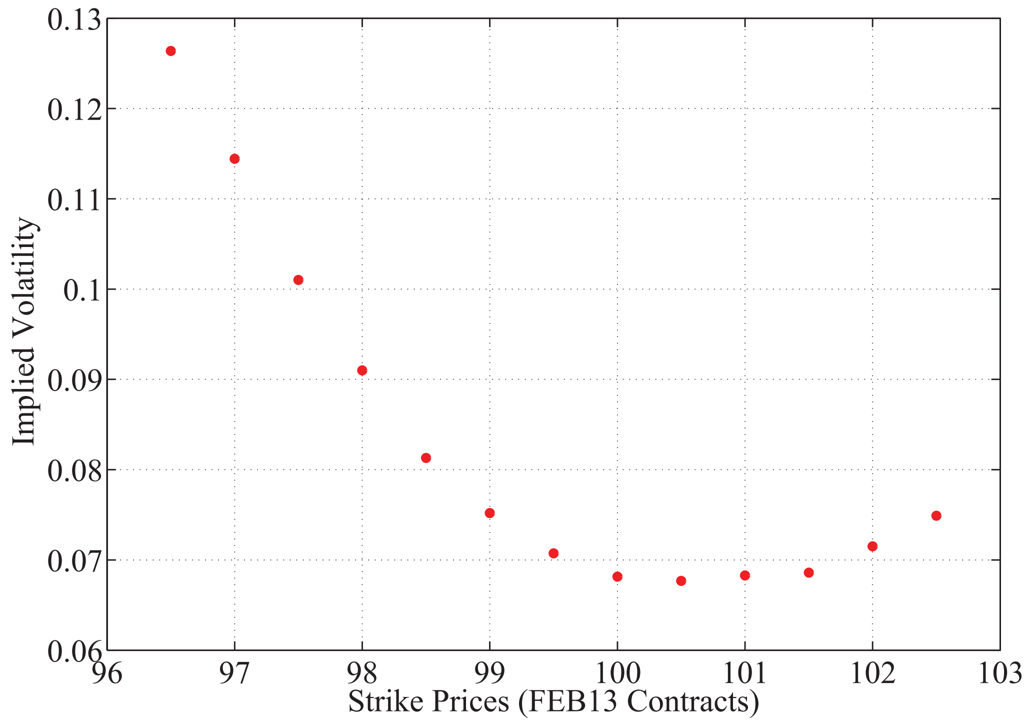

Figure 6: The implied volatility for February 2013 contracts on January 29, 2013.

We can re-express option pricing problems as an expectation over a probability distribution, called the risk neutral distribution function, and in simple models, like the BSM model, we can exactly solve for the risk neutral distribution function. If we have option price data, generically, we can obtain the risk neutral distribution function through relationships that involve the derivative of option price function with respect to strike (although such a procedure does not immediately tell us how to replicate the option's payoff). By using the volatility smile inferred from option prices listed on the exchange, we can generate a continuous function of option prices with respect to strike, which allows us to differentiate the curve and elicit the risk neutral distribution function. In the foreign exchange context, we can use the 95% confidence region of the implied risk neutral distribution function to give us insight into the trading range of the US dollar.

The Implied Volatility with Black-Scholes Interpolation method for determining the risk neutral distribution function for the Canada-US dollar exchange rate proceeds as follow:

- Using the Garman-Kohlhagen [15] extension of the BSM equation, derive the implied volatility smile from the call option data for each maturity month using the center of the bid-ask spread as the option price.

- Use time averaged foreign and domestic LIBOR for each maturity date in the option chain.
- Since foreign exchange volatility smiles tend to exhibit symmetry, parameterize the volatility smile using a second degree polynomial fit as a function of strike price.
- Using the volatility smile polynomial, construct an option pricing function with respect to strike prices.
- Construct the risk neutral distribution function using the numerically differentiated option pricing function.
- Repeat the procedure for the put option data. (We do not assume that the Exchange data satisfies put-call parity – we are using stale quotes with large bid/ask spreads.)
- Use the average of the implied distribution functions from the calls and the puts as the risk neutral distribution function for the market.

The bid-ask spread in the exchange listed prices are wide (up to 100%, depending on moneyness[9]) and thus extracting higher order features in the smile by fitting a higher degree polynomial runs the risk of fitting noise. As a result, we restrict ourselves to modelling with a quadratic smile. We can see in figure 7, that a quadratic fit describes the data well. Once we have a parameterization of the volatility smile, we can calculate option prices as a function of strike. We should stress that in reality, a quadratic fit is an inadequate representation of the market – we could not use this scheme to offer competitive derivative prices to the market. In option markets, market participants heavily trade the skew and the symmetry of the implied volatility curve – the smile is never perfectly symmetric as implied by a quadratic fit. To elicit these features, we need better data than the Canadian Derivatives Exchange listings.[10] We will return to obtaining better sources of data in the conclusions.

While the empirical approach yields the risk neutral distribution function at each maturity month, it does so only on the strike price interval given by the data. In contract months with more than six months left to maturity, the strike price interval will in general not cover the distribution function beyond the 80% quantile or less than the 20% quantile.[11] To estimate the 95% and 5% quantile levels in cases where the strike price interval terminates prematurely, we extrapolate by fitting the Gumbel extreme value distribution to the tails

[9] Moneyness measures the relative position of the asset with respect to the strike price, often quoted as the log-ratio of the strike to the forward price.

[10] Traders make use of composite instruments called risk Reversals and butterflies to trade the asymmetric and symmetric parts of the smile respectively. The Canadian Derivatives Exchange does not list the prices of these assets.

[11] In foreign exchange option trading, $\Delta = 10$ and $\Delta = 25$, which roughly estimate the probability (in percentage) of ending in-the-money, are the most liquid positions. Contracts with long maturity dates (~ 1 year), do not have much data beyond these positions on the Canadian Derivatives Exchange.

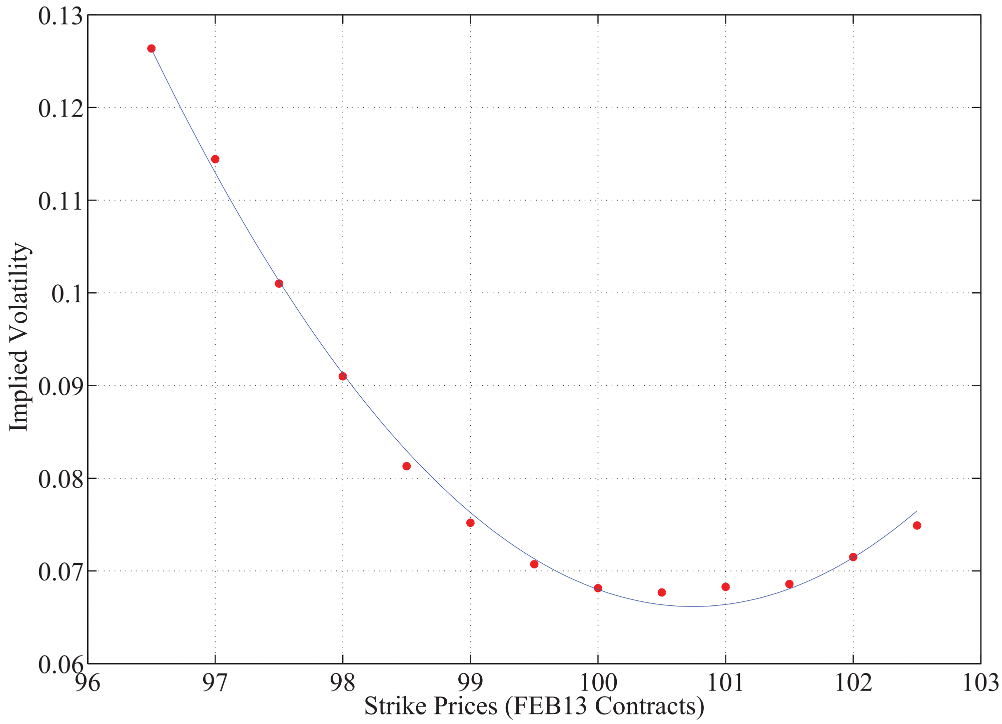

Figure 7: The implied volatility for February 2013 contracts on January 29, 2013 with quadratic fit.

of the implied risk neutral distribution (5% of the distribution in each of the left and right tail). The Gumbel distribution is a well-motivated extreme value distribution which forms the domain of attraction for the tails of a wide class of distribution functions. Since we confine ourselves to a small portion of the tail derived from the option pricing function, we use a simple least squares fit to the Gumbel distribution by using a Gumbel transformation on the tails of the implied distribution.

In figure 8, we see the 95% trading range of the US dollar across the option chain given on January 29, 2013. We summarized the results in table 3.

2.2 A two dimensional lognormal mixture model

The implied volatility approach offers a simple method for ascertaining the trading range of the US dollar under the risk neutral distribution. Unfortunately, the method suffers from the need to extrapolate the distribution function when market data does not contain option contracts with strike prices within the 95% confidence region. While the Gumbel extreme value distribution extension gives us a controlled extrapolation technique, we should also

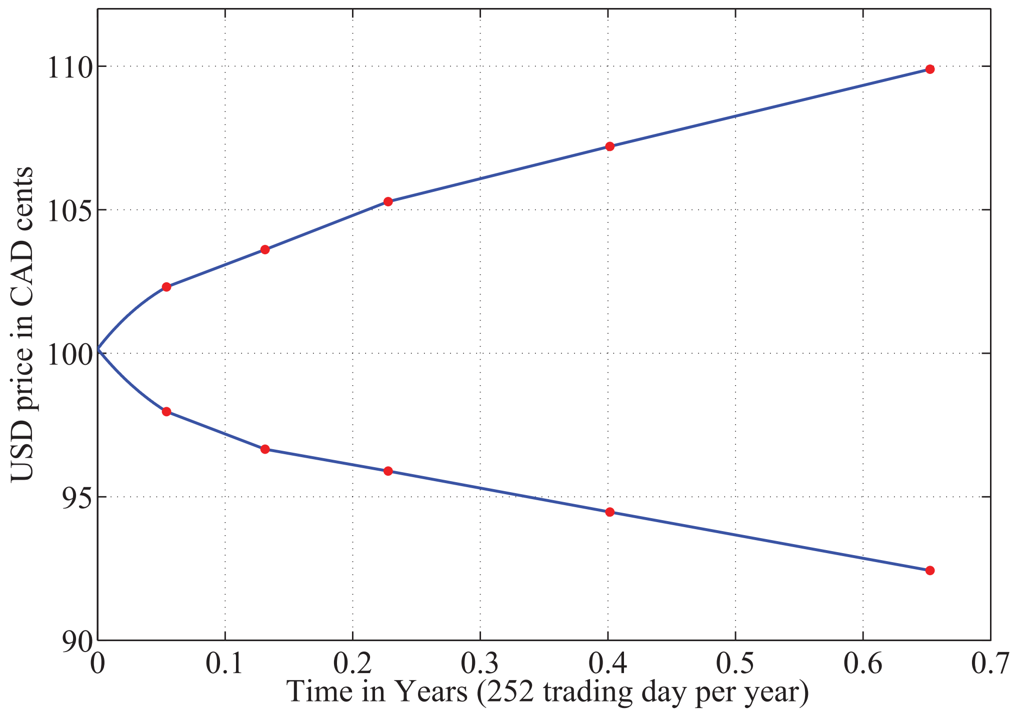

Figure 8: The 95% trading contour of the USD generated by the implied volatility model for January 29, 2013.

look for methods that explicitly contain the tail of the distribution within the estimation procedure. A lognormal mixture model offers us a method of approximating the entire distribution using five parameters for each contract month.

The volatility smile shows us that the BSM model, with its normal distribution for log-returns, does not fully capture price dynamics. Instead of giving up entirely on the log-normal assumption, we describe the risk neutral density as a weighted sum of lognormal distributions – a mixture model – under which the BSM model attains as the zeroth order expansion. To keep the number of parameters to a minimum (five parameters), we implement a lognormal mixture model with two weighted lognormal distributions in modelling

Table 3: USD trading range 95% confidence region (Implied Vol Model), January 29, 2013

95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Upper	102.3	103.6	105.3	107.2	109.9
Lower	98.0	96.7	95.9	94.5	92.4

Table 4: USD trading range 95% confidence region (Mixture Model), January 29, 2013

95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Upper	102.5	103.9	105.4	107.9	110.5
Lower	97.8	96.7	95.6	93.8	91.9

the risk neutral distribution. Since we depart from the simple lognormal distribution, the model has the ability to capture smile dynamics.

The implementation of the model requires us to fit the five parameters of the model to each contract month in the option chain – each contract month has its own set of parameters. Since the objective function in the optimization is in general not convex, we rely on non-derivative based search methods. The estimation method proceeds as follows:

- Create the option pricing function as an expectation over the mixture model for both calls and puts.
- Use time averaged foreign and domestic LIBOR rates for each contract month in the option chain and include the implied LIBOR forward price in the data set.
- Determine the best fit for the five parameters for each contract month by minimizing the squared differences of the option chain price at the midpoint of the bid-ask spread, weighted by the inverse of the absolute value of the bid-ask spread difference.
- Use the resulting mixture model as the risk neutral distribution.

We weight each squared difference by the inverse of the bid-ask difference to underweight options with less well known prices. We use two non-derivative based search methods to locate the minimum: The Nelder-Mead simplex method and Simulated Annealling. We take the best fit from the results of both methods.

In figure 9 we see the trading range of the US dollar as inferred from the mixture model for the January 29, 2013 option price chain. We list the results for each contract month in table 4. Notice the similarity to the results in table 3. We see that the estimate of the 95% confidence limit differ by less than 1% between the two models.

2.3 The Heston Model

Thus far, we have used models that attempt to capture smile dynamics in an ad hoc fashion. In the implied volatility model case, we use a volatility polynomial approximation, and in the mixture model case we fit two weighted lognormal distributions. Both models

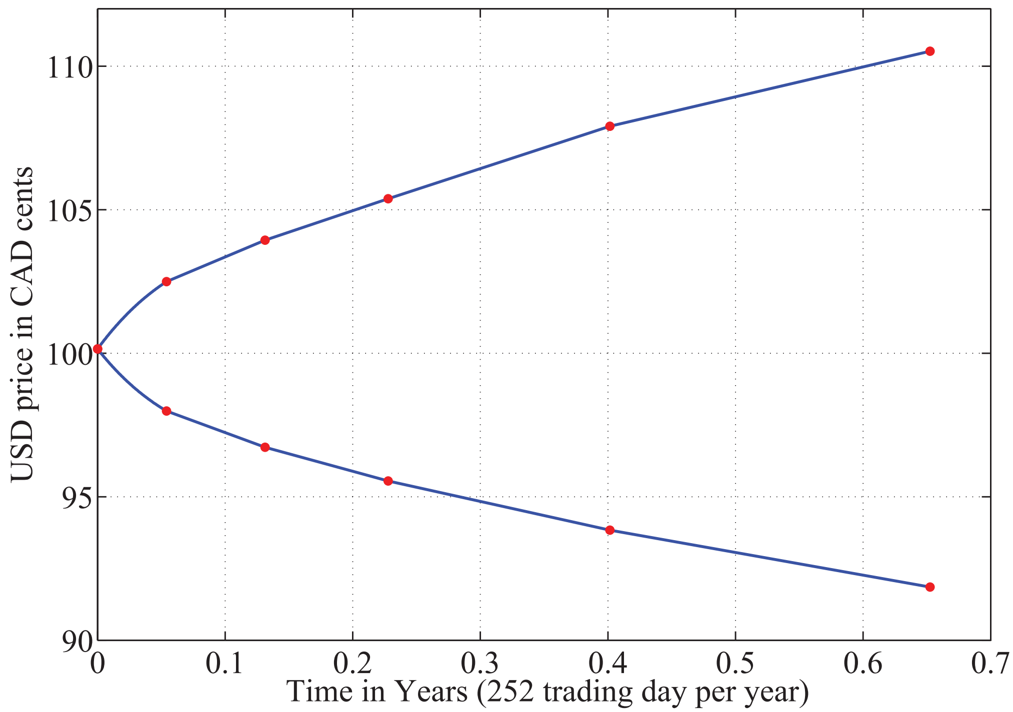

Figure 9: The 95% trading contour of the USD generated by the mixture model.

describe the data only at each contract month, thus requiring interpolation of the 95% confidence limit as a function of time. We have not employed a full market model with these descriptions.

To overcome these shortcomings, we implement the Heston stochastic volatility model [19] which promotes the volatility parameter of the BSM model to a stochastic variable. Stochastic volatility in the Heston model follows a square root diffusion process which has the ability to capture the smile dynamics observed in the market and the model offers a closed form solution for call and put options.

The Heston Model has five parameters which we estimate across all contract months by using a global fit based non-derivative search methods (Nelder-Mead and Simulated Annealling). We minimize the squared difference between the option data and the calculated option price weighted by the inverse of the bid-ask spread.

The Heston modelling technique proceeds as follows:

- Create option pricing function from the closed form solution of the Heston model.
- Use time averaged foreign and domestic LIBOR for each contract month in the op-

tion chain as an input to the Heston model for option pricing.

- Determine the best fit for the five parameters by minimizing the squared differences of the option chain prices at the midpoint of the bid-ask spread, weighted by the inverse of the absolute value of the bid-ask spread difference.
- Use the resulting risk neutral density function to give the 95% confidence limits for the trading range of the US dollar.

The Heston model allows for smooth interpolation since we have the pricing solution across all time as opposed to evaluations only in contract months. We display the trading range of the US dollar as inferred from the Heston model for the option chain in figure 10 and we list the results for each contract month in table 5. Again, notice the similarity to the results in tables 3 and 4. We see that the estimate of the 95% confidence limit differ by less than 1% between the all models.

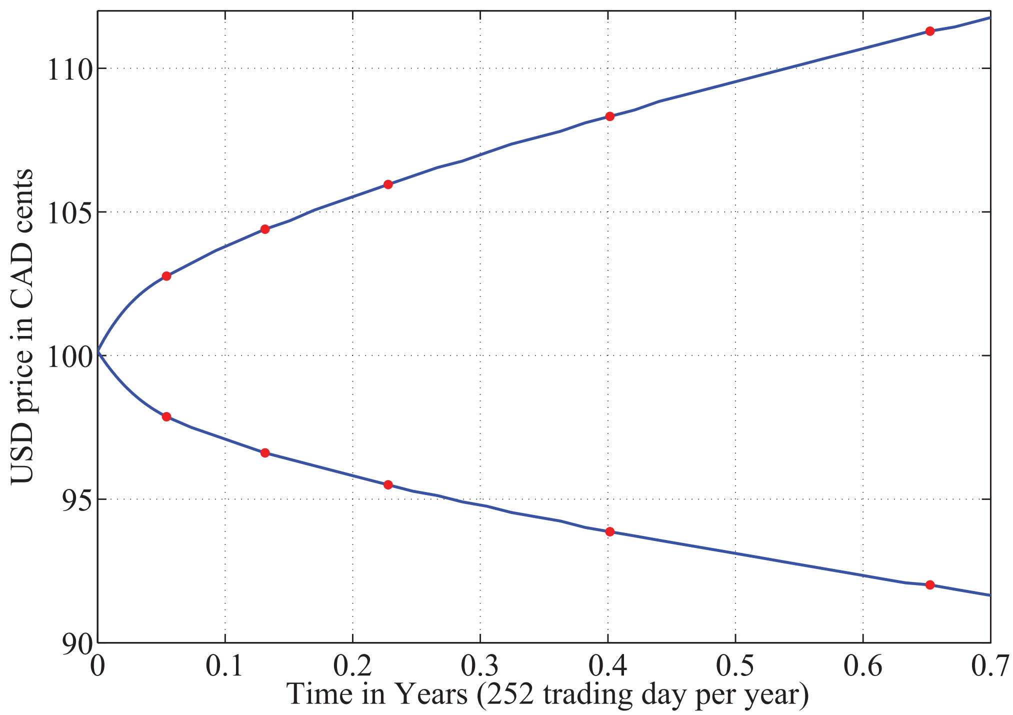

Figure 10: The 95% trading contour of the USD generated by the Heston model.

2.4 Model Average

Each of the three market models represents a method to elicit the risk neutral distribution implied by derivative prices. In principle, the market-consistent risk neutral distribution

Table 5: USD trading range 95% confidence region (Heston Model), January 29, 2013

95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Upper	102.8	104.4	106.0	108.3	111.3
Lower	97.9	96.6	95.5	93.9	92.0

helps market specialists price exotic instruments which do not have simple closed form solutions. For our needs, we only require the 95% confidence limit which we set by averaging the three models. While this averaging technique lacks the accuracy for pricing exotic instruments – which we cannot achieve by using the exchange data alone – it does give us insight into the trading range of the US dollar under the implied risk neutral distribution.

In table 6, we summarize the results of each method. Notice that the limits of the trading range within the 95% confidence region differ by less than 1% across the three models. We display the average 95% trading range envelope in figure 11. Using the techniques in this paper, ADM(Fin CS) can get daily updated US dollar trading ranges from the daily close market prices list on the Canadian Derivatives Exchange.

Table 6: USD trading range 95% confidence region, January 29, 2013

	Contract Months				
95% Trading Range	FEB 13	MAR 13	APR 13	JUN 13	SEP 13
Implied volatility model					
Upper	102.3	103.6	105.3	107.2	109.9
Lower	98.0	96.7	95.9	94.5	92.4
Lognormal mixture model					
Upper	102.5	103.9	105.4	107.9	110.5
Lower	97.8	96.7	95.6	93.8	91.9
Heston stochastic volatility model					
Upper	102.8	104.4	106.0	108.3	111.3
Lower	97.9	96.6	95.5	93.9	92.0
Model Average					
Upper	102.5	104.0	105.5	107.8	110.6
Lower	97.9	96.7	95.7	94.1	92.1

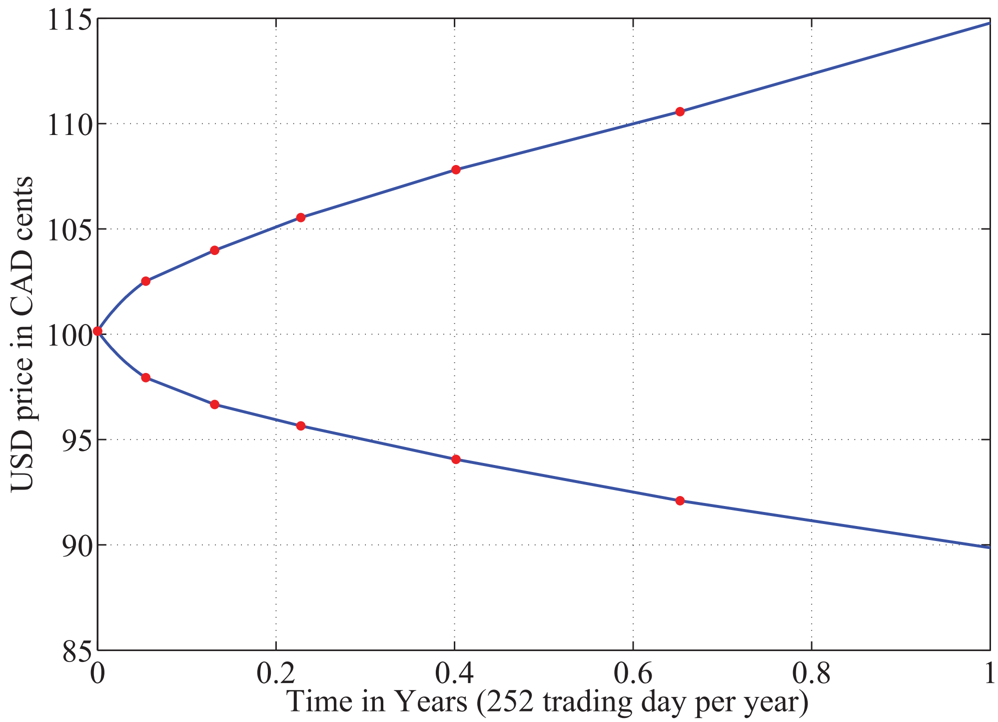

Figure 11: The 95% trading contour of the USD generated by averaging the the implied volatility model, the mixture model, and the Heston model using data from January 29, 2013.

3 Discussion

Using the average of three different financial quantitative models, we offer the 95% trading envelope of the US dollar on a one year horizon based on call and put option prices traded on the Canadian Derivative Exchange. ADM(Fin CS) can use the methods of this paper to receive daily updates on the exchange rate trading range with the additional possibility of including the results in the ADM(Fin CS) quarterly foreign exchange update document.

The models we use in this paper extract the risk neutral distribution function from exchange listed option price data. While the risk neutral distribution ensures arbitrage free derivative prices, the real world follows its own distribution. In asset pricing, the real world requires compensation for holding risk. From the foreign exchange perspective, the expectations model (see [22] for details) tells us that a high interest rate difference between two currencies signals an expected devaluation of the currency with the higher interest rate. A carry trade strategy, under which a market participant tries to earn the interest rate difference between two currencies, exposes the agent to the risk of devaluation. In the late 1990s we saw this effect in East Asia as interest rates in East Asian countries were much higher than the yield on US treasuries. Under the expectations model interpretation, the market had priced the probability of a dramatic currency devaluation and the extra return for holding interest bearing assets in the East Asian market came from holding risk. In 1997, the Asian Financial Crisis demonstrated that concerns for currency devaluation had been well placed. In general, it is not possible to earn excess profits on a risk adjusted basis using a carry trade strategy – the apparent extra yield comes from holding the risk associated with a currency devaluation. On the other hand, in circumstances where the interest rate in the foreign country is higher than usual relative to the domestic interest rate, there is evidence for the possibility of earning an excess return (again, see [22] for details). This pricing puzzle [21] might reflect a risk adjustment for compensating rare but dramatic devaluations – a phenomenon referred to as Peso problems[12]. For countries with a long history of stable currencies, like Canada and the United States, the risk neutral distribution over a one year period offers insight into the actual trading range of the currencies, Peso problems aside.

Estimating the risk neutral distribution beyond one year introduces new problems in the form of interest rate modelling. In our analysis, we have modelled the exchange rate using average interest rates as inputs, taken from LIBOR posted with [8]. Beyond one year, interest rate modelling becomes important. The most liquid foreign exchange instruments have maturities of less than one year and the effect of interest rate changes do not affect option prices significantly with such short durations. At maturities longer than one year, interest rates changes have important pricing implications and modelling becomes more difficult with long dated illiquid contracts. We do not model interest rates in this paper.

We must emphasize that our estimate of the risk neutral distribution produces a crude picture. We do not have access to over-the-counter market data. Our data is contaminated by stale quotes on option prices and deposit rates. Furthermore, we have no access to the market volatility matrix, information on risk reversal and butterflies, nor do we have access to institutional specific deposit rates. Finally, the Exchange has the US dollar as its only foreign currency for derivative trading. To overcome these shortcomings, we recommend that ADM(Fin CS) staff obtain access to more detailed financial data, which will give DND a much more accurate picture of the global foreign exchange market, which will include exchange rate data across all currencies. With over-the-counter data, we have access to the implied risk neutral distribution based on the most recent market data yielding a much more accurate description of the trading ranges than the results of this study. More detailed fi-

[12] In the early 1970s, Mexican interest rates were higher than US interest rates even though Mexico had pegged the Peso to the US dollar more than a decade earlier. Thus, it was possible for Americans to earn substantially higher interest rates relative to the domestic market by holding Mexican interest bearing assets and converting back to US dollars at the pegged rate. From a risk adjustment perspective, the market had priced the possibility of a currency collapse – small probabilities of substantial devaluations can in principle generate large cross currency interest rate differences. Eventually, the Peso collapsed and wiped out the supposed excess return that American investors could have earned under a carry trade strategy. Such pricing anomalies are called Peso problems. See [16] for more details.

nancial data would also allow us to run counterfactual hedging scenarios, mark-to-market, using actual over-the-counter market prices matched to DND's foreign exchange obligations.

With the models presented in this paper and data from the Canadian Derivatives Exchange, ADM(Fin CS) staff has insight into the trading range of the US dollar over a one year horizon. By observing changes in the trading range under the risk neutral distribution, ADM(Fin CS) staff can better understand the evolution of market risks and properly weigh the benefits of using a foreign exchange hedge. As a first application of these results, ADM(Fin CS) staff can use the derived trading range in DND's foreign exchange reporting documents and internal monitoring services. If senior decision makers move toward a hedging strategy, ADM(Fin CS) staff can use the results from this paper to help ascertain when hedging foreign exchange risk becomes appropriate, given the market conditions. ADM(Fin CS) staff could use changes in the size of the derived trading range to signal a time to enter a zero cost structure or other structured product. Setting internal bounds on the US dollar trading range in conjunction with foreign currency payment dates can help ADM(Fin CS) staff recognize when the market senses more risk than usual and understand which type of hedging programs would best suit the DND's needs.

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Annex A: Model details

In this paper, we use market price data of call and put options traded on the Canadian Derivative exchange to calculate the implied risk neutral distribution function. As we will see in detail throughout this annex, we can write the price of a call option as an expectation over a risk neutral probability measure, namely

$$c(K,T) = e^{-r_d T} \int_0^{\infty} (y-K)^{+} f(y,T;S_0)\, dy, \tag{A.1}$$

where r_d denotes the domestic interest rate, K is the strike price of the call option, and T gives the maturity date,

Differentiating A.1 with respect to the strike price, K, we find the distribution function relationship,

$$\int_0^{x} f(y,T;S_0) dy = F(x) = 1 + e^{r_d T} \frac{\partial c(x,T)}{\partial x}. \tag{A.2}$$

Similarly, for put options we have,

$$p(K,T) = e^{-r_d T} \int_0^{\infty} (K-y)^{+} f(y,T;S_0)\, dy, \tag{A.3}$$

yielding,

$$\int_0^{x} f(y,T;S_0) dy = F(x) = e^{r_d T} \frac{\partial p(x,T)}{\partial x}. \tag{A.4}$$

Since we require differentiation of the option price curve with respect to strike prices, we need a market model calibrated to market data that will interpolate over a continuum of strike prices.

A.1 The Black-Scholes-Merton model

Before examining the modelling specifics of the paper, let us review pricing a European call option on a non-dividend paying asset. The Black-Scholes-Merton (BSM) model assumes that the asset price follows a geometric random walk,

$$dS(t) = \alpha S(t)\, dt + \sigma S(t)\, dW(t), \tag{A.5}$$

where $dW(t)$ denotes the differential element for Brownian motion, α represents the asset specific growth rate, and σ gives the asset's volatility. We assume that the parameters of the model are constant and that the prevailing continuous compounding riskless interest rate is given by r. In the geometric random walk model, the log-returns are normally distributed. While real assets do not exactly follow a geometric random walk, empirically, the model represents an acceptable approximation to the actual return distribution for economic discussion.

We wish to hedge the short position of a single European call option contract. To hedge the derivative, we make use of the two assets at our disposal – a money market account and the underlying asset. We begin by splitting a portfolio worth $X(t)$, at time t, between the asset and the money market account. At time t, we have $\Delta(t)$ shares of the asset with the rest of the portfolio deposited in a money market account that yields the riskless interest rate. Our portfolio, $X(t)$, evolves over a small time increment as,

$$\begin{aligned} dX(t) &= \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))\,dt \\ &= rX(t)dt + \Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t), \end{aligned} \tag{A.6}$$

where we have used A.5 in the last line. Notice that the discounted asset price, $f(S,t) = e^{-rt}S(t)$ evolves according to,

$$\begin{aligned} d(e^{-rt}S(t)) &= df(S,t) \\ &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}dS(t)dS(t) \\ &= (\alpha - r)e^{-rt}S(t)\,dt + \sigma e^{-rt}S(t)\,dW(t). \end{aligned} \tag{A.7}$$

Applying the same technique to the discounted portfolio, $e^{-rt}X(t)$, we find

$$d(e^{-rt}X(t)) = \Delta(t)d(e^{-rt}S(t)). \tag{A.8}$$

As expected, we see that the discounted asset grows at the rate $(\alpha - r)$, and the changes in the discounted portfolio come only from changes in the discounted stock price.

The European call option on the asset has a payoff function at time T that reads,

$$c(S(T),T) = [S(T) - K]^+, \tag{A.9}$$

which means that $c(S(T),T) = S(T) - K$ so long as $S(T) > K$, otherwise $c(S(T),T) = 0$. To hedge our short position in this contract, we need our portfolio process, $X(t)$, to match the valuation of the call option for all time. That is, we need $X(t)$ to become the replicating portfolio. First, recognize that the call option obeys the relationship,

$$\begin{aligned} dc(S(t),t) &= \frac{\partial c}{\partial t}dt + \frac{\partial c}{\partial S}dS + \frac{1}{2}\frac{\partial^2 c}{\partial S^2}(dS)^2 \\ &= \left[\frac{\partial c}{\partial t} + \alpha S(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial S^2}\right]dt \\ &\quad + \sigma S(t)\frac{\partial c}{\partial S}dW(t), \end{aligned} \tag{A.10}$$

which yields the discounted call price differential,

$$\begin{aligned} d(e^{-rt}c(S(t),t)) &= e^{-rt}\left[-rc + \frac{\partial c}{\partial t} + \alpha S(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial S^2}\right]dt \\ &\quad + e^{-rt}\sigma S(t)\frac{\partial c}{\partial S}dW(t). \end{aligned} \tag{A.11}$$

To turn $X(t)$ into the replicating portfolio, we require the matching condition,

$$d(e^{-rt}X(t)) = d(e^{-rt}c(S(t),t)) \text{ for all } t \in [0,T) \tag{A.12}$$

with $X(0) = c(S(0),0)$. Equating the two differentials, we find

$$\begin{aligned} &\Delta(t)(\alpha - r)S(t)\,dt + \Delta(t)\sigma S(t)\,dW(t) \\ &= \left[-rc + \frac{\partial c}{\partial t} + \alpha S(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial S^2}\right] dt \\ &+\sigma S(t)\frac{\partial c}{\partial S}\,dW(t), \end{aligned} \tag{A.13}$$

which, the $dW(t)$ and dt terms respectively yield,

$$\Delta(t) = \frac{\partial c}{\partial S} \quad \text{for all } t \in [0,T), \tag{A.14}$$

and

$$rc = \frac{\partial c}{\partial t} + rS(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial S^2} \quad \text{for all } t \in [0,T). \tag{A.15}$$

The last equation, A.15, arising from the dt term, gives us a continuous partial differential equation that tells us how the price of the call option evolves in time while A.14 gives us the hedging rule that determines how much of our wealth to split between the asset and the money market account over time. With these rules, we have our replicating portfolio and hence our hedge. Putting everything together, we arrive at the Black-Scholes-Merton equation for the price of a call option on a non-dividend paying asset,

$$\frac{\partial c}{\partial t} + rS(t)\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2(t)\frac{\partial^2 c}{\partial S^2} = rc \quad \text{for all } t \in [0,T),\ S \geq 0, \tag{A.16}$$

with the terminal condition

$$c(S,T) = [S-K]^+. \tag{A.17}$$

Notice that A.16 does not contain any reference to the drift parameter α – the drift parameter cancelled in the differential matching condition. The replicating portfolio does not depend on subjective opinions of the growth term, nor any estimate of it. That is, the hedge is priced under the risk neutral measure, which ensures the absence of arbitrage. The BSM model provides arbitrage free prices with a unique risk neutral measure, and the price of the option is the required capital to start the hedge. If a market model has a risk neutral measure, then the model will not admit arbitrage. Furthermore, if in a market model all derivatives can be hedged, then the model is said to be complete and the risk neutral measure is unique. Incomplete models that do not allow arbitrage do not have a unique risk neutral measure and as a result practitioners must calibrate them with respect to market prices. We will discuss incomplete models further when we develop the Heston model later in this annex.

We see that the price of the European call option depends only on the riskless interest rate, r, the volatility of the asset, σ, the initial stock price, $S(t_0)$, the strike price, K, and the time to maturity, T. The volatility is the only parameter not directly observable in the market. To find the price of the call option, we must solve (A.16). The rich mathematical structure of stochastic calculus contains a remarkable theorem – the Feynman-Kac Lemma – that connects partial differential equations to stochastic differential equations which we can use to solve the pricing problem. The lemma, turns the pricing problem into an expectation of the call option's payoff over the risk neutral measure,

$$c(S,t) = \mathbb{E}\left[e^{-r(T-t)}(S(T)-K)^+ \middle| \mathscr{F}(t) \right]. \tag{A.18}$$

The solution is,

$$c(S,t) = S\mathscr{N}(d_+) - Ke^{-r(T-t)}\mathscr{N}(d_-), \quad 0 \le t < T,\ S > 0, \tag{A.19}$$

where $S = S(t)$,

$$d_\pm = \frac{1}{\sigma\sqrt{(T-t)}}\left[\log\frac{S}{K} + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right], \tag{A.20}$$

and where $\mathscr{N}$ denotes the cumulative standard normal.

While pricing the European put option also follows with the same analysis applied to the payoff $(K-S)^+$, we can make use of put-call parity which links the prices of call and put options to the forward contract implied by holding a long call and a short put.

We can adapt the BSM model to a foreign exchange setting. Under the risk neutral measure, the exchange rate follows,

$$dS(t) = (r_d - r_f)S(t)\,dt + \sigma S(t)\,dW(t) \tag{A.21}$$

where r_d and r_f denote the domestic and foreign interest rate, and σ labels the volatility. In the presence of the yield curve, in this paper, we make the replacements,

$$\begin{aligned} r_d(T) &\to \frac{1}{T}\int_0^T r_d(t)dt \\ r_f(T) &\to \frac{1}{T}\int_0^T r_f(t)dt, \end{aligned} \tag{A.22}$$

$$\tag{A.23}$$

that is, we use the average interest rate over the time period. Generally, the price of any derivative attached to this foreign exchange market model satisfies the solution of the BSM equation,

$$\begin{aligned} \frac{\partial}{\partial t}\nu(t,x) - r_d\nu + (r_d - r_f)x\frac{\partial}{\partial x}\nu(t,x) + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2}\nu(t,x) &= 0 \\ \nu(T,x) &= F \end{aligned} \tag{A.24}$$

where F is the terminal payoff. In the case of call and put options,

$$F(T) = [\phi(S(T) - K)]^+ \tag{A.25}$$

where $\phi = \pm 1$ for calls and puts respectively. Using the Feynman-Kac lemma, we can write the solution as an expectation over the risk neutral measure, namely,

$$v(t, x, K, T, \sigma, r_d, r_f, \phi) = \exp(-r_d(T-t))\mathbb{E}([\phi(S-K)]^+) \tag{A.26}$$

where we have explicitly included the full parameter dependence in the option price. Solving A.26, we find,

$$v(t, x, K, T, \sigma, r_d, r_f, \phi) = \phi e^{r_d \tau}\left[fN(\phi d_+) - KN(\phi d_-)\right] \tag{A.27}$$

where

$$\tau = (T - t) \tag{A.28}$$

$$f = xe^{(r_d - r_f)\tau} \tag{A.29}$$

$$d_\pm = \frac{\log \frac{f}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}} \tag{A.30}$$

$$\mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2}\, dt. \tag{A.31}$$

The implied volatility model requires us to invert A.27 to solve for the volatility parameter σ. Since,

$$\frac{\partial v}{\partial \sigma} > 0 \tag{A.32}$$

we have a unique solution: $\sigma \to v(t, x, K, T, \sigma, r_d, r_f, \phi)$ – which we can find numerically through the Newton-Raphson method.

A.2 The mixture model

The lognormal mixture model (see for example [18]) assumes that the risk neutral density function takes the form,

$$f(x, T; S_0) = \omega_1 L_1(x; \alpha_1, \beta_1) + \omega_2 L_2(x; \alpha_2, \beta_2) \tag{A.33}$$

where the weights satisfy,

$$\omega_1 + \omega_2 = 1; \quad \omega_{1,2} > 0, \tag{A.34}$$

and where each lognormal density term in A.33 is given by,

$$L_i(x; \alpha_i, \beta_i) = \frac{1}{x\sqrt{2\pi\beta_i^2}} \exp\left(-\frac{1}{2}\beta_i^2(\ln x - \alpha_i)^2\right). \tag{A.35}$$

Given that $f(x,T;S_0)$ represents the risk neutral density, by the Feynman-Kac lemma we have,

$$v(K,T,\phi) = e^{-r_d T} \int_0^{\infty} [\omega L_1(x;\alpha_1,\beta_1) + (1-\omega) L_2(x;\alpha_2,\beta_2)] [\phi(x-K)]^+ \, dx \quad \text{(A.36)}$$

where again $\phi = \pm 1$ for call and put options respectively. From the market data, we acquire a vector of call option prices $(c_1, c_2, \ldots, c_n)$ and a vector of put option prices $(p_1, p_2, \ldots, p_m)$, all with the same maturity date, which use to the find the model parameters $\theta = \{\alpha_1,\, \alpha_2,\, \beta_1 > 0,\, \beta_2 > 0,\, 0 \leq \omega \leq 1\}$ through

$$\begin{aligned} \min_{\theta} \; & \sum_i^n g_i \left(c_i - c(K_i,T)\right)^2 + \sum_i^m h_i \left(p_i - c(K_i,T)\right)^2 \\ & + \left(\omega e^{\alpha_1 + \beta_1^2/2} + (1-\omega) e^{\alpha_2 + \beta_2^2/2} - e^{(r_d - r_f)T} S_0\right)^2. \end{aligned} \quad \text{(A.37)}$$

where g_i and h_i denote the inverse of the square of the bid-ask spread of each option. The last term in A.37 represents the additional objective arising from the price of a forward contract. We determine the parameters of the model for each contract month.

A.3 The Heston model

The Heston model [19] is a stochastic volatility model represented by the equations,

$$dS(t) = \mu S(t)\, dt + \sqrt{v} S(t)\, dW_1 \quad \text{(A.38)}$$

$$dv = \kappa(\theta - v) dt + \sigma \sqrt{v} dW_2 \quad \text{(A.39)}$$

$$dW_1\, dW_2 = \rho\, dt \quad \text{(A.40)}$$

We can use the Heston model in the foreign exchange setting. First, consider a general contingent claim with price $U(t,v,S)$ at time t, with terminal price $g(T,S) = U(T,v,S)$. The replicating portfolio takes the form,

$$dX = \Delta dS + \Gamma dZ + r_d (X - \Gamma Z)\, dt - (r_d - r_f) \Delta\, dt \quad \text{(A.41)}$$

In A.41, S is the underlying asset and Z is some other derivative or asset. The replicating portfolio requires the second asset in this case because we have a second source of uncertainty – the Brownian motion driving the volatility process. In general, to hedge a derivative, we must have as many assets as we have sources of uncertainty. Using Ito's Lemma with A.41 and rearranging, we find,

$$\begin{aligned} & \Delta(\mu - (r_d - r_f))\, dt + \Delta \sqrt{v} S\, dW_1 + r_d U\, dt - r_d \Gamma Z\, dt + \tilde{Z}\, dt \\ & + \Gamma \sigma \sqrt{v} \frac{\partial Z}{\partial v} dW_2 + \Gamma \sqrt{v} S \frac{\partial Z}{\partial S} dW_1 = \tilde{U}\, dt + \sqrt{v} \sigma \frac{\partial U}{\partial v} dW_2 + \sqrt{v} S \frac{\partial U}{\partial S} dW_1 \end{aligned} \quad \text{(A.42)}$$

where,

$$\tilde{V} = \frac{\partial V}{\partial t} + \kappa(-v)\frac{\partial V}{\partial v} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \frac{1}{2}^2\frac{\partial^2 V}{\partial S^2} + \rho\sigma v\frac{\partial^2 V}{\partial v\partial S}. \tag{A.43}$$

Matching the dW_1 and dW_2 coefficients, we find the hedging rules,

$$\Gamma = \frac{\partial U/\partial v}{\partial Z/\partial v} \tag{A.44}$$

$$\Delta = \frac{\partial U}{\partial S} - \frac{\partial U/\partial v}{\partial Z/\partial v}\frac{\partial Z}{\partial S}, \tag{A.45}$$

and matching the dt terms, we find,

$$\frac{1}{\partial Z/\partial v}\left(\frac{\partial Z}{\partial t} + \kappa(\theta - v)\frac{\partial Z}{\partial v} + (r_d - r_f)S\frac{\partial Z}{\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 Z}{\partial v^2} + \frac{1}{2}^2\frac{\partial^2 Z}{\partial S^2} + \rho\sigma v\frac{\partial^2 Z}{\partial v\partial S} - r_d Z\right)$$
$$= \frac{1}{\partial U/\partial v}\left(\frac{\partial U}{\partial t} + \kappa(\theta - v)\frac{\partial U}{\partial v} + (r_d - r_f)S\frac{\partial U}{\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}^2\frac{\partial^2 U}{\partial S^2}\right.$$
$$\left. + \rho\sigma v\frac{\partial^2 U}{\partial v\partial S} - r_d U\right). \tag{A.46}$$

Since A.46 represents the equality of two functions with arguments (t, v, S), we must have that both sides equals some function, $\Lambda(t, v, S)$. We will assume that $\Lambda(t, v, S) = \lambda v$, which gives us the Heston partial differential equation for the value of contingent claim,

$$\frac{\partial U}{\partial t} + (r_d - r_f)S\frac{\partial U}{\partial S} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial v^2} + \frac{1}{2}^2\frac{\partial^2 U}{\partial S^2} + \rho\sigma v\frac{\partial^2 U}{\partial v\partial S} - r_d U$$
$$+[\kappa(\theta - v) - \lambda v]\frac{\partial U}{\partial v} = 0. \tag{A.47}$$

Switching to log-coordinates, $x = \ln(S)$, the Heston partial differential equation becomes,

$$\frac{\partial U}{\partial t} + \frac{1}{2}\frac{\partial^2 U}{\partial x^2} + \left(r_d - r_f - \frac{1}{2}v\right)\frac{\partial U}{\partial x} + \rho\sigma v\frac{\partial^2 U}{\partial v\partial x} + \frac{1}{2}\sigma^2 v\frac{\partial^2 U}{\partial^2 v} - r_d U$$
$$+[\kappa(\theta - \mu) - \lambda v]\frac{\partial U}{\partial v} = 0. \tag{A.48}$$

We can solve the Heston partial differential equation by appealing to the Feynman-Kac lemma.

Regardless of the market model, we can write the call price as,

$$c(T, K) = e^{-r_d\tau}\mathbb{E}[(S_T - K)^+]$$
$$= e^{-r_d\tau}\mathbb{E}[S_T\mathbb{1}_{S_T>K}] - e^{-r_d\tau}\mathbb{E}[\mathbb{1}_{S_T>K}], \tag{A.49}$$

where $\tau = T - t$, the time left to maturity.

Notice that we break down the call price into two expectations, each one involving the indicator function. If we call the original measure P, we see that the second term in A.49 is simply the discounted $\mathbb{P}$-measure probability that S_T exceeds K, namely

$$e^{-r_d \tau}\mathbb{E}[\mathbb{1}_{S_T>K}] = e^{-r_d \tau}\mathbb{P}(S_T > K). \tag{A.50}$$

The first term represents a more complicated problem. Consider the random variable,

$$Z(T) = \frac{D(T)S_T}{D(t)S_t}, \tag{A.51}$$

where $D(t)$ is the discount rate, and let us suppose for the moment that S_t does not pay a dividend. We see that $Z(T)$ is an almost surely positive martingale under the $\mathbb{P}$ measure with expectation 1. Thus, we can use the object as a Radon-Nikodým derivative for a measure change,

$$\begin{aligned} e^{-r_d \tau}\mathbb{E}_{\mathbb{P}}[S_T \mathbb{1}_{S_T>K}] &= \mathbb{E}_{\mathbb{P}}[D(T)S_T \mathbb{1}_{S_T>K}] \\ &= D(t)S_t\mathbb{E}_{\mathbb{P}}\left[\mathbb{1}_{S_T>K}\frac{D(T)S_T}{D(t)S_t}\right] \\ &= D(t)S(t)\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{S_T>K}] \\ &= e^x\mathbb{E}_{\mathbb{Q}}[\mathbb{1}_{S_T>K}] = e^x\mathbb{Q}(S_T > K), \end{aligned} \tag{A.52}$$

(A.53)

where in the last line, we re-express in log-space. In the case of a continuous dividend or a foreign interest rate, the result becomes,

$$e^{-r_d \tau}\mathbb{E}_{\mathbb{P}}[S_T \mathbb{1}_{S_T>K}] = e^{-r_f \tau}e^x\mathbb{Q}(S_T > K). \tag{A.54}$$

Thus, we can write the call option price using two measures,

$$c(T,K) = e^{-r_f \tau}e^x\mathbb{Q}(S_T > K) - Ke^{-r_d \tau}\mathbb{P}(S_T > K). \tag{A.55}$$

We can now apply the Heston partial differential equation to A.55. Defining $\mathbb{P} = P_1$ and $\mathbb{Q} = P_2$, switching the direction of time, and substituting A.55 into A.48, we find the relationship

$$\begin{aligned} &-\frac{\partial P_j}{\partial \tau} + \rho\sigma\nu\frac{\partial^2 P_j}{\partial x \partial \nu} + \frac{1}{2}\frac{\partial^2 P_j}{\partial x^2} + \frac{1}{2}\nu\sigma^2\frac{\partial^2 P_j}{\partial \nu^2} + (r_d - r_f + u_j\nu)\frac{\partial P_j}{\partial x} \\ &+(a - b_j\nu)\frac{\partial P_j}{\partial \nu} = 0, \end{aligned} \tag{A.56}$$

with $j = 1,2$; $u_1 = 1/2$; $u_2 = -1/2$; $a = \kappa\theta$; $b_1 = \kappa + \lambda - \rho\sigma$; $b_2 = \kappa + \lambda$. To solving A.59, we once again go through the Feynman-Kac Lemma. Consider some function $f(x,t)$ with terminal condition

$$f(x,T) = e^{i\phi x_T} = e^{i\phi \ln(S_T)}. \tag{A.57}$$

If we demand that $f(x,t)$ satisfies A.59, which is differential operator that gives the evolution of the expectation of the indicator function, then by the Feynman-Kac lemma we have,

$$f(x,t) = \mathbb{E}[e^{i\phi x_T}]. \tag{A.58}$$

Thus, $f(x,t)$, with A.57, is the Fourier Transform of P_j,

$$-\frac{\partial f_j}{\partial \tau} + \rho\sigma\nu\frac{\partial^2 f_j}{\partial x \partial \nu} + \frac{1}{2}\frac{\partial^2 f_j}{\partial x^2} + \frac{1}{2}\sigma^2\frac{\partial^2 f_j}{\partial \nu^2} + (r_d - r_f + u_j\nu)\frac{\partial f_j}{\partial x}$$
$$+(a - b_j\nu)\frac{\partial f_j}{\partial \nu} = 0, \text{(A.59)}$$

with the terminal condition $f_j(x,T) = \exp(i\phi x_T)$. Assume that the Fourier Transform has the form,

$$f_j(\phi; x, \nu) = \exp\left[C_j(\tau,\phi) + D_j(\tau,\phi)\nu + i\phi x\right]. \tag{A.60}$$

Assembling the parts upon substitution we find two ordinary differential equations in τ,

$$\frac{dD_j}{d\tau} = \phi D_j - \frac{1}{2}\phi^2 + \frac{1}{2}\sigma^2 D_j^2 + u_j i\phi - b_j D_j, \tag{A.61}$$

$$\frac{dC_j}{d\tau} = (r_d - r_f)i\phi + aD_j, \tag{A.62}$$

with $D_j(0,\phi) = C_j(0,\phi) = 0$. The first equation in this set is of Riccati type. The solution, while a bit tedious, follows directly from solving the Riccati equation,

$$D_j = \frac{b_j - \rho\sigma i\phi + d_j}{\sigma^2}\left(\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right) \tag{A.63}$$

$$C_j = (r_d - r_f)i\phi\tau + \frac{a}{\sigma^2}\left[(b_j - \rho\phi + d_j)\tau - 2\ln\left(\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right)\right], \tag{A.64}$$

where

$$d_j = \sqrt{(\rho\phi - b_j)^2 - \sigma^2(2u_j i\phi - \phi^2)} \tag{A.65}$$

$$g_j = \frac{b_j - \rho\phi + d_j}{b_j - \rho\phi - d_j}. \tag{A.66}$$

Finally, we need to invert the Fourier Transforms, $f_{1,2}$, to recover $P_{1,2}$ in A.55. Recall the Fourier Transform pair,

$$\phi(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt \tag{A.67}$$

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\phi(\omega)\, d\omega \tag{A.68}$$

Notice, that by using the transform pair, we can write,

$$F(x) = \int_{-\infty}^{x} f(t)dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{x} e^{-i\omega t} \phi(\omega)\, dt\, d\omega. \tag{A.69}$$

Making the change of variable $z = x - t$, we have,

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{-i\omega x} \phi(\omega) e^{i\omega}\, dz\, d\omega = \int_{-\infty}^{\infty} e^{i\omega x} \phi(\omega) \mathscr{F}(h(z))\, d\omega, \tag{A.70}$$

where the last term denotes the Fourier Transform of the Heaviside step function. Continuing, we find,

$$F(x) = \int_{-\infty}^{\infty} e^{i\omega x} \phi(\omega) \left(\frac{1}{\sqrt{2\pi}} \frac{i}{\omega} + \sqrt{\frac{\pi}{2}} \delta(\omega) \right) d\omega, \tag{A.71}$$

where $\delta(\omega)$ is the usual Dirac-delta function. Carrying out the integration and recalling that $\phi(0) = 1/\sqrt{2\pi}$ when $f(t)$ represents a normalized probability density, we arrive at the inversion formula [20],

$$F(x) = \frac{1}{2} + \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \mathscr{R} \left(\frac{e^{-i\omega x} \phi(\omega)}{i\omega} \right) d\omega \tag{A.72}$$

In probability theory, authors often define the Fourier Transform as,

$$\phi(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt \tag{A.73}$$

with a full factor of $1/2\pi$ appearing with the inverse Fourier Transform. Under this convention, we have the more traditional formula for the inversion integral,

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \mathscr{R} \left(\frac{e^{-i\omega x} \phi(\omega)}{i\omega} \right) d\omega. \tag{A.74}$$

Applying this result to A.55, we have,

$$C(K,T) = e^{x} e^{-r_f \tau} P_1 - e^{-r_d \tau} K P_2 \tag{A.75}$$

$$P_j = \frac{1}{2} + \frac{1}{\pi} \int_{0}^{\infty} \mathscr{R} \left(\frac{e^{i\phi \ln(K)} f_j(\phi; \nu, x)}{i\phi} \right) d\phi. \tag{A.76}$$

A similar result holds for the put option.

The Heston Model is not complete in the sense that we do not have the necessary asset to hedge the uncertainty coming from the stochastic volatility. We already assumed the functional form $\Lambda(t, \nu, S) = \lambda \nu$, which allowed us to transform to the risk neutral measure.

Our solution still contains the unknown parameter, λ, but the call and put option solutions in the Heston model have the symmetry,

$$\kappa \to \kappa + \lambda, \tag{A.77}$$

$$\theta \to \frac{\kappa}{\kappa + \lambda}\theta, \tag{A.78}$$

which preserves the option price. Thus, we can set $\lambda = 0$ by a parameter redefinition. Nevertheless, to hedge the short position in the option, we still require the second asset correlated to the stochastic volatility. We set $\lambda = 0$ in our parameter fits. Furthermore, we impose the restriction,

$$2\kappa\theta \geq \sigma^2, \tag{A.79}$$

which ensures that the volatility process remains positive.

To perform the fit to data, we acquire a vector of market call and put prices, $(c_1, c_2, \ldots, c_n)$, $(p_1, p_2, \ldots, p_m)$, and we minimize with respect to $\Omega = \{\kappa, \theta, \rho, \sigma, \nu\}$,

$$\min_{\Omega} \sum_{i}^{n} \omega_i \left(c_i - c(K_i, T_i)\right)^2 + \sum_{i}^{m} \xi_i \left(p_i - p(K_i, T_i)\right)^2, \tag{A.80}$$

where ω_i, ξ_i denote the weights given by the inverse square of the bid-ask spread.

List of Acronyms

ADM(Mat)	Assistant Deputy Minister (Materiel)
ADM(Fin CS)	Assistant Deputy Minister (Finance and Corporate Services)
ATM	At-the-money
BSM	Black-Scholes-Merton
CAD	Canadian Dollar
CORA	Centre for Operational Research and Analysis
DSFC	Directorate Strategic Finance and Costing
DMGOR	Directorate Materiel Group Operational Research
DND	Department of National Defence
DRDC	Defence Research and Development Canada
GARCH	Generalized Autoregressive Conditional Hetroskedasticity
GDP	Gross Domestic Product
LIBOR	London Interbank Offer Rate
OTC	Over-the-counter
USD	United States Dollar
VaR	Value at Risk

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ADM(Fin CS) and senior decision makers at the Department of National Defence (DND) require insight into financial risks stemming from foreign exchange obligations in procurements and program delivery. We implement three popular derivative based quantitative financial models which provide the conditional Canada-US exchange rate trading range, under the risk neutral distribution, within a 95% confidence region up to a one year horizon. ADM(Fin CS) can use the model inferred trading range to help decide on a hedging rule in connection with foreign exchange budget obligations. Our results give a useful thumbnail sketch of the underlying probability distribution and confidence regions but, to gain a better understanding of foreign exchange market conditions, we require access to over-the-counter derivative data. Finally, ADM(Fin CS) staff can use the derived trading range in DND's foreign exchange reporting documents and internal monitoring services.

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