Derivation of an explicit expression for the Fournier-Forand phase function in terms of the mean cosine.

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Mean Cosine Multiple Scattering Function

1. Radiative transfer theory makes extensive use of the mean cosine of the phase function to compute the evolution of the light field under scattering.

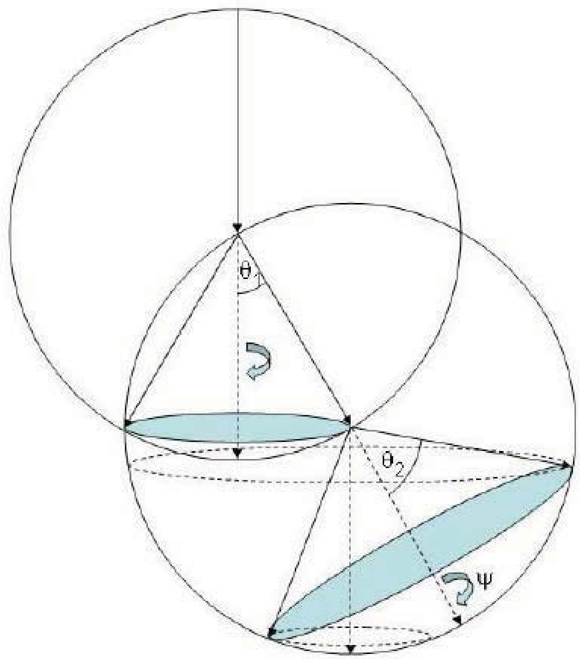
2. A simple analytic expression for the mean-cosine of the Fournier-Forand phase function is derived.

3. This expression and the power law- index of refraction relationship of Mobley[3] are used to explicitly parameterize the Fournier-Forand phase function by its mean cosine in a similar manner to the Henyey-Greenstein[4] function.

4. This function is then used to approximate the multiple scattering distribution and the results compared to the direct computation of this distribution both in the case no absorption and in the case of finite absorption.

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?



$$\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos\psi$$

Note: graph from J. Piskozub and D. McKee, "Effective scattering phase functions for the multiple scattering regime", Opt. Express 19(5), 4786-4794 (2011)

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?

Any scattering phase function can be expanded in a Legendre polynomial series as follows:

$$p(\theta) = \sum_{n=0}^{n=\infty} \left(n + \frac{1}{2} \right) f_n P_n(\cos\theta)$$

$$f_n = \int_{-1}^{1} p(\theta) P_n(\cos\theta) d(\cos\theta)$$

When there is no absorption, it has been shown that the resulting scattering phase function after m collisions becomes simply:

$$p_m(\theta) = \sum_{n=0}^{n=\infty} \left(n + \frac{1}{2} \right) f_n^{\,m} P_n(\cos\theta)$$

Which is also a Legendre series where all the single scattering coefficients are simply taken to the power of the number of collisions!

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?

Since by definition we have the following for the first and second Legendre polynomials:

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

We obtain the following results for the corresponding coefficients of the Legendre expansion of any phase function:

$$f_0 = \int_{-1}^{1} p(\theta)\, d(\cos\theta) = N = 1$$

$$f_1 = \int_{-1}^{1} p(\theta)\cos\theta\, d(\cos\theta) = \langle\cos\theta\rangle = g$$

The first coefficient is merely the normalization factor of the phase function and the second coefficient is the mean cosine.

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?

Since, with no absorption, the resulting scattering phase function after m collisions is:

$$p_m(\theta) = \sum_{n=0}^{n=\infty} \left(n + \frac{1}{2} \right) f_n^m P_n(\cos\theta)$$

The mean cosine after m collisions is therefore given by the single scattering mean cosine to the power m:

$$g_m = g^m$$

This result has been verified numerically for the Fournier-Forand and Henye-Greenstein functions by Piskozub and McKee

Mean Cosine Multiple Scattering Function

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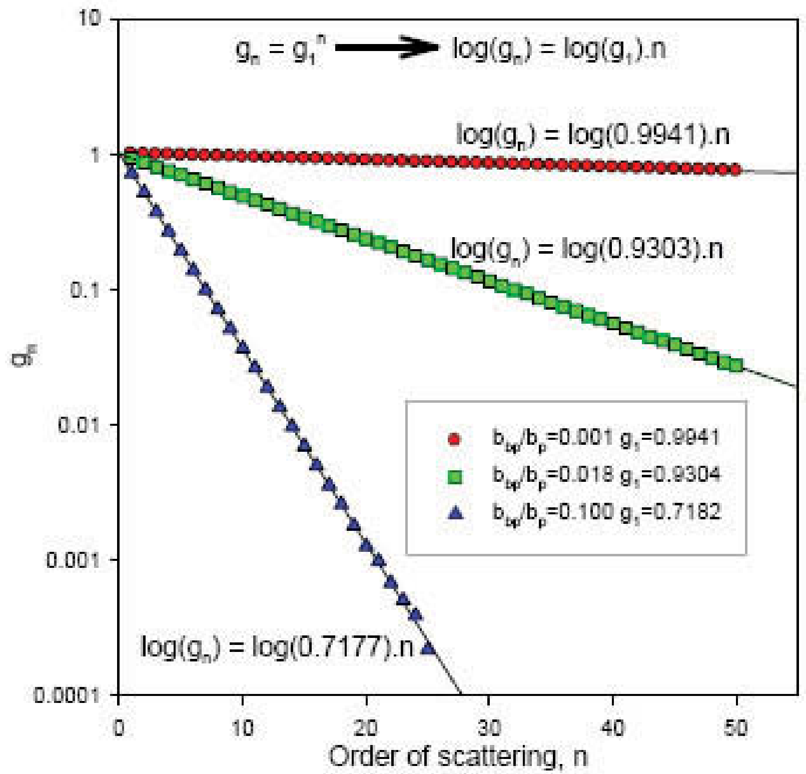


Note: graph from J. Piskozub and D. McKee, "Effective scattering phase functions for the multiple scattering regime", Opt. Express 19(5), 4786-4794 (2011)

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?

In the presence of absorption, it is possible to obtain an analogous result as follows. We will use the albedo (the ratio of scattering to scattering + absorption) as the controlling parameter.

$$\omega = \frac{b}{a+b} = \frac{b}{c}$$

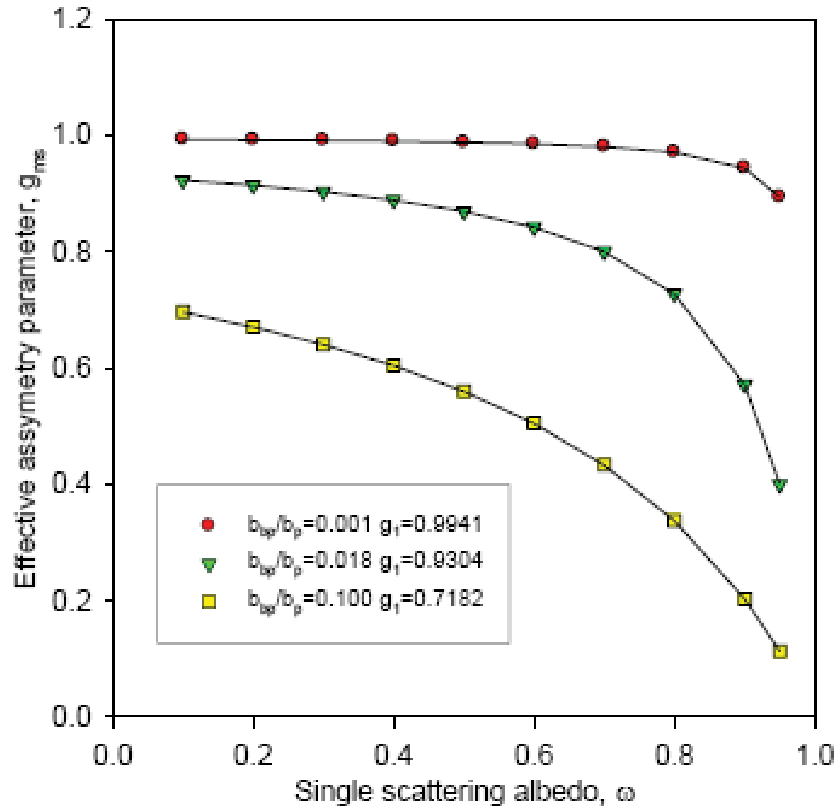
If we assume that absorption is independent of scattering angle even after multiple collisions, the asymptotic value of the mean cosine is given by:

$$g_s = \frac{\sum_{m=1}^{m=\propto} g_m \omega^m}{\sum_{m=1}^{m=\propto} \omega^m} = \frac{g(1-\omega)}{(1-g\omega)}$$

This result has also been verified numerically for the Fournier-Forand and Henye-Greenstein functions by Piskozub and McKee

Mean Cosine Multiple Scattering Function

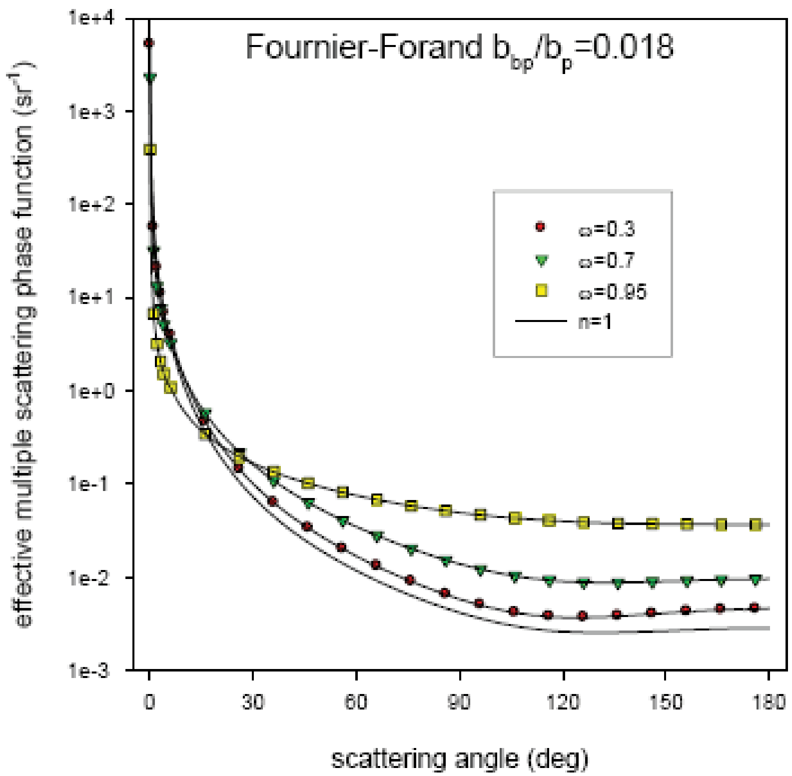
Why is the mean cosine important in radiative transfer theory?



Note: graph from J. Piskozub and D. McKee, "Effective scattering phase functions for the multiple scattering regime", Opt. Express 19(5), 4786-4794 (2011)

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?



These are some of the corresponding Fournier-Forand multiple scattering phase functions as computed by Piskozub and McKee

Note: graph from J. Piskozub and D. McKee, "Effective scattering phase functions for the multiple scattering regime", Opt. Express 19(5), 4786-4794 (2011)

Mean Cosine Multiple Scattering Function

Why is the mean cosine important in radiative transfer theory?

We now note a truly remarkable property of the HG phase function. It is self similar under multiple scattering!

$$p^{HG}(\theta) = \frac{1}{2}\frac{1-g^2}{\left(1-2g\cos\theta+g^2\right)^{3/2}} = \sum_{n=0}^{n=\infty}\left(n+\frac{1}{2}\right)g^n P_n(\cos\theta)$$

$$p_m^{HG}(\theta) = \sum_{n=0}^{n=\infty}\left(n+\frac{1}{2}\right)\left(g^n\right)^m P_n(\cos\theta) = \frac{1}{2}\frac{1-g^{2m}}{\left(1-2g^m\cos\theta+g^{2m}\right)^{3/2}}$$

This self-similarity may mean that the HG phase function may be the asymptotic multiple scattering state of other phase functions!

There is at this time no equivalent result for the FF function. However, if we wish to begin to investigate the transition from single to multiple scattering of the FF function it would be very helpful to first parameterize it in terms of its mean cosine.

Mean Cosine Multiple Scattering Function

In order to obtain an expression for the mean cosine of the FF function we need to evaluate the following two integrals in sequence:

$$g(x) = \langle \cos(\theta) \rangle = 2\pi \int_0^{\pi} p(x,\theta)\cos(\theta)\sin(\theta)d\theta$$

$$g = \int_0^{\infty} g(x)\, dx$$

Mean Cosine Multiple Scattering Function

We therefore need to first derive the scattering phase function from first principles. We start with the single particle phase function approximation. We want this function to be normalized to unity.
Note that:

$$2\pi\int_0^\pi f(\theta)\sin(\theta)d\theta = 1$$

$$u(\theta) = 2\sin\left(\frac{\theta}{2}\right)$$

$$\cos(\theta) = 1 - \frac{u(\theta)^2}{2}$$

$$\sin(\theta)d\theta = u(\theta)d\,u(\theta)$$

$$x = \frac{2\pi r}{\lambda}$$

Mean Cosine Multiple Scattering Function

First represent the single particle Airy function angular diffraction pattern by the following approximation:

$$f(x,\theta) = \frac{N_0}{\left[1 + \frac{u^2 x^2}{3}\right]^2}$$

Normalizing to unity implies that:

$$N_0 = \frac{1}{4\pi}\left(1 + \frac{4x^2}{3}\right)$$

The single particle normalized phase function is therefore:

$$f(x,\theta) = \frac{1}{4\pi}\frac{\left(1 + \frac{4x^2}{3}\right)}{\left[1 + \frac{u^2 x^2}{3}\right]^2}$$

Mean Cosine Multiple Scattering Function

To get the full phase function we then need to integrate over all x accounting for the scattering efficiency Q

$$Q = \frac{2(n-1)^2 x^2}{1+(n-1)^2 x^2}$$

If we assume an inverse power (Junge) particle size distribution we obtain the following equation:

$$\int_0^\infty f(x,\theta)\, Q(n,x)\, \pi x^2 \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{N_1}{x^\mu}\right)\left(\frac{\lambda}{2\pi}\right)^{-\mu}\left(\frac{\lambda}{2\pi}\right) dx$$

To normalization we need to perform the following operations:

$$\int_0^\infty \left[2\pi \int_0^\pi \sin(\theta) f(x,\theta)\, d\theta\right] Q(n,x)\, \pi x^2 \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{N_1}{x^\mu}\right)\left(\frac{\lambda}{2\pi}\right)^{-\mu}\left(\frac{\lambda}{2\pi}\right) dx = 1$$

Mean Cosine Multiple Scattering Function

The normalization factor becomes:

$$N_1 = \left(\frac{(n-1)\,2\pi}{\lambda}\right)^{3-\mu} \frac{1}{\pi^2}\cos\left(\frac{\mu\pi}{2}\right)$$

The normalized phase function before integration then becomes:

$$p(x,\theta) = \cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu}\ \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right] f(x,\theta)$$

$$p(x,\theta) = \cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu}\ \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right] \left\{ \frac{1}{4\pi} \frac{\left(1+\frac{4x^2}{3}\right)}{\left[1+\frac{u^2 x^2}{3}\right]^2} \right\}$$

$$u(\theta) = 2\sin\left(\frac{\theta}{2}\right)$$

The above expression is the one we need to use to evaluate related parameters such as the mean cosine (asymmetry) factor

Mean Cosine Multiple Scattering Function

Expressing the cosine in terms of our variables we obtain:

$$\cos(\theta) = 1 - \frac{u(\theta)^2}{2}$$

Because of our normalization the unity factor above simply integrates to one and we only need to evaluate the second term. Performing the angular integral we obtain:

$$g(x) = \langle \cos(\theta) \rangle = 1 + \frac{3\ln(27) + x^2(4 + \ln(81)) - (3 + 4x^2)\ln(3 + 4x^2)}{8x^4}$$

Integrating this over x is analytic but leads to a complex result involving $_2F_1$ functions. We can somewhat simplify the expressions by first using the following variable replacement.

$$z = \frac{3(n-1)^2}{4}$$

Mean Cosine Multiple Scattering Function

The mean cosine for a Fournier-Forand function is then given by:

$$g(\mu,n) = 1 + 4z^{\frac{5-\mu}{2}}$$

$$\left(\frac{z\ {}_2F_1\left[1, \frac{3-\mu}{2}; \frac{5-\mu}{2}; z\right]}{(\mu-3)} + \frac{\left(10 - 2\mu - (\mu-3)(\mu-1)\ z\ {}_2F_1\left[1, \frac{5-\mu}{2}; \frac{7-\mu}{2}; z\right]\right)}{(\mu-5)(\mu-3)(\mu-1)} \right) +$$

$$2z\left(-1 + (1-z)\left(\ln\left[\frac{(1-z)}{z}\right] - \pi \tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

This formula can be simplified by expanding the $_2F_1$ functions in a highly convergent series for small values of *z*.

Mean Cosine Multiple Scattering Function

After performing the expansion we obtain the following expression for the mean cosine:

$$g(\mu,n)=1-8z^{\frac{5-\mu}{2}}\left(\frac{1}{(\mu-3)(\mu-1)}+\frac{z}{(\mu-5)(\mu-3)}+\frac{z^2}{(\mu-7)(\mu-5)}+\frac{z^3}{(\mu-9)(\mu-7)}+\ldots\right)+$$

$$2z\left(-1+(1-z)\left(\ln\left[\frac{(1-z)}{z}\right]-\pi\tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

This expression is actually valid for the complete Fournier-Forand function since the contribution to the mean cosine of the second additive term of the function is identically zero because it is symmetrical about 90 degrees.

$$\left(\frac{\left(1-\delta_\pi^\nu\right)}{16\pi\left(\delta_\pi-1\right)\delta_\pi^\nu}\right)2\pi\int_0^\pi \cos(\theta)\left[3\cos(\theta)^2-1\right]\sin(\theta)\,d\theta=0$$

Mean Cosine Multiple Scattering Function

In order to parameterize the Fournier-Forand function in terms of the mean cosine, we need to reduce the expression above to a single parameter. To do this we can use the relationship found by Mobley et al between index and Junge power law. This relationship can be approximated as:

$$(\mu - 3) = 6(n - 1)$$

Note that the Fournier-Forand formula is only strictly valid between the limits , which imply an absolute limit in index of:

$$0 \leq n - 1 \leq 1/3$$

However one should be somewhat conservative and restrict the range so as to not approach the limits of validity too closely. This issue should be looked at in more detail but a suitable choice for the limits for now would be:

$$3.25 \leq \mu \leq 4.75$$

Mean Cosine Multiple Scattering Function

With the above constraints the formula for the mean cosine can now be approximated as:

$$1-g(n)=8z^{1-3(n-1)}$$

$$\left(\frac{1}{6(n-1)(6(n-1)+2)}+\frac{z}{6(n-1)(6(n-1)-2)}+\frac{z^2}{(6(n-1)-2)(6(n-1)-4)}+\frac{z^3}{(6(n-1)-4)(6(n-1)-6)}\right)-$$

$$2z\left(-1+(1-z)\left(\ln\left[\frac{(1-z)}{z}\right]-\pi\tan\left[3(n-1)\pi+\frac{3}{2}\pi\right]\right)\right)$$

We can easily obtain a simple approximate expression for (1-g) in terms of (n-1) by performing a modified variable power Pade first order approximation and inverting the resulting function.

$$(1-g)=\frac{23(n-1)^{5/2}}{1-7.5(n-1)^{5/2}}$$

Mean Cosine Multiple Scattering Function

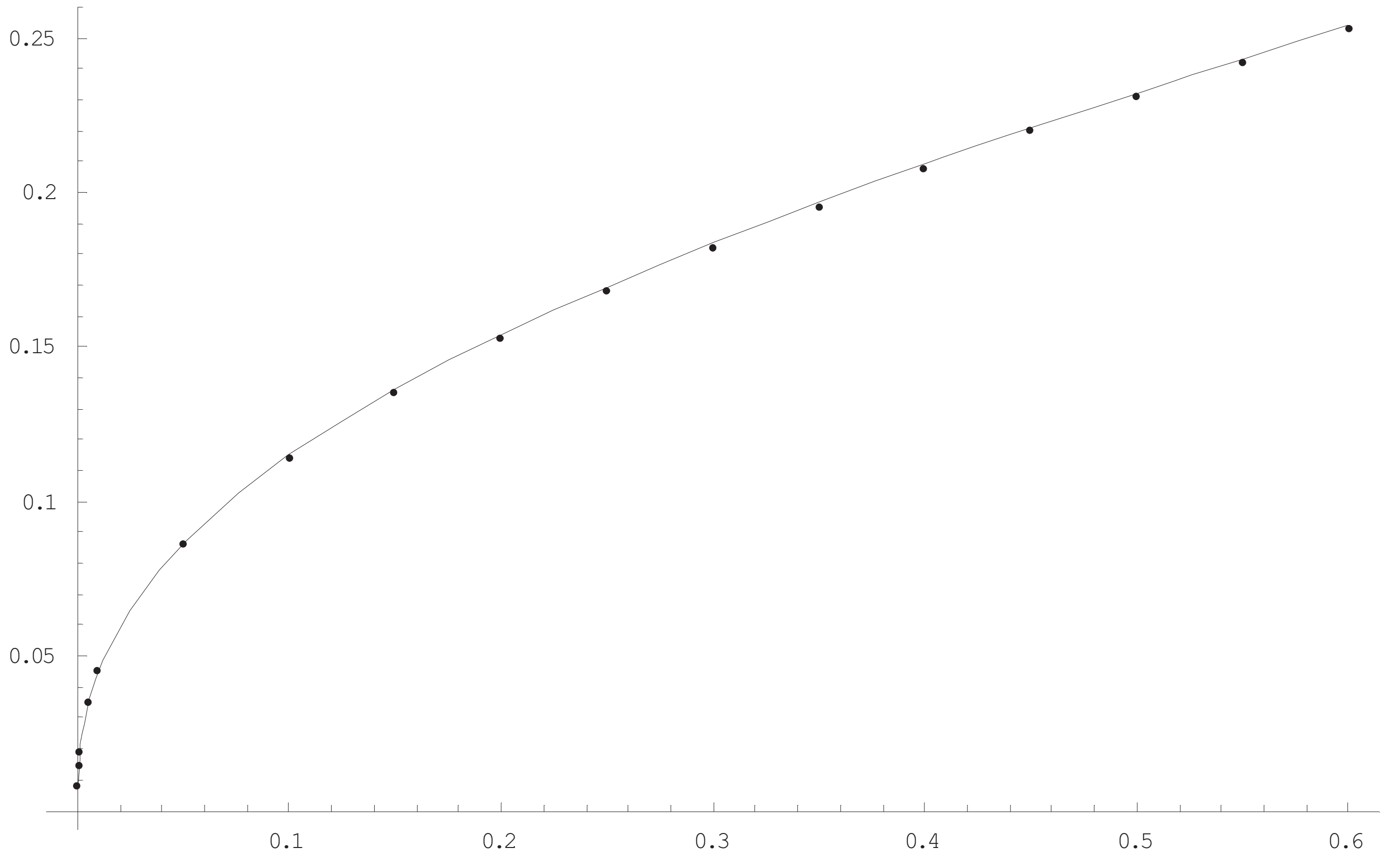
Inverting the previous formula we obtain:

$$(n-1)=\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{2/5}$$

The formula is valid from an index of 1.01 to 1.25 which corresponds to a range for *g* of :

$$0.0002 \le (1-g) \le 0.6$$

Mean Cosine Multiple Scattering Function



Plot of (n-1) in terms of (1-g). The dots are the exact solution using the full expression with the Mobley substitution and the curve is given by the inversion of the Pade approximation.

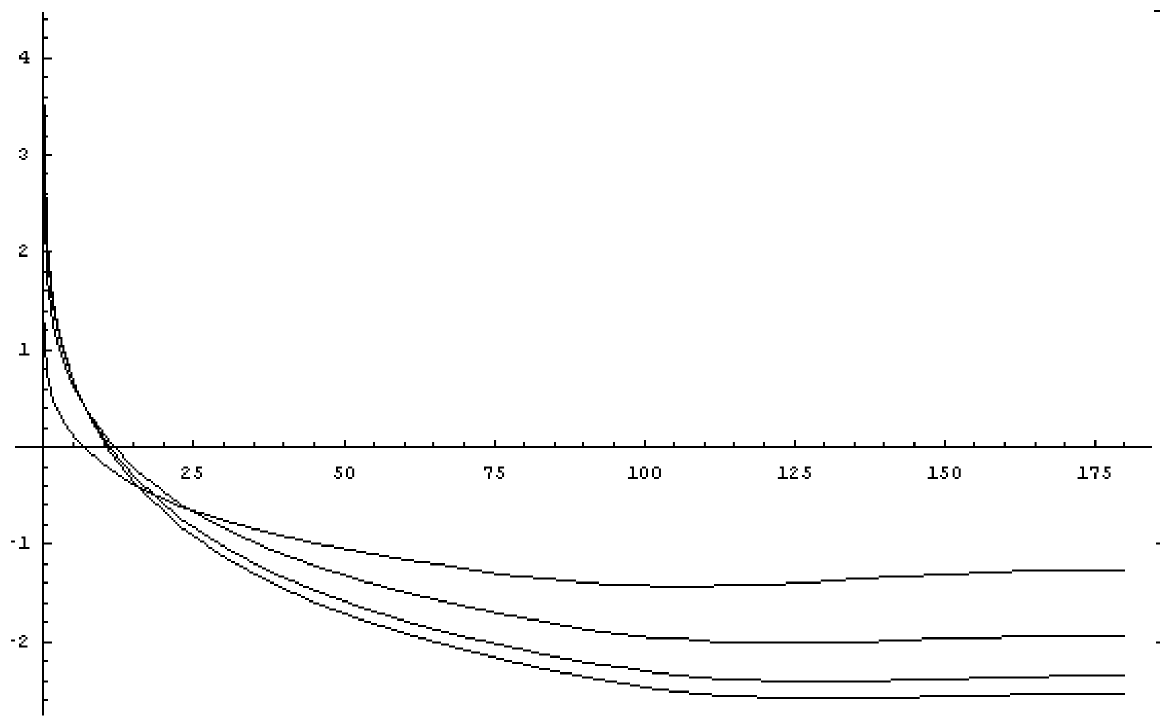
Mean Cosine Multiple Scattering Function

We can now use Mobley's relation along with this inverse Pade formula to rewrite the Fournier-Forand function in terms of the mean cosine. We simply need to perform the following replacements in the variables of the FF function.

$$\nu = -3\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{2/5} \quad \text{and} \quad \delta = \frac{4u(\theta)^2}{3}\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{-4/5}$$

After the above replacement, the FF function for multiple scattering can be approximated by simply inputting into the formula the appropriate value of g . This is a valid approach for a low number of collision. However, as mentioned previously we have no guarantee that the resulting function will model accurately the multiple scattering function as in the case of the HG function.

Mean Cosine Multiple Scattering Function



Graph of the FF multiple scattering limit for finite albedo. The cases are the same as those investigated by J. Piskozub and D. McKee. g=0.9304 which corresponds to bb/b=0.018 for all cases. From top to bottom the albedo values vary from 0.95 to 0.7 to 0.3. The bottom curve is the single scattering phase function itself with g=0.9304. There is a good fit to the exact results of Piskozub and McKee.

Mean Cosine Multiple Scattering Function

It should be noted that we can obtain an exact expression for the Legendre expansion of the additive term of the FF function. The zero order term is zero since the term was specifically designed not to modify the overall normalization of the FF function. As noted previously the first order term is also zero by symmetry. The only non-zero term is the second order term. The final result is given below.

$$\gamma_1 = \left(\frac{1}{2\pi}\right)\frac{1}{10}\left(\frac{\left(1-\delta_\pi^\nu\right)}{\left(\delta_\pi - 1\right)\delta_\pi^\nu}\right)$$

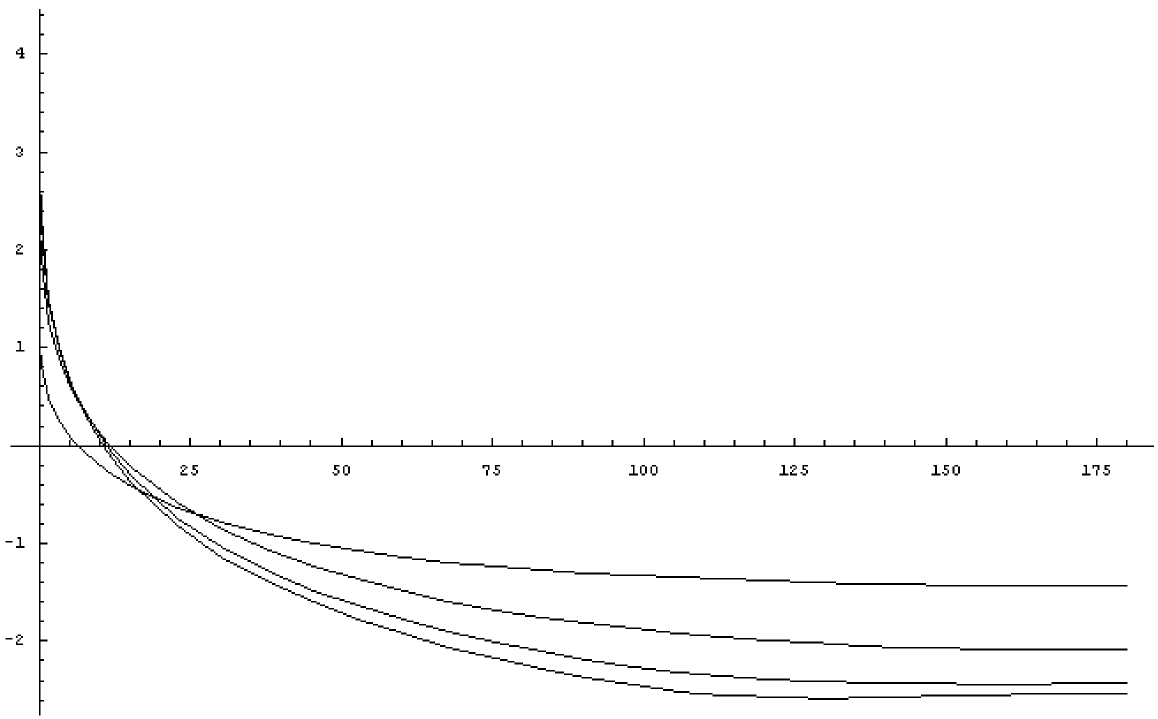
For the case of m order scattering with no absorption we have:

$$p_m(\theta) = \frac{5}{4}\left(\gamma_1\right)^m\left(3\cos^2\theta - 1\right)$$

For the asymptotic case with finite absorption we have:

$$p_s(\theta) = \frac{5}{4}\left[\frac{\gamma_1(1-\omega)}{(1-\gamma_1\omega)}\right]\left(3\cos^2\theta - 1\right)$$

Mean Cosine Multiple Scattering Function



Graph of the FF multiple scattering limit for finite albedo for the exact additive term . The cases are the same as those investigated by J. Piskozub and D. McKee. g=0.9304 which corresponds to bb/b=0.018 for all cases. From top to bottom the albedo values vary from 0.95 to 0.7 to 0.3. The bottom curve is the single scattering phase function itself with g=0.9304. The fit to the exact results of Piskozub and McKee is significantly improved when compared to the approximate additive term expression .

Mean Cosine Multiple Scattering Function

PLANNED FUTURE WORK

Investigate the behavior of the higher order terms of the Legendre expansion of the FF function.

Verify if the coefficients of the expansion coefficients of the FF function for multiple scattering tend to a limit and if in this limit they become identical to the coefficients of the HG function.

Check if the approach to this asymptotic case can be approximated analytically.