# Derivation of an explicit expression for the Fournier-Forand phase function in terms of the mean cosine.

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An analytic expression is derived for the mean cosine of the Fournier-Forand[1-2] phase function. This expression and the power law- index of refraction relationship of Mobley[3] are used to parameterize the Fournier-Forand phase function by its mean cosine in a similar manner to the Henyey-Greenstein[4] function.

Radiative transfer theory makes extensive use of the mean cosine of the phase function to compute the evolution of the light field under scattering. In this paper an analytic expression for the mean-cosine of the Fournier-Forand phase function in terms of Hypergeometric functions is derived. This expression reduces to a simple, easy to evaluate, very strongly convergent series that is a function of both mean index of refraction and the Junge size distribution inverse power law exponent. By judicious use of the relationship between the index of refraction and the power law proposed by Mobley it becomes possible to parameterize completely the Fournier-Forand phase function in terms of its mean cosine. This result is more complex but still analogous to the situation that prevails with the Henye-Greenstein[5] phase function where the asymmetry factor is identical to the mean cosine. The implications of using this expression for the mean-cosine in the multiple scattering regime of radiative transfer are explored along the lines recently suggested by Piskozub and McKee[6].

In order to obtain an expression for the mean cosine we need to evaluate the following two integrals in sequence:

$$g(x) = \langle \cos(\theta) \rangle = 2\pi \int_0^{\pi} p(x,\theta)\cos(\theta)\sin(\theta)d\theta$$

$$g = \int_0^{\infty} g(x)\,dx$$

We therefore need to first derive $p(x,\theta)$ from first principles. We start with the single particle phase function approximation. We want this function to be normalized to unity.

$$2\pi \int_0^{\pi} f(\theta)\sin(\theta)d\theta = 1$$

Note that:

$$u(\theta) = 2\sin\left(\frac{\theta}{2}\right)$$

$$\cos(\theta) = 1 - \frac{u(\theta)^2}{2}$$

$$\sin(\theta)d\theta = u(\theta)d\,u(\theta)$$

$$x = \frac{2\pi r}{\lambda}$$

Note also that, as in the case of the derivation of the full Fournier-Forand phase function, we approximate the single particle Airy function angular diffraction pattern by the following approximation:

$$f(\theta) = \frac{N_0}{\left[1 + \frac{u^2 x^2}{3}\right]^2}$$

This implies that:

$$\frac{1}{N_0} = 2\pi \int_0^2 \frac{1}{\left[1 + \frac{u^2 x^2}{3}\right]^2} u \; du$$

$$\frac{1}{N_0} = \frac{4\pi}{\left(1 + \frac{4x^2}{3}\right)}$$

$$N_0 = \frac{1}{4\pi}\left(1 + \frac{4x^2}{3}\right)$$

So the single particle normalized phase function approximation is

$$f(x,\theta) = \frac{1}{4\pi} \frac{\left(1 + \frac{4x^2}{3}\right)}{\left[1 + \frac{u^2 x^2}{3}\right]^2}$$

To get the full phase function we then need to integrate over all x accounting for $Q_{scat}$

$$Q_{scat} = \frac{2(n-1)^2 x^2}{1 + (n-1)^2 x^2}$$

This gives the following integral:

$$\int_0^\infty f(x,\theta)\, Q(n,x)\, \pi x^2 \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{N_1}{x^\mu}\right)\left(\frac{\lambda}{2\pi}\right)^{-\mu} \left(\frac{\lambda}{2\pi}\right) dx$$

To normalize we need to perform the following operations

$$\int_0^\infty \left[2\pi \int_0^\pi \sin(\theta) f(x,\theta) d\theta\right] Q(n,x)\, \pi x^2 \left(\frac{\lambda}{2\pi}\right)^2 \left(\frac{N_1}{x^\mu}\right)\left(\frac{\lambda}{2\pi}\right)^{-\mu} \left(\frac{\lambda}{2\pi}\right) dx = 1$$

Since we have already normalized as a function of angle

$$\left[2\pi \int_0^\pi \sin(\theta) f(x,\theta) d\theta\right] = 1$$

Therefore:

$$\frac{1}{N_1} = \left(\frac{\lambda}{2\pi}\right)^{3-\mu} \int_0^\infty Q(n,x)\,\pi\,x^2 \left(\frac{1}{x^\mu}\right) dx$$

$$\frac{1}{N_1} = \left(\frac{\lambda}{2\pi}\right)^{3-\mu} \int_0^\infty \left[\frac{2(n-1)^2 x^2}{1+(n-1)^2 x^2}\right] \pi\,x^2 \left(\frac{1}{x^\mu}\right) dx = \left(\frac{\lambda}{2\pi}\right)^{3-\mu} 2\pi(n-1)^2 \int_0^\infty \left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right] dx$$

$$\int_0^\infty \left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2/4}\right] dx = (n-1)^{\mu-5} \frac{\pi}{2} \frac{1}{\cos\left(\frac{\mu\pi}{2}\right)}$$

$$N_1 = \left(\frac{(n-1)\,2\pi}{\lambda}\right)^{3-\mu} \frac{1}{\pi^2} \cos\left(\frac{\mu\pi}{2}\right)$$

The normalized phase function before integration can therefore be written as:

$$\left(\frac{(n-1)\,2\pi}{\lambda}\right)^{3-\mu} \frac{1}{\pi^2} \cos\left(\frac{\mu\pi}{2}\right)\left(\frac{\lambda}{2\pi}\right)^{3-\mu} \int_0^\infty \left[2\pi\int_0^\pi \sin(\theta) f(x,\theta)\,d\theta\right] \left[\frac{2(n-1)^2 x^2}{1+(n-1)^2 x^2}\right] \pi\,x^2 \left(\frac{1}{x^\mu}\right) dx = 1$$

$$\cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu} \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right]\left[2\pi\int_0^\pi \sin(\theta) f(x,\theta)\,d\theta\right]$$

$$2\pi\int_0^\pi p(x,\theta)\sin(\theta)\,d\theta = \cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu} \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right]\left[2\pi\int_0^\pi \sin(\theta) f(x,\theta)\,d\theta\right]$$

$$p(x,\theta) = \cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu} \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right] f(x,\theta)$$

The following expression will be normalized to 1 when integrated over angle and an inverse power law from 0 to infinity. It is convenient to evaluate related parameters such as the asymmetry parameter g

$$p(x,\theta) = \cos\left(\frac{\mu\pi}{2}\right)(n-1)^{5-\mu} \frac{2}{\pi}\left[\frac{x^{4-\mu}}{1+(n-1)^2 x^2}\right]\left\{\frac{1}{4\pi}\frac{\left(1+\frac{4x^2}{3}\right)}{\left[1+\frac{u^2 x^2}{3}\right]^2}\right\}$$

$$u(\theta) = 2\sin\left(\frac{\theta}{2}\right)$$

Note that the first term in the expression for the cosine in terms of our angular variable u is unity

$$\cos(\theta) = 1 - \frac{u(\theta)^2}{2}$$

Because of the normalization of our phase function expression we do not need to integrate that term as the result will be 1. We will now concentrate our efforts on the second term.
Let's first perform the angular integral.
We obtain:

$$g(x) = \langle \cos(\theta) \rangle = 1 + \frac{3\ln(27) + x^2(4 + \ln(81)) - (3 + 4x^2)\ln(3 + 4x^2)}{8x^4}$$

Integrating over x is analytic but leads to a complex result involving ${}_2F_1$ functions.

$$g(\mu,n) = 1 + \frac{3}{32}(n-1)^{5-\mu} 2^{\mu} 3^{\frac{5-\mu}{2}}$$

$$\left( \frac{(n-1)^2 \; {}_2F_1\left[1, \frac{3-\mu}{2}; \frac{5-\mu}{2}; \frac{3}{4}(n-1)^2\right]}{(\mu-3)} + \frac{1}{3} \frac{\left(40 - 8\mu - 3(\mu-3)(\mu-1)(n-1)^2 \; {}_2F_1\left[1, \frac{5-\mu}{2}; \frac{7-\mu}{2}; \frac{3}{4}(n-1)^2\right]\right)}{(\mu-5)(\mu-3)(\mu-1)} \right) +$$

$$\left(\frac{3(n-1)^2}{8}\right)\left(-4 + \left(4 - 3(n-1)^2\right)\left(\ln\left[\frac{\left(4 - 3(n-1)^2\right)}{3(n-1)^2}\right] - \pi \tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

An approximate form can be obtained directly from the exact result by expanding the ${}_2F_1$ Hypergeometric functions in series for small arguments.

$$g(\mu,n) = 1 - \frac{3}{32}(n-1)^{5-\mu} 2^{\mu} 3^{\frac{5-\mu}{2}} \left( \frac{8}{3(\mu-3)(\mu-1)} + \frac{4(n-1)^2}{(\mu-7)(\mu-3)} \right) +$$

$$\left(\frac{3(n-1)^2}{8}\right)\left(-4 + \left(4 - 3(n-1)^2\right)\left(\ln\left[\frac{\left(4 - 3(n-1)^2\right)}{3(n-1)^2}\right] - \pi \tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

The formulas can be simplified by using the following variable replacement:

$$z = \frac{3(n-1)^2}{4}$$

We then obtain:

$$g(\mu,n) = 1 + 4z^{\frac{5-\mu}{2}}$$

$$\left( \frac{z \; {}_2F_1\left[1, \frac{3-\mu}{2}; \frac{5-\mu}{2}; z\right]}{(\mu-3)} + \frac{\left(10 - 2\mu - (\mu-3)(\mu-1)\, z \; {}_2F_1\left[1, \frac{5-\mu}{2}; \frac{7-\mu}{2}; z\right]\right)}{(\mu-5)(\mu-3)(\mu-1)} \right) +$$

$$2z\left(-1 + (1-z)\left(\ln\left[\frac{(1-z)}{z}\right] - \pi \tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

And the two-term approximation becomes:

$$g(\mu,n) = 1 - 8z^{\frac{5-\mu}{2}} \left( \frac{1}{(\mu-3)(\mu-1)} + \frac{z}{(\mu-5)(\mu-3)} \right) +$$

$$2z\left(-1 + (1-z)\left(\ln\left[\frac{(1-z)}{z}\right] - \pi \tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

The rest of the series terms can be added if required as follows:

$$g(\mu,n)=1-8z^{\frac{5-\mu}{2}}\left(\frac{1}{(\mu-3)(\mu-1)}+\frac{z}{(\mu-5)(\mu-3)}+\frac{z^2}{(\mu-7)(\mu-5)}+\frac{z^3}{(\mu-9)(\mu-7)}+\ldots\right)+$$
$$2z\left(-1+(1-z)\left(\ln\left[\frac{(1-z)}{z}\right]-\pi\tan\left[\frac{\mu\pi}{2}\right]\right)\right)$$

This expression is actually valid for the complete Fournier-Forand function since the contribution to the mean cosine of the second additive term is identically zero because it is symmetrical about $\theta=\pi/2$.

$$\left(\frac{(1-\delta_\pi^\nu)}{16\pi(\delta_\pi-1)\delta_\pi^\nu}\right)2\pi\int_0^\pi\cos(\theta)\,\left[3\cos(\theta)^2-1\right]\,\sin(\theta)d\theta=0$$

In order to parametrize the Fournier-Forand function in terms of the mean cosine, we need to reduce the expression above to a single parameter. To do this we can use the relationship found by Mobley et al between index and Junge power law. This relationship can be approximated as:

$$(\mu-3)=6(n-1)$$

Note that the Fournier-Forand formula is only strictly valid between the limits $3\le\mu\le5$, which imply an absolute limit in index of $0\le n-1\le1/3$. However one should be somewhat conservative and restrict the range so as to not approach the limits of validity too closely. This issue should be looked at in more detail but a suitable choice for the limits for now would be $3.25\le\mu\le4.75$

If we substitute the Mobley expression in the equation for the mean cosine we obtain:

$$1-g(n)=8z^{1-3(n-1)}$$
$$\left(\frac{1}{6(n-1)(6(n-1)+2)}+\frac{z}{6(n-1)(6(n-1)-2)}+\frac{z^2}{(6(n-1)-2)(6(n-1)-4)}+\frac{z^3}{(6(n-1)-4)(6(n-1)-6)}\right)-$$
$$2z\left(-1+(1-z)\left(\ln\left[\frac{(1-z)}{z}\right]-\pi\tan\left[3(n-1)\pi+\frac{3}{2}\pi\right]\right)\right)$$

We can easily obtain a simple approximate expression for $(n-1)$ in terms of $(1-g)$ by performing a modified variable power Pade first order approximation and inverting the resulting function. By this method we first obtain a formula valid from an index of 1.01 to 1.25 which corresponds to a range for *g* of $0.0002\le(1-g)\le0.6$:

$$(1-g)=\frac{23(n-1)^{5/2}}{1-7.5(n-1)^{5/2}}$$

Inverting this formula we obtain:

$$(n-1)=\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{2/5}$$

We can now use Mobley's relation along with this formula to rewrite the Fournier-Forand function in terms of the mean cosine. We simply need to perform the following replacements in the variables of the FF function.

$$\nu=-3\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{2/5} \quad \text{and} \quad \delta=\frac{4u(\theta)^2}{3}\left(\frac{(1-g)}{23-7.5(1-g)}\right)^{-4/5}$$

This gives a form suitable for use with the multiple scattering formulas recently derived and numerically verified by Piskozub and McKee[6]. They show that for order m of multiple scattering with a single scattering albedo of 1 the mean cosine is given by:

$$g_m=g^m$$

They also show that the mean cosine of the asymptotic distribution for a substance with a single scattering albedo $\omega$ is given by:

$$g_{ms}=\frac{g(1-\omega)}{(1-g\,\omega)}$$

We can therefore use the parameterized version of the Fournier-Forand to compute the multiply scattered distribution but we need to keep in mind that the limits on *g* imposed by the range of validity of the phase function also apply to both $g_m$ and $g_{ms}$.

The approach above needs to be tested further since there is no evidence at this time that the Fournier-Forand function is self similar under the iterated finite convolution operation required to compute multiple scattering. As examples, the Gaussian is a self-similar function under convolution over an infinite range and the Henyey-Greenstein function has been proven to be self-similar under the multiple scattering convolution operations. It is therefore highly unlikely that the final asymptotic stationary state under high albedo conditions, even though it must have the appropriate $g_m$ and $g_{ms}$, be described by a parameterized FF function. The full problem of going from the parameterized single scattering form to the correct asymptotic distribution is still open and will be the subject of further work. We now have the possibility the Henyey-Greenstein function could be used as the asymptotic multiple scattering limit of the Fournier-Forand function with the single scattering mean cosine given by the Fournier-Forand form derived above. However, under the moderate albedo conditions encountered in ocean waters, the present approach should give accurate results in the important and difficult to evaluate transition range of 1 to 10 scattering events.

**References:**


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