Weibull-based Time-phasing of Budget Expenditures

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Abstract

The assessment of financial positions and subsequent reallocation of resources between various capabilities or business lines is at the centre of any private or public sector budgeting process. This in-year budget utilization process is particularly crucial in public sector organizations due to financial regulations that require unspent budget dollars to be “lapsed”. Lapsed funding implies the loss of funding for a given fiscal year, i.e., that money is permanently unavailable to the organization. In order to facilitate the optimal reallocation of budgets among capabilities in-year, a probability-based approach was developed in this study to model expenditure rhythm. Expenditure rhythm is the time-phasing of budget expenditures over a given fiscal year. The methodology was demonstrated in a use case analysis. A sensitivity analysis was conducted to address the impact of model parameters on the results. The proposed model provides financial managers the ability to respond whenever the expenditures are higher or lower than expected.

Résumé

L’analyse de la situation financière et de la réaffectation des ressources entre différents secteurs d’activités ou capacités qui s’ensuit est au centre du processus budgétaire de tous les organismes des secteurs privé et public. Le processus d’affectation du budget en cours d’année financière est particulièrement crucial dans les organisations du secteur public, en raison des règlements financiers qui exigent que les fonds non utilisés soient déclarés « périmés ». Les fonds périmés se traduisent par un manque à gagner dans une année financière donnée; autrement dit, l’organisation perd définitivement ces sommes. L’étude visait à mettre au point une méthode fondée sur la probabilité permettant de modéliser le rythme des dépenses dans le but d’optimiser la réaffectation des budgets entre les capacités en cours d’année. Une analyse de cas d’utilisation a servi à démontrer la méthodologie. Nous avons réalisé une analyse de sensibilité afin d’étudier l’incidence des paramètres du modèle sur les résultats. Grâce au modèle proposé, les gestionnaires des finances sont en mesure d’intervenir lorsque les dépenses sont supérieures ou inférieures aux prévisions.

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Executive summary

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Abderrahmane Sokri; DRDC CORA TM: 2012-224; Defence R&D Canada – CORA.

Expenditure time-phasing is the distribution of budget expenditures over a given fiscal year. Developing an expenditure time-phasing is essential to the success of any budget management. This study provides an expenditure time-phasing methodology to produce a predictable expenditure profile.

Background

The assessment of financial positions and subsequent reallocation between various capabilities or business lines is at the centre of any private or public sector budgeting process. This in-year budget utilization process is particularly crucial in public sector organizations due to financial regulations that require unspent budget dollars to be "lapsed". Lapsed funding implies the loss of funding for a given fiscal year, i.e., that money is permanently unavailable to the organization. In order to facilitate the optimal reallocation of budgets among DND funds and cost centres in-year, the Department of National Defence (DND) is looking for appropriate ways to assign public funds to military projects.

Aim

This study has two objectives:

- Develop a methodology for predicting expenditure profiles of Canadian military expenditures; and
- Enhance the understanding of expenditure time-phasing both theoretically and practically.

Methodology

This modelling procedure involves a three-stage process of data preparation, parameter estimation and model application or forecasting.

1. To summarize data and highlight seasonal variations, we used a particular table where each row represents the proportions of expenditures observed over the 13 periods of the fiscal year, and each column represents a period (month). To diagnose expenditure profile, average proportions of expenditures are calculated and accumulated.
2. Using these observed proportions, we estimated a theoretical model that fits the observed pattern.
3. To forecast the cumulative budget expenditures, the theoretical model is scaled by the annual budget to convert it into dollars.

Application

The developed methodology was applied to many DND funds and cost centres. An example using Royal Canadian Air Force (RCAF) data was used in this paper for illustration purpose. For the RCAF, the model was estimated and the amounts of expenditures to occur in the next fiscal year were forecasted.

Results indicate that, for the next year, it is forecasted to spend about 2% in April, 5% in May, 6% in June, 8% in July, 9% in August, September and October, 8% in November and December, 7% in January, 6% in February, and 18% in March. The 6% remaining is an adjustment. As shown in Figure 4, the actual expenditures in FY 2010-11 were slightly deviating from the expected budget tempo in April and May. The amount spent in the other months was in the 95% Confidence Interval. Armed with this information, budget authorities would have advised RCAF to adjust expenditures. The model provides financial managers the ability to respond whenever the expenditures are higher or lower than expected.

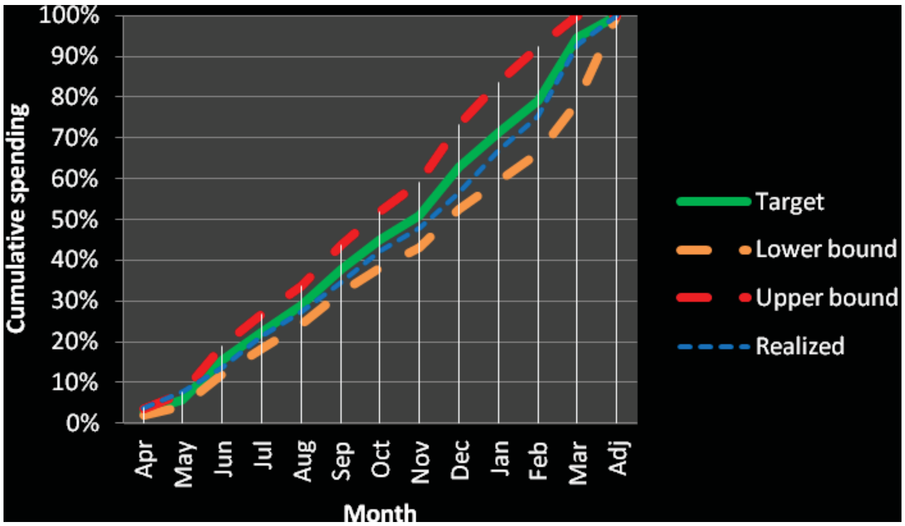

Figure 4: Spent proportions for RCAF in FY 2010-11

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Le rythme des dépenses correspond à l'échelonnement des dépenses budgétaires au cours d'une année financière donnée. L'échelonnement des dépenses est essentiel à l'efficacité de la gestion budgétaire. La présente étude propose une méthode d'échelonnement des dépenses permettant de prévoir le profil des dépenses.

Contexte
L'analyse de la situation financière et de la réaffectation des ressources entre différents secteurs d'activités ou capacités qui s'ensuit est au centre du processus budgétaire de tous les organismes des secteurs privé et public. Le processus d'affectation du budget en cours d'année financière est particulièrement crucial dans les organisations du secteur public, en raison des règlements financiers qui exigent que les fonds non utilisés soient déclarés « périmés ». Les fonds périmés se traduisent par un manque à gagner dans une année financière donnée; autrement dit, l'organisation perd définitivement ces sommes. Le ministère de la Défense nationale (MDN), afin d'optimiser la réaffectation des budgets entre ses fonds et ses centres de coûts en cours d'année, s'efforce de trouver des moyens appropriés pour affecter les fonds publics aux projets militaires.

Objet
L'auteur de la présente étude vise deux objectifs :

- Mettre au point une méthodologie permettant de prévoir le profil des dépenses militaires du Canada;
- Favoriser une meilleure compréhension de l'échelonnement des dépenses sur les plans théorique et pratique.

Méthodologie
La procédure de modélisation comporte trois étapes, soit la préparation des données, l'estimation des paramètres et l'application du modèle (ou la prévision).
1. Pour résumer les données et illustrer les variations saisonnières, nous avons employé un tableau particulier dans lequel chaque ligne représente les proportions des dépenses constatées au cours des 13 périodes de l'année financière, et chaque colonne correspond à une période (mois). Pour établir le profil des dépenses, les proportions moyennes des dépenses ont été calculées et additionnées.

2. Au moyen des proportions constatées, nous avons estimé un modèle théorique qui correspond à la tendance observée.
3. Le modèle théorique a été pondéré par la valeur du budget afin de le convertir en dollars. Ce qui a permis de prévoir les dépenses budgétaires cumulatives.

Application

La méthodologie mise au point a été appliquée à de nombreux fonds et centres de coûts du MDN. Dans le présent document, nous avons employé des données de l'Aviation royale canadienne (ARC) pour illustrer notre propos. Dans le cas de l'ARC, nous avons fait une estimation du modèle et une prévision du montant des dépenses de l'année financière à venir.

Les résultats révèlent que, pour la prochaine année, l'ARC devrait dépenser 2 p. 100 en avril, 5 p. 100 en mai, 6 p. 100 en juin, 8 p. 100 en juillet, 9 p. 100 en août, en septembre et en octobre, 8 p. 100 en novembre et en décembre, 7 p. 100 en janvier, 6 p. 100 en février et 18 p. 100 en mars. Les 6 p. 100 résiduels représentent un ajustement. Comme le montre la figure 4, les dépenses réelles de l'année financière 2010-2011 se sont légèrement écartées du rythme budgétaire prévu en avril et mai. Les montants dépensés pendant les autres mois se trouvaient dans l'intervalle de confiance de 95 p. 100. Armés de telles données, les responsables des budgets auraient conseillé à l'ARC d'ajuster ses dépenses. Grâce au modèle proposé, les gestionnaires des finances sont en mesure d'intervenir lorsque les dépenses sont supérieures ou inférieures aux prévisions.

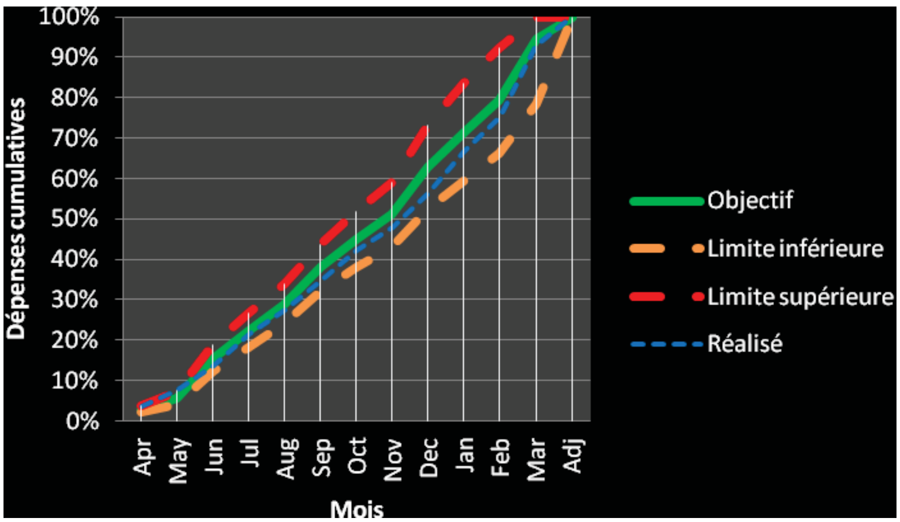

Figure 4 : Proportions des dépenses de l'ARC – AF 2010-2011

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1. Introduction

1.1 Background

The assessment of financial positions and subsequent reallocation between various capabilities or business lines is at the centre of any private or public sector budgeting process. This in-year budget utilization process is particularly crucial in public sector organizations due to financial regulations that require unspent budget dollars to be "lapsed". Lapsed funding implies the loss of funding for a given fiscal year. That money is permanently unavailable to the organization. The Department of National Defence (DND) is looking for appropriate ways to allocate funds to its different cost centres with respect to this additional budget constraint.

Two main things characterize the different cost centres spending:

- the propensity to spend has not always been regular; and
- the risk of lapsed funding has been omnipresent.

To find a solution to this problem, the Director Budget (DB) asked the Defence Economics Team to define an adequate way to allocate funds to the different cost centres over the fiscal year. In particular, the DB is seeking help in modelling expenditure rhythm in order to design a predictable expenditure profile.

1.2 Aim

This study has two objectives:

- Develop a methodology for characterizing profiles of Canadian military expenditures; and
- Enhance the understanding of expenditure time-phasing both theoretically and practically.

1.3 Methodology

There is considerable literature on the nature of public-sector budgeting and end-of-year spending. These articles tend to frame the problem in a principal-agent framework [1] or describe the perverse incentives of central agencies and departments [2]. However, this report focuses on the budget time-phasing and not the behavioural reasons behind budget expenditures.

Time-phasing is one of the well-founded measures to respond to this situation. Time-phasing is a technique of organizing and scheduling expenditures within a fiscal year to meet the planned objectives of the organization. This modelling procedure involves a three-stage process of data preparation, parameter estimation, and model application or forecasting.

- To summarize data and highlight seasonal variations, we used a Buys-Ballot table. In this table, each row represents a year's worth of proportions spent over the 13 months of the fiscal year, and each column represents a month. A cell (i,j) of this table contains the proportion of the expenditure made during the year i at the month j. Monthly averages are calculated and accumulated to obtain an observed cumulative-expenditure versus time curve.

- To model the expenditure profile, a Weibull distribution that fits the observed cumulative-expenditure curve is considered. The model was estimated by minimizing the sum of squared errors between the theoretical and empirical Cumulative Distribution Functions (CDFs). To handle the infinite tail issue, the minimization was constrained by the completion time.
- To forecast the cumulative budget expenditures, the obtained Cumulative Distribution Function (CDF) is scaled by the annual budget.

In the literature, three functional forms have been used to forecast budget profiles, namely, Rayleigh, Beta and Weibull distributions. Different research studies [3, 4, 5, 6] have identified three reasons why the Weibull distribution is preferred:

- The Weibull distribution includes the Rayleigh distribution as a special case;
- It has more parameters to accommodate variation among spending profiles with a small sacrifice of simplicity [6];
- The Beta distribution is flexible, but there is little justification for its parameter inputs [3].

Due to inflation, the purchasing power changes over the years. To remove the effects of inflation and allow easy comparisons between periods, current dollar values need to be converted into inflation-adjusted dollars or constant dollars [3]. In this study, constant dollars are computed using the GDP deflator, and the year 2000 was chosen as the baseline.

1.4 Document structure

This paper is organized into five sections. Following the introduction, Section 2 provides a comprehensive review of literature on government spending management. Section 3 sets up the proposed model and describes its mathematical formulation. In Section 4, an example using RCAF expenditures was used to illustrate the methodology. In Section 5, some concluding remarks are made.

2. Literature review

Managing public expenditures has been an active research area in the last few decades, and has considered a myriad of topics such as expenditure forecasting [7], the sustainability of budget policy [8], government spending and economic growth [9], the economic impact of military expenditures [10], equity in the public funding [11], etc.

The forecasting subset of this literature can be divided along methodological lines into three major models:

- The first category uses the Box-Jenkins [12] stochastic time-series method and other traditional econometric techniques. The unknown parameters are estimated to arrive at coefficients which best fit the selected model (13, 14, 15).

- The second class uses the decomposition technique developed by Persons [16]. In this group the observed time series is decomposed into its salient features such as trend, seasonal and irregular components (17, 18). The decomposition technique is not strictly a forecasting method, but it is a useful tool for understanding time series better, and therefore it assists in forecasting [19].

- The third type, to which this paper belongs, uses Norden theory [20] to assess a budget profile. The simplicity of this functional approach added to its flexible nature makes it very attractive to analysts. The Weibull distribution, in particular, is the most flexible. There is almost always a model which provides an adequate description to the data.

The first two forecasting methods would not be useful for this research question for two reasons:

- With the expenditure and budget series expressed in values, these two models could not take significant budget cuts into account in their forecasts;
- They could be used to model the expenditure rate series (expressed in proportions), but they could not provide reliable predictions for 52 weeks, for example.

The third methodology is, however, able to provide forecasts for any time of year (days, weeks, months, etc.). The Weibull distribution has been originally used in time-to-failure measurements, when the failure probability varies over time. It is now used in many fields such as medical research and cost reduction analysis [21], but we know of no paper that used this approach to estimate departmental budgetary expenditures.

The Norden theory [20] was mainly used to study the defence research and development (R&D) budgets. These programs generally outlay over several years, and the challenge for government is usually to meet each fiscal year expenditure requirement [5]. Since the seminal paper by Norden which demonstrated that the Rayleigh function could model manpower requirements for R&D programs, this literature has grown considerably. Brown et al. [5], for example, showed that the Rayleigh distribution suffers from theoretical limitations that make the Weibull function a better model for R&D program expenditures. The authors analyzed 128 completed program budget profiles and used the Weibull in lieu of the Rayleigh model to show that initial budget profile projections would be improved on average by 60%. Burgessa [6] presented four quality metrics to evaluate the suitability of various functional forms, such as Rayleigh and Weibull curves, to fit individual historical program profiles. The author used a case study on military and intelligence satellite acquisition programs to demonstrate his approach. For more details on the forecasting methods and their applications see Makridakis et al. [19].

Few other papers adopt the Buys-Ballot [22] table to forecast public expenditures. The literature has mostly used this decomposition technique as a tool for understanding time series. Iwueze and Nwogu [23], for example, provide an estimation procedure based on the Buys-Ballot table for time series decomposition. Simulated examples showed that the descriptive model outperformed the autoregressive integrated moving average (ARIMA) model in terms of forecasts. In this paper, we try to find the outlay pattern (the targeted fixed percentage of funds to be spent each time) by combining the Norden theory and the Buys-Ballot decomposition.

3. The Model

Consider a time series of departmental proportions of monthly spending during T years. Let e_{ij} be the percentage of funds spent during the year i $(i=1,\ldots,T)$ at the month j $(i=1,\ldots,13)$. To summarize the series and highlight monthly variation, we put the data into a Buys-Ballot table (Table 1).

Table 1: Buys-Ballot Table for monthly expenditures

Year	Month (j)							
(i)	1	2	…	j	…	13	$\bar{e}_{i\cdot}$	$\sigma_{i\cdot}$
1	e_{11}	e_{12}	…	e_{1j}	…	e_{113}	$\bar{e}_{1\cdot}$	$\sigma_{1\cdot}$
2	e_{21}	e_{22}	…	e_{2j}	…	e_{213}	$\bar{e}_{2\cdot}$	$\sigma_{2\cdot}$
…	…	…	…	…	…	…	…	…
i	e_{i1}	e_{i2}	…	e_{ij}	…	e_{i13}	$\bar{e}_{i\cdot}$	$\sigma_{i\cdot}$
…	…	…	…	…	…	…	…	…
T	e_{T1}	e_{T2}	…	e_{Tj}	…	e_{T13}	$\bar{e}_{T\cdot}$	$\sigma_{T\cdot}$
$\bar{e}_{\cdot j}$	$\bar{e}_{\cdot 1}$	$\bar{e}_{\cdot 2}$	…	$\bar{e}_{\cdot j}$	…	$\bar{e}_{\cdot 13}$	$\bar{e}_{\cdot\cdot}$	—
$\sigma_{\cdot j}$	$\sigma_{\cdot 1}$	$\sigma_{\cdot 2}$	…	$\sigma_{\cdot j}$	…	$\sigma_{\cdot 13}$	—	$\sigma_{\cdot\cdot}$

In this table, the main entries in the center are the spent proportions (e_{ij}), each row represents a year's worth of data observed over the *12+1* budget months of the fiscal year[1], and each column represents a particular month's data observed over the T year span of the data. The *i-th* year's average and standard deviation are respectively denoted by $\bar{e}_{i\cdot}$ and $\sigma_{i\cdot}$, while the month's average and standard deviation are denoted by $\bar{e}_{\cdot j}$ and $\sigma_{\cdot j}$. The overall observation average and standard deviation are denoted by $\bar{e}_{\cdot\cdot}$ and $\sigma_{\cdot\cdot}$. Let

$$G(j) = \sum_{t=1}^{j} \bar{e}_{\cdot t} \tag{1}$$

be the observed average value of accumulated proportions at the end of the month j ($j=1,2,\ldots,13$). The value of this observed CDF lies between 0 and 1. Assuming a nonlinear evolution, and denoting by $t \geq 0$ the time, the accumulation of these proportions can be modelled by the following Weibull CDF,

$$F(t) = 1 - e^{-\alpha t^{\beta}}, \tag{2}$$

[1] An additional budget month (13) was introduced to make adjustments to accounts once the year has ended.

where $\alpha > 0$ and $\beta > 0$ represent respectively the scale parameter and the shape parameter. The parameter α determines the steepness of the CDF and the parameter β allows proportions to evolve at other than linear rates. For example, if $\beta = 2$ in (2), then the Weibull model is known as the Rayleigh cumulative distribution function [5]. The Weibull distribution is the leading method in the world for fitting work in progress [21]. *F(t)* can be regarded as the proportion of expenditures spent before the end of the month *t*. It is a continuous monotonically nondecreasing function from 0 to 1.

In order to fit the statistical model to the data set, the appropriate parameters are estimated using non-linear optimization by minimizing the sum of squared errors between the theoretical and actual CDFs. Specifically,

$$Min\left(F(t) - G(t)\right)^2. \tag{3}$$

The Weibull model begins at time zero and has an infinite but thin right tail. To be realistic, a point must be determined that can be considered as the end of the fiscal year [4]. In this study, the minimization was constrained by the completion time, that is

$$F(13) = 1. \tag{4}$$

Once the parameters to fit the empirical distribution are estimated, the modeled cumulative expenditures at time *t* then can be expressed as follows,

$$X(t) = B\left(1 - e^{-at^b}\right), 0 < t < 13 \tag{5}$$

where *a* and *b* are the estimated parameters of α and β respectively, and *B* is the annual budget (budget authority). Subtracting cumulative expenditures of the previous month from the cumulative amount of the month under consideration gives the monthly expenditure amounts. This simple but effective formula allows managers to predict the cumulative expenditures at any time of the year.

4. Application

The best way to understand the application of this theoretical model is to demonstrate its use on some real data. In this section, military expenditures are analyzed for the particular cost centre of the Royal Canadian Air Force (RCAF), the appropriate parameters are estimated, and an expenditure time-phasing is provided. A sensitivity analysis is carried out to assess the impact of varying the parameters a and b on the predicted expenditure profile.

4.1 Data

There are two distinct data types analyzed in this study. The first is the expenditures divided into capabilities or spending units. This data is also aggregated according to specific corporate or agency level managed funds such as military compensation and capital spending. The second details final budget allocations by capabilities and funds. The defence expenditures used in this study were provided by DB. These data represent the spending and budget time series of 42 cost centres and 10 funds. The data sets are available as monthly averages from April 2001-02 to March 2010-11. To use the Weibull methodology, monthly expenditures were converted into inflation-adjusted dollars. The corresponding proportions were put into the Buys-Ballot [22] tables and averaged to get the cumulative spending for each month. To demonstrate the methodology, a case study using the RCAF data is provided.

Since the fiscal year 2001–02, the RCAF expenditures (expressed in 2000 constant dollars) rose from $ 842.85 million to $903.66 million in 2010-11, showing an average annual growth rate of 0.78%. The amount spent by this cost centre in FY 2010-11 represented 5.09 % of the DND expenditures. Inflation adjustment was accomplished by dividing monthly current expenditures by the GDP deflator.

The date on which goods were received or services were rendered dictates the expenditures imputation. Goods and services received on or after 1 April are recorded as new-year expenditures. Those received before the 1st April are paid from old-year funds. Two additional budget months (13 and 14) were introduced to make adjustments to accounts once the year has ended. The difference between the periods 13 and 14 depends on the processing date. Old-year transactions processed from 1st to 18th April are posted to the period 13. Those processed from 1st to 4th May are posted to the period 14. The posting date of all these transactions is 31 Mar of the old-year. The period 14 was not taken into account in this study, because many of its values are negative (revenue). The period 13 expenditures (Adj) were deflated using the annual average of the GDP deflator.

Table 2 contains the proportions spent each month by the RCAF between FY 2001-02 and FY 2010-11. Each column in this table represents the percentage spent (all accounts) in a specific month over these ten years.

Table 2: Buys-Ballot Table for the RCAF - Proportions of expenditures

FY	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Adj	Ratio[2]
2001-02	2%	3%	12%	5%	7%	6%	6%	7%	9%	11%	5%	19%	9%	90%
2002-03	3%	1%	15%	6%	7%	7%	7%	4%	14%	5%	7%	12%	12%	82%
2003-04	2%	4%	11%	9%	6%	12%	8%	5%	16%	5%	9%	18%	-4%	81%
2004-05	5%	1%	12%	6%	7%	10%	7%	7%	6%	11%	7%	18%	3%	82%
2005-06	2%	4%	9%	6%	7%	8%	4%	9%	15%	7%	10%	5%	15%	85%
2006-07	4%	1%	10%	7%	7%	10%	8%	4%	15%	7%	8%	20%	-1%	92%
2007-08	1%	4%	10%	6%	5%	11%	6%	7%	12%	11%	9%	18%	1%	92%
2008-09	4%	2%	8%	8%	7%	8%	9%	7%	14%	8%	8%	9%	7%	96%
2009-10	3%	3%	6%	9%	6%	10%	9%	6%	7%	11%	8%	17%	5%	94%
2010-11	4%	4%	6%	8%	6%	7%	8%	6%	8%	10%	9%	18%	7%	93%
Average	3%	3%	10%	7%	7%	9%	7%	6%	12%	9%	8%	15%	5%	89%
Max	5%	4%	15%	9%	7%	12%	9%	9%	16%	11%	10%	20%	15%	96%
Min	1%	1%	6%	5%	5%	6%	4%	4%	6%	5%	5%	5%	-4%	81%
CDF	3%	6%	16%	22%	29%	38%	45%	51%	63%	71%	79%	95%	100%	-

The overlay plots in Figure 1 show the monthly spending for each year (i.e. row by row plotting of the data). This graph reveals a seasonality of these plots with repeating patterns in terms of peaks and troughs. For example, June, December and March appear to be high spending months, while May continues to be low spending year over year. This seasonality reflects the business planning processes in the department. In the first months of the fiscal year, the department only spends statutory activities such as pay. At this time the annual budget has not yet been approved. By June, the approval to spend yields a spike in spending. By November, supplementary funding to cover pay raises and other incremental operational costs is provided to the department by Treasury Board. This amount is spent in December. The March peak is a direct consequence of "use it or lose it" principle.

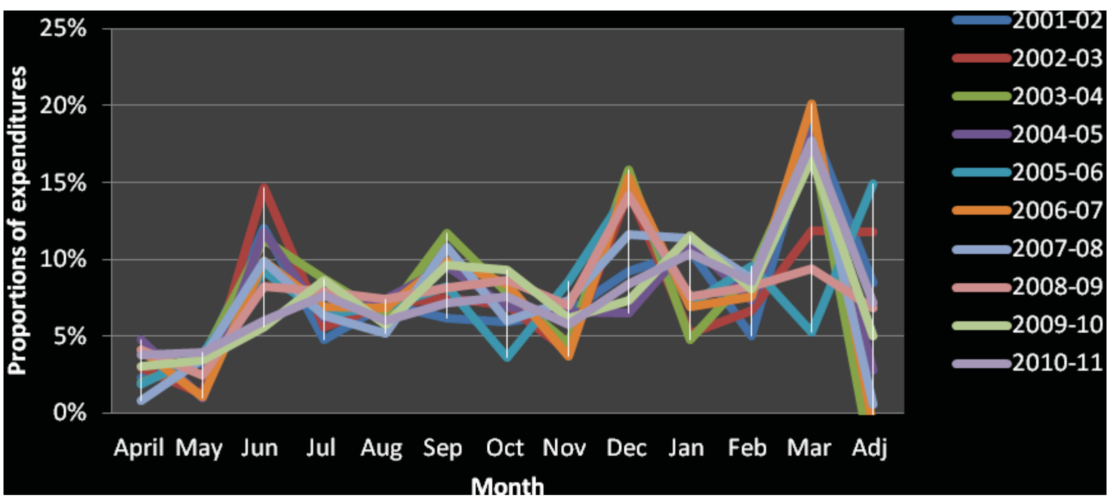


Figure 1: Months-by-year RCAF proportions of expenditures

[2] Ratio = Total expenditures divided by the corresponding budget

The basic trend in the data can be revealed by the yearly averages ($\bar{e}_i$.) as a function of the years. Analyzing the trend would be inappropriate in this case, because the sum of proportions is always 100% and the yearly average is always 7.69% (assuming no lapsed funds).

If all of the disbursements were done on the first of the month, then the empirical CDF would be accurate and sufficient. But if one is also interested in daily or weekly forecasts, it would be easier and more effective to use a functional form.

4.2 Estimation

To choose the Weibull CDF that best fits monthly historical expenditures, we first compute the observed cumulative distribution of the monthly proportions of spending. As shown in Table 2, from FY 2001-02 to FY 2010-11, RCAF spent at the moderate rate of 3% of its budget through the first two months of April and May. From May to June, RCAF real expenditures increased in average by 10%. RCAF cumulative spending grew 12% from November to December. While March consistently had the largest part of total expenditures (15%), May was the weakest month over the ten previous years (3%). In this study each year is given equal weight in the estimation. A sensitivity analysis was also carried out to assess the year's relative importance in the estimation. The analysis indicated that the expenditure profile is insensitive to the year's weights. When the weight of a year increases, every month of that year increases at the same rate leaving the part of each month almost unchanged.

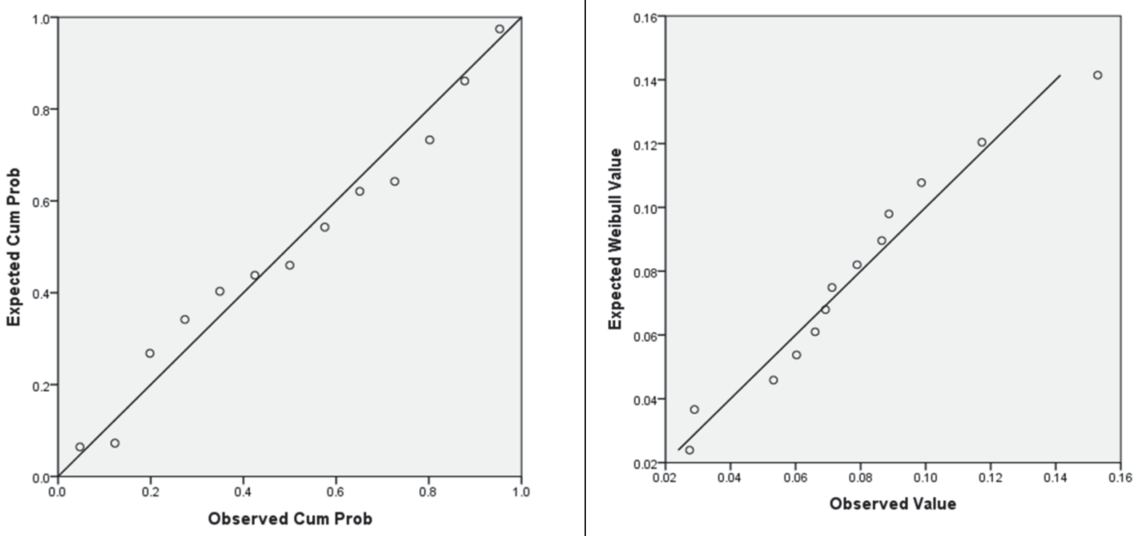
To investigate whether the data are actually from a Weibull distribution, three common goodness-of-fit tests were used: Chi-Square test, Kolmogorov-Smirnov (K-S) test, and Anderson-Darling (A-D) test. Results indicate quantitatively and qualitatively that the data reasonably follow a Weibull distribution.

Table 3 displays for each test the fit statistic and the critical value at $\alpha = 1\%$ level of significance. The fit statistic measures how well the Weibull distribution fits the input data. It measures the deviation of the Weibull distribution from the observations. The smaller the fit statistic is, the better the fit. This table shows that for each test, the fit statistic is below the critical value, indicating that the observations follow the specified distribution.

Table 3: Fit comparison for data

	Fit Statistic	Critical Value
Chi-Sq Statistic	1.08	13.82
K-S Statistic	0.10	0.27
A-D Statistic	0.21	0.98

These findings were also corroborated by the P-P (or Probability-Probability) and Q-Q (or Quantile-Quantile) plots. As shown in Figure 2, the plots are nearly linear and the data points are not seriously deviated from the fitted line. This result indicates that the fit is good and the proportions of expenditures are indeed from a Weibull distribution.



(a) Weibull P-P Plot

(b) Weibull Q-Q Plot

Figure 2: Weibull P-P and Q-Q plots of spending proportions

As shown in Equation 3, Weibull shape and scale parameters are estimated by minimizing the sum of the squared difference between the actual expenditure CDF and the expenditure CDF generated by the Weibull function. As the Weibull model has an infinite right tail, the minimization was constrained by the completion time. The constrained nonlinear search leads to a scale parameter of approximately 0.0188 and a shape parameter of about 1.807. These parameters govern the slope and shape of the cumulative expenditure curve as time increases. They could be obtained by any nonlinear search packages.

4.3 Time-phasing

Once the parameters are derived, we can obtain cumulative expenditure profile and the amount of expenditures expected to occur at any time during the next fiscal year. An outlay profile allowing the expenditures to progress at an appropriate rate reads as follows,

$$\hat{F}(t) = 1 - e^{-at^b}, \quad 0 < t < 13, \tag{6}$$

where t is the forecast horizon, a=*0.0188*, and b=*1.807.* Since the Weibull model has an infinite right tail, an observation must be selected to be considered as the end of the fiscal year [4]. To make the function (6) reach 1, the forecast of any discrete time ($t=1, 2, \ldots, 13$) was replaced by its corresponding value in the observed CDF. Equation (6) assumes no lapsed funds by $t=13$.

The targeted fixed percentage of funds to be spent over the next year is given in Table 3 and illustrated by the baseline scenario of Figure 3.

The cumulative expenditures are obtained by multiplying the targeted percentages by the annual budget. If we assume, for example, that the initial annual budget is $900 million, the monthly and cumulative values of expenditures to be incurred are given in Table 3 (columns 5 and 6 respectively). The monthly expenditure amount is calculated by subtracting the cumulative expenditures of the previous month from the cumulative amount of the current month.

Table 4: Monthly forecasted expenditure profile

Month	Theoretical CDF (in %)			Expenditures (in $M)	
	Target	Lower	Upper	Cumulative	Monthly
Apr	1.86%	1.10%	2.62%	16.78	16.78
May	6.37%	5.74%	7.00%	57.36	40.58
Jun	12.80%	11.50%	14.10%	115.23	57.87
Jul	20.58%	19.56%	21.60%	185.21	69.98
Aug	29.17%	27.85%	30.49%	262.49	77.28
Sep	38.08%	36.40%	39.76%	342.74	80.25
Oct	46.92%	44.78%	49.06%	422.26	79.52
Nov	55.34%	53.63%	57.05%	498.09	75.82
Dec	63.11%	60.07%	66.15%	568.03	69.94
Jan	70.08%	68.12%	72.04%	630.68	62.65
Feb	76.15%	73.73%	78.57%	685.32	54.64
Mar	94.49%	90.95%	98.03%	850.42	165.10
Adj	100.00%	100.00%	100.00%	900.00	49.58

For the next year, the theoretical model forecasts spending about 2% in April, 5% in May, 6% in June, 8% in July, 9% in August, September and October, 8% in November and December, 7% in January, 6% in February, and 18% in March. The 6% remaining is an adjustment.

4.4 Parametric analysis

A parametric analysis was conducted to illustrate what would happen to the expenditure distributions if the key parameter values were different from the baseline scenario. It was performed by freezing all the variables at the baseline values except the one analyzed. Figure 3, in which the green curve provides the results of the baseline scenario, contains the impact of varying the Weibull scale (a) and shape (b) parameters.

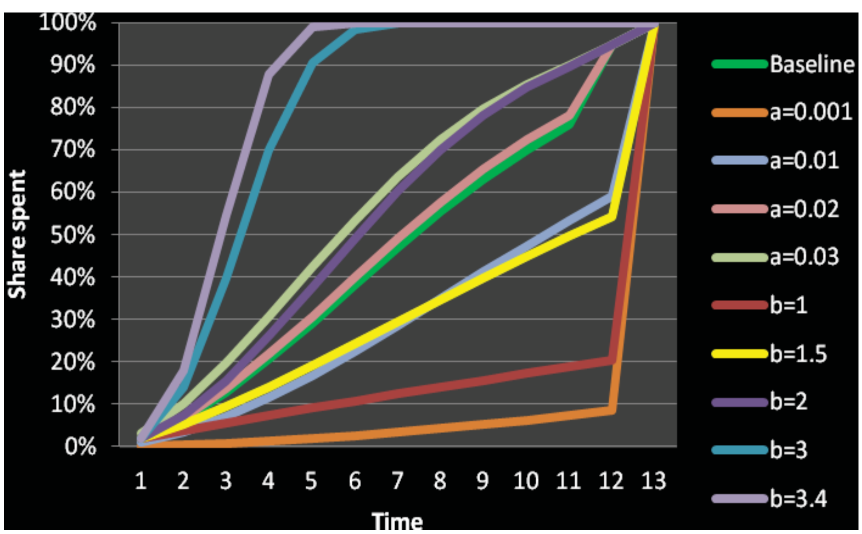


Figure 3: Effects of changing the scale (*a*) and shape (*b*) parameters

This figure mainly highlights the following key points:

- Increasing the value of either *a* or *b* leads to higher expenditures in the first months.
- While varying the scale parameter will only change the steepness of the CDF, it is important to note that the impact of varying *b* on the CDF is much more pronounced. The larger the shape parameter, the shorter is the expenditure duration. This can be attributed to the flexibility of the Weibull distribution. For instance, if *b = 1*, the Weibull distribution becomes an exponential distribution, if *b = 2*, it produces the Rayleigh distribution, and if *b* is between 3 and 4 it approximates the normal distribution.

4.5 Out-of-sample simulation

The objective in this subsection is to perform a resampling test of this methodology. The in-sample period corresponds to the sample used to re-estimate the model. In this simulation, the in-sample data were observed over the last nine years. The out-of-sample test is conducted over one forecasting horizon (the tenth year). We compare the forecasted spending generated by the model to the realized expenditures. We are particularly interested in whether the proportions of the realized expenditures are different to those of the forecasted time-phasing. Table 5 shows the out-of-sample simulation results.

Table 5: Out-of-sample data and forecasts for 2010-11 based on prior years

Month	Realized	Expected		
		Target	Lower	Upper
Apr	3.74%	2.89%	2.09%	3.69%
May	3.90%	2.74%	1.96%	3.52%
Jun	6.03%	9.87%	8.05%	11.69%
Jul	7.69%	6.92%	6.07%	7.77%
Aug	6.04%	6.60%	6.10%	7.10%
Sep	7.19%	8.87%	7.74%	10.00%
Oct	7.58%	7.12%	6.04%	8.20%
Nov	5.78%	6.03%	5.02%	7.04%
Dec	8.38%	11.73%	9.39%	14.07%
Jan	10.35%	8.65%	6.94%	10.36%
Feb	8.59%	7.89%	7.02%	8.76%
Mar	17.59%	15.29%	12.15%	18.43%
Adj	7.14%	5.32%	1.58%	9.06%

This simulation only compares the distributional differences between two related series. It does not determine whether the time-phasing is suitable or not. The model provides financial managers the ability to respond whenever the expenditures are higher or lower than expected. As shown in Figure 4, the actual expenditures in FY 2010-11 were slightly deviating from the expected budget tempo in April and May. The amount spent in the other months was in the 95% Confidence Interval. Armed with this information, budget authorities would have advised RCAF to adjust their expenditures when necessary. The evolution of observed expenditures and the two bounds of spending for RCAF is shown in Figure 4.

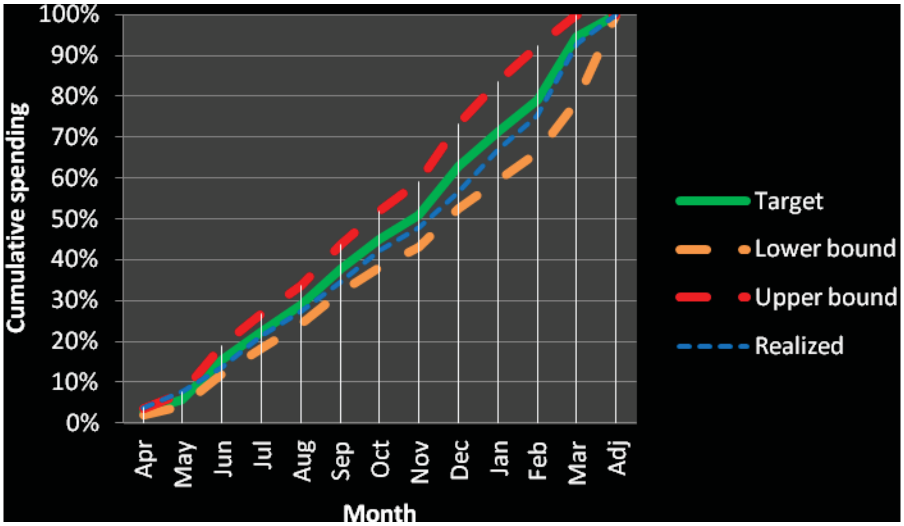


Figure 4: Spent proportions for RCAF in FY 2010-11

5. Conclusion

Expenditure modelling plays an important role in financial management, particularly during times of economic crisis. Because of its sensitivity to the economic cycle, time-phasing of public spending in terms of dollar values is extremely difficult.

This study presents a new hybrid forecasting methodology that makes use of a Buys-Ballot decomposition technique and the Weibull model for cumulative spend. This methodology is based on the concept of cumulative expenditure profile and uses a nonlinear iterative method to derive model parameters.

The proposed methodology was applied to many funds and cost centres. An example using the RCAF data was used in this paper for illustration. For this cost centre, the model parameters were derived, the cumulative expenditure profile was set up and the amounts of expenditures to occur in the next fiscal year were forecasted. A sensitivity analysis was also conducted to address the impact of the model parameters on the results.

The model provides financial managers the ability to respond whenever expenditures are higher or lower than expected. For example, the RCAF case, illustrated in this study report for FY 2010-11, shows that actual expenditures were at times deviating from the expected budget tempo. Armed with this information, budget authorities would have advised RCAF to adjust expenditures. This methodology could be used to identify areas for improvement in budgetary management by any other department within the federal government. It could be employed with both short and long time-series. Future work might consider a more general framework involving more than one cost centre at a time with possible reallocation between them.

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List of abbreviations/acronyms

A-D	Anderson-Darling (Test)
ARIMA	Autoregressive Integrated Moving Average
CDF	Cumulative Distribution Function
DB	Director Budget
DND	Department of National Defence
FY	Fiscal Year
GDP	Gross Domestic Product
K-S	Kolmogorov-Smirnov (Test)
P-P	Probability-Probability (Plot)
Q-Q	Quantile-Quantile (Plot)

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The assessment of financial positions and subsequent reallocation of resources between various capabilities or business lines is at the centre of any private or public sector budgeting process. This in-year budget utilization process is particularly crucial in public sector organizations due to financial regulations that require unspent budget dollars to be “lapsed”. Lapsed funding implies the loss of funding for a given fiscal year, i.e., that money is permanently unavailable to the organization. In order to facilitate the optimal reallocation of budgets among capabilities in-year, a probability-based approach was developed in this study to model expenditure rhythm. Expenditure rhythm is the time-phasing of budget expenditures over a given fiscal year. The methodology was demonstrated in a use case analysis. A sensitivity analysis was conducted to address the impact of model parameters on the results. The proposed model provides financial managers the ability to respond whenever the expenditures are higher or lower than expected.

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Weibull distribution, Time-phasing, Budget expenditures, Resource allocation.