Defence Research and Development Canada
Recherche et développement pour la défense Canada

DEFENCE R&D DÉFENSE

Max-Min SNR: an optimum approach to array antenna signal processing for GPS anti-jamming

Problem statements, theoretical results, and solution strategies

A. Yasotharan

Defence R&D Canada – Ottawa

Technical Memorandum
DRDC Ottawa TM 2011-202
December 2011

Canada

Max-Min SNR: an optimum approach to array antenna signal processing for GPS anti-jamming

Problem statements, theoretical results, and solution strategies

A. Yasotharan

Defence R&D Canada – Ottawa

Technical Memorandum

DRDC Ottawa TM 2011-202

December 2011

Principal Author

Original signed by A. Yasotharan

A. Yasotharan

Approved by

Original signed by Bill Katsube

Bill Katsube
Head/Communications and Navigation Electronic Warfare

Approved for release by

Original signed by Chris McMillan

Chris McMillan
Chief Scientist & Head/Document Review Panel

Abstract

Global Positioning System (GPS) receivers are extremely vulnerable to jamming. One mitigation method is to use a CRPA (Controlled Reception Pattern Antenna) which is an array antenna whose signal outputs are weighted-and-summed and fed into the GPS receiver. Ideally, the weights must be chosen so as to null the jammers and receive with high strength all satellite signals. This is a challenging problem if the GPS receiver has one common input port through which all satellite signals are received. In conventional CRPA, the weights are calculated without considering where the satellites or jammers are located. In this approach, there is limited control over the reception pattern of the array antenna.

In this report, we propose a class of optimal approaches that assume knowledge of the directions of arrival (DOA) of the visible satellites. Within this, three variations are presented: 1) maximize satellites' SINR (Signal-to-Interference-plus-Noise Ratio) assuming that jammers-plus-noise covariance is known, 2) maximize satellites' SNR (Signal-to-Noise Ratio) while nulling the jammers, assuming that jammer DOAs are known, 3) a hybrid of these. The maximization is in the 'max-min SINR' and 'max-min SNR' senses, where the 'min' is over the satellites. We show that all three variations can be reduced to the same generic form where a 'max-min SNR' problem is solved as if there were no jammers. We show a connection between the 'max-min SNR' and 'max weighted-average SNR' problems. We derive several results, including Karush-Kuhn-Tucker conditions, to facilitate the solution of the generic problem, and propose solution strategies.

Résumé

Les récepteurs GPS (système de positionnement global) sont extrêmement vulnérables au brouillage. L'une des façons d'atténuer ce problème est d'utiliser une antenne à diagramme de rayonnement contrôlé (Controlled Reception Pattern Antenna — CRPA), qui est une antenne réseau dont les signaux émis sont pondérés, sommé et acheminer au récepteur GPS. Idéalement, les pondérations choisies doivent permettre d'éliminer le brouillage et de recevoir tous les signaux satellites à pleine puissance, ce qui pose un défi si le récepteur GPS n'a qu'un seul port d'entrée courant pour recevoir tous les signaux satellites. Dans une CRPA classique, les pondérations sont calculées sans tenir compte de l'emplacement des satellites ou des brouilleurs, ce qui offre un contrôle limité sur le diagramme de rayonnement de réception de l'antenne réseau.

Dans le présent rapport, on propose un ensemble de méthodes optimales fondées sur l'hypothèse que les directions d'arrivées des satellites visibles sont connues. Dans cet ensemble, on décrit trois variations. La première consiste à maximiser le rapport ≪ signal sur brouillage plus bruit ≫ (S/BB) en prenant pour acquis que la covariance ≪ brouilleurs plus bruit ≫ est connue, la deuxième, à maximiser le rapport ≪ signal sur bruit ≫ (S/B) en neutralisant les brouilleurs et en prenant pour acquis que les directions d'arrivées des brouilleurs sont connues et la dernière, à employer une méthode hybride combinant les deux premières. La maximisation est dans un contexte ≪ S/BB max-min ≫ et ≪ S/B max min ≫, où le ≪ min

≫ est supérieur à celui des satellites. Dans le rapport, on montre que les trois variations peuvent être réduites à une seule forme générique d'après laquelle un problème ≪ S/B max-min ≫ peut être résolu comme s'il n'y avait pas de brouilleurs. On montre également un lien entre les problèmes de ≪ S/B max min ≫ et de ≪ S/B par pondération moyenne maximale ≫. Plusieurs résultats sont ainsi obtenus, y compris des conditions de Karush Kuhn Tucker, afin de faciliter la résolution du problème générique et de proposer des stratégies de solution.

Executive summary

Max-Min SNR: an optimum approach to array antenna signal processing for GPS anti-jamming

A. Yasotharan; DRDC Ottawa TM 2011-202; Defence R&D Canada – Ottawa; December 2011.

Background: Global Positioning System (GPS) receivers are extremely vulnerable to jamming. One mitigation method is to use a CRPA (Controlled Reception Pattern Antenna) which is an array antenna whose signal outputs are weighted-and-summed and fed into the GPS receiver. Ideally, the weights must be chosen so as to null the jammers and receive with high strength all satellite signals. This is a challenging problem if the GPS receiver has one common input port through which all satellite signals are received. In conventional CRPA, the weights are calculated without considering where the satellites or jammers are located. In this approach, there is limited control over the reception pattern of the array antenna.

Principal results: We propose a class of optimal approaches that assume knowledge of the directions of arrival (DOA) of the visible satellites. Within this, three variations are presented: 1) maximize satellites' SINR (Signal-to-Interference-plus-Noise Ratio) assuming that jammers-plus-noise covariance is known, 2) maximize satellites' SNR (Signal-to-Noise Ratio) while nulling the jammers, assuming that jammer DOAs are known, 3) a hybrid approach where the satellites' SINR is maximized while nulling a subset of jammers. The maximization is in the 'max-min SINR' and 'max-min SNR' senses, where the 'min' is over the satellites. We show that all three variations can be reduced to the same generic form where a 'max-min SNR' problem is solved as if there were no jammers. We show a connection between the 'max-min SNR' and 'max weighted-average SNR' problems. We derive several results, including Karush-Kuhn-Tucker conditions, to facilitate the solution of the generic problem, and propose solution strategies.

Significance of results: The results provide a theoretical framework for optimally receiving satellite signals in the presence of jammers. If there is sufficient DOA-separation between the satellites and jammers, the proposed approaches would allow more accurate pseudo-range measurements which in turn would provide more accurate measurements of GPS user position and velocity. If a satellite and a jammer have nearly the same DOA, then this satellite may weigh down the SINR of the other satellites. The framework would help to identify such satellites and exclude them so that the other satellites may achieve high SINR.

Future work: 1) The proposed solution strategies should be implemented and tested, 2) An efficient method should be developed for adapting to changes of problem parameters such as number and DOA of satellites and jammers and jammers-plus-noise covariance, 3) The sensitivity of the solution to problem parameters should be studied.

Sommaire

Max-Min SNR: an optimum approach to array antenna signal processing for GPS anti-jamming

A. Yasotharan ; DRDC Ottawa TM 2011-202 ; R & D pour la défense Canada – Ottawa ; décembre 2011.

Introduction : Les récepteurs GPS (système de positionnement global) sont extrêmement vulnérables au brouillage. L'une des façons d'atténuer ce problème est d'utiliser une antenne à diagramme de rayonnement contrôlé (Controlled Reception Pattern Antenna — CRPA), qui est une antenne réseau dont les signaux émis sont pondérés, sommé et acheminer au récepteur GPS. Idéalement, les pondérations choisies doivent permettre d'éliminer le brouillage et de recevoir tous les signaux satellites à pleine puissance, ce qui pose un défi si le récepteur GPS n'a qu'un seul port d'entrée courant pour recevoir tous les signaux satellites. Dans une CRPA classique, les pondérations sont calculées sans tenir compte de l'emplacement des satellites ou des brouilleurs, ce qui offre un contrôle limité sur le diagramme de rayonnement de réception de l'antenne réseau.

Résultats : Dans le présent rapport, on propose un ensemble de méthodes optimales fondées sur l'hypothèse que les directions d'arrivées des satellites visibles sont connues. Dans cet ensemble, on décrit trois variations. La première consiste à maximiser le rapport ≪ signal sur brouillage plus bruit ≫ (S/BB) en prenant pour acquis que la covariance ≪ brouilleurs plus bruit ≫ est connue, la deuxième, à maximiser le rapport ≪ signal sur bruit ≫ (S/B) en neutralisant les brouilleurs et en prenant pour acquis que les directions d'arrivées des brouilleurs sont connues et la dernière, à employer une méthode hybride selon laquelle le rapport S/BB des satellites est maximisé tout en neutralisant un sous-ensemble de brouilleurs. La maximisation est dans un contexte ≪ S/BB max-min ≫ et ≪ S/B max min ≫, où le ≪ min ≫ est supérieur à celui des satellites. Dans le rapport, on montre que les trois variations peuvent être réduites à une seule forme générique d'après laquelle un problème ≪ S/B max-min ≫ peut être résolu comme s'il n'y avait pas de brouilleurs. On montre également un lien entre les problèmes de ≪ S/B max min ≫ et de ≪ S/B par pondération moyenne maximale ≫. Plusieurs résultats sont ainsi obtenus, y compris des conditions de Karush Kuhn Tucker, afin de faciliter la résolution du problème générique et de proposer des stratégies de solution.

Portée : Les résultats constituent un cadre théorique pour recevoir de façon optimale les signaux satellites en présence de brouilleurs. Si les directions d'arrivée sont suffisamment espacées entre les satellites et les brouilleurs, les méthodes proposées peuvent permettre des mesures de pseudo distance plus exactes et, par conséquent, des mesures plus exactes de la position et de la vitesse de l'utilisateur du GPS. Dans le cas où un satellite et un brouilleur ont presque la même direction d'arrivée, le premier peut entraîner une diminution de la pondération du rapport S/BB d'autres satellites. Le cadre proposé aiderait à identifier de

telles situations et à exclure les satellites en cause, afin que d'autres puissent atteindre un rapport S/BB plus élevé.

Recherches futures : 1) Les stratégies de solution proposées doivent être appliquées et mises à l'essai ; 2) Une méthode efficace doit être élaborée pour assurer une adaptation aux changements de paramètres de problèmes (nombre de satellites et de brouilleurs, directions d'arrivée de ceux ci, covariance « brouilleurs plus bruit », etc.) ; 3) La sensibilité de la solution aux paramètres de problémes doit être étudiée.

This page intentionally left blank.

Table of contents

List of figures

Acknowledgements

I would like to thank Dr. Sichun Wang (CNEW) for reviewing this report and providing many comments that helped to improve its readability.

I would also like to thank Mr. Bruce Liao (CNEW) and Mr. Jeff Bird (CNEW) for their comments on the front matter and introduction.

1 Introduction

The Global Positioning System (GPS) allows users to obtain position, velocity and time information from signals transmitted by satellites. These signals, when received on earth, are very weak, and therefore, ordinary GPS receivers are extremely vulnerable to interference and jamming [1] (chapter 10), [2], [3]. As civilian infrastructures and military operations become increasingly reliant on the GPS, and as jamming equipment proliferate, the need to protect GPS receivers becomes extremely important. This need is being addressed by Defence Research & Development Canada's Navigation Warfare (NAVWAR) program.

Array antennas, combined with signal processing, have the capability to discriminate signals based on Direction of Arrival (DOA). Therefore, array antenna signal processing is a promising approach to mitigating jammers. When the outputs of an array antenna are combined through a *weight-and-sum* operation, the gain with which a signal or jammer is received depends on the DOA. This functional dependance on DOA - called the *reception pattern* - can be shaped, to an extent, to have nulls towards jammmers and high values towards the satellites, by appropriately choosing the weights. This is the idea behind the Controlled Reception Pattern Antenna (CRPA).

There are different types of weight-and-sum operation. The simplest type is where antenna outputs measured only at the same time instant across the array are included. This is called *spatial combining* (SC) to differentiate it from *space-time combining* (STC) or *space-frequency combining* (SFC) where array outputs measured over more than one time instant are included in the weight-and-sum. SC is sufficient for small arrays. STC or SFC is needed if the array is large or if there is significant multipath. In this report, we only consider SC.

To put this report in proper context, we have to distinguish between the 'legacy GPS receiver' which was originally meant to be used with a single-element antenna and the 'advanced GPS receiver' meant to be used with a beam-steering array antenna. More technically, a legacy GPS receiver is one which has one common input port through which all satellite signals are received, and an advanced GPS receiver is one which has multiple input ports, each of which can be allocated to a specific satellite. When protecting a legacy GPS receiver with a CRPA, the same set of weights will apply to all satellites, because the result of the weight-and-sum is fed into the common input port of the receiver. In contrast, when protecting an advanced GPS receiver with a beam-steering CRPA, each satellite will have its own set of weights and the result of the weight-and-sum will be fed into the receiver input port allocated to that satellite. Advanced GPS receivers are under development [4].

This report is primarily about protecting legacy GPS receivers, a large number of which are in the possesion of the Canadian Forces and its allies.

One approach to calculating the weights in a CRPA for a legacy receiver is based on *power minimization*, where the weights are chosen so as to minimize the total output power, subject to some normalizing constraint. It requires knowledge of only the statistics (covariance matrix) of the outputs of the array antenna. It does not require knowledge of locations of jammers or satellites. The weights and hence the reception pattern depend on the jammer

DOAs and power levels, but not on satellite DOAs because satellite signals are too weak to influence the weights. Moreover, the method does not guarantee uniformly high gain in all directions away from the jammers, and is susceptible to forming parasitic nulls.

The set of weights that can null a combination of jammers is not unique in general. If there are M antenna elements and L jammers, where $L < M$, there are $(M - L)$ *degrees of freedom* in choosing a set of weights that will null all the jammers. It is highly desirable to be able to exploit these degrees of freedom and maximize the array gains towards the satellites, rather than settle for the one set of weights given by power minimization, for example. This is the motivation behind this report.

In this report, we assume to have knowledge of the DOAs of the visible satellites,[1] and propose a class of optimal approaches to calculate the array weights for a legacy GPS receiver. Within this class, we propose three variations that aim to achieve high gain towards the satellites while mitigating or nulling the jammers:

1. Maximize the minimum SINR (Signal-to-Interference-plus-Noise Ratio) of the satellites, assuming knowledge of the total covariance matrix of the jammers and noise. Thus the jammers are mitigated but not perfectly nulled.
2. Maximize the minimum SNR (Signal-to-Noise Ratio) of the satellites, while (perfectly) nulling the jammers, assuming knowledge of the DOAs of the jammers.
3. This is a hybrid of the first and second. In this, we (perfectly) null a subset of jammers, based on knowledge of their DOAs, and maximize the minimum SINR of the satellites, based on knowledge of the covariance matrix of the remaining (non-nulled, mitigated) jammers and noise.

The above mentioned assumptions will be justified in Section 4.

Although, the above optimal approaches were developed with the legacy GPS receiver in mind, the approaches are applicable to a GPS receiver that lies in-between a legacy receiver and an advanced receiver (separate input for every visible satellite). If we have a GPS receiver with multiple inputs, each of which can be allocated to a group of satellites rather than just one satellite, then we can optimally calculate a set of weights for each group of satellites (equivalently, each receiver input) using the proposed approach. Such a receiver will likely be cheaper to manufacture than a receiver with a separate input for every visible satellite. Moreover, such a receiver will be adequate in situations where the array antenna, due to space or cost limitation, does not have enough elements to form a sharp beam towards each satellite, but has enough elements to form broad beams towards different sectors of the sky.

The overall contribution of this report is to provide a theoretical framework for optimally choosing the weights of an array antenna for protecting a legacy GPS receiver via the weight-and-sum operation. The specific contributions are described next, together with the organization of the report.

---

1. This assumption is also made in beam-steering array antennas used with advanced GPS receivers.

1.1 Contributions and Organization of the Report

Section 2 presents the array antenna signal processing theory relevant to this report. Section 3 describes the GPS anti-jamming system where the signal processing approaches developed in this report will be applied. Section 4 states the assumptions on knowledge about satellites and jammers made in subsequent sections.

The Sections 5, 6, and 7 state optimization problems whose solutions will constitute optimal approaches to choosing the weights of an array antenna. In each section, the optimization problem is reduced to a generic mathematical form which is treated in the rest of the report. This generic form is what we would get if there were no jammers and we maximize the minimum SNR. Therefore we call the generic problem 'Max-Min SNR' problem.

Section 8 presents several theoretical results to facilitate the solution of the 'Max-Min SNR' problem. These include Karush-Kuhn-Tucker (KKT) conditions for optimality, a canonical form for optimal weights, and a method of solving the problem assuming the active set (satellites which will have minimum SNR) is known. Section 8 also presents a solution strategy which is combinatorial in nature and finds the active set.

Section 9 states variations of the above optimization problems by replacing 'minimum over satellites' with 'weighted-average over satellites', and presents their generic form - 'Maximum Weighted-Average SNR' problem - and its solution. [2] While the latter solution is of practical interest as a method of choosing the array antenna's weights, it has a theoretical use in developing an alternative solution strategy for the 'Max-Min SNR' problem.

Based on the 'Maximum Weighted-Average SNR' solution, Section 10 presents another optimization problem - 'Minimize the Largest Eigenvalue' problem - whose optimal objective value is shown to be an upper bound to the optimal objective value of the 'Max-Min SNR' problem. Section 10 then presents a theorem which says that under a condition the solution of the 'Minimize the Largest Eigenvalue' problem directly gives the solution of the 'Max-Min SNR' problem. The 'Minimize the Largest Eigenvalue' problem is a convex problem and hence efficiently solvable. Based on these, Section 10 presents a strategy for solving the 'Max-Min SNR' problem.

Section 11 draws conclusions and makes suggestions for further work. Tedious mathematical derivations and proofs are given in the Annexes.

2. The weights with which the SNRs of satellites are averaged should be distinguished from the weights applied to the array antenna outputs.

2 Array Antenna Signal Model

In this section, the basic theory behind array antenna signal processing is presented. More advanced and detailed treatment can be found in [5], for example.

2.1 Steering Vector Model for Narrowband Signals

Consider an antenna array with M identical and isotropic elements and assume that there is no mutual electromagnetic coupling between the antenna elements. A vector consisting of measurements of complex envelopes (I and Q signals) taken across the array at the same instant is called a *snapshot*. When there is only one narrowband signal, or interference, coming from direction θ, a snapshot at time t will have the form

$$\mathbf{x}(t) = a(t)\mathbf{v}(\theta) \tag{1}$$

where $a(t)$ is the complex envelope seen at the first element and the vector $\mathbf{v}(\theta)$ is the *steering vector*

$$\mathbf{v}(\theta) = \left[1, e^{j\tau_2(\theta)}, \ldots, e^{j\tau_M(\theta)}\right]^T \tag{2}$$

where $\tau_n(\theta)$ is based on the delay at the n^{th} element w.r.t. the first element.

If there are L signals impinging on the array from directions $\theta_1, \theta_2, \ldots, \theta_L$, then the snapshot $\mathbf{x}(t)$ will have the form

$$\mathbf{x}(t) = \sum_{i=1}^{L} a_i(t)\mathbf{v}(\theta_i). \tag{3}$$

In practice the snapshot $\mathbf{x}$ will also contain additive noise which is assumed to be zero-mean i.i.d. (independent and identically distributed) over antenna elements and snapshots.

The above steering vector model can be extended to an array of non-indentical and non-isotropic elements in a straightforward manner. See [5] for details.

2.2 Weight-and-Sum Operation, Nulling

Let $[w_1, w_2, \ldots, w_M]$ be a vector of complex-valued weights and denote

$$\mathbf{w} = [w_1^*, w_2^*, \ldots, w_M^*]^T. \tag{4}$$

Let $\mathbf{x} = [x_1, x_2, \ldots, x_M]^T$ be a snapshot of the array. Then the weight-and-sum operation

$$\sum_{n=1}^{M} w_n x_n \tag{5}$$

can be formally written, in linear algebra terms, as the inner product

$$\mathbf{w}^H \mathbf{x}. \tag{6}$$

The effect of this weight-and-sum operation on a signal, or interference, coming from direction θ is that its complex-valued amplitude is multiplied by

$$\mathbf{w}^H \mathbf{v}(\theta) \tag{7}$$

where $\mathbf{v}(\theta)$ is the steering vector. Thus the magnitude of the quantity of (7), treated as a function of θ, is called the *reception pattern* of the array.

If an interference is coming from direction θ, then it can be nulled by choosing $\mathbf{w}$ such that

$$\mathbf{w}^H \mathbf{v}(\theta) = 0. \tag{8}$$

In linear algebra terms, we say that $\mathbf{w}$ is orthogonal to $\mathbf{v}(\theta)$. In choosing such a weight vector $\mathbf{w}$, we have $M-1$ degrees of freedom. If L independent interferences have to be nulled, where $L < M$, then (8) must be satisfied for every interference, and in choosing such a $\mathbf{w}$, we have $M-L$ degrees of freedom.[3] The available degrees of freedom can be used to maximize the reception in some particular direction or even to shape the reception pattern over a region of interest. Thus by appropriately choosing $\mathbf{w}$, we can *control* the reception pattern, which is the idea behind Controlled Reception Pattern Antenna (CRPA).

2.3 SINR or SNR after Weight-and-Sum

Suppose a desired signal and some interferences are impinging on the array antenna. Then the array snapshot $\mathbf{x}(t)$ will have the form

$$\mathbf{x}(t) = a(t)\mathbf{s} + \mathbf{y}(t) \tag{9}$$

where $\mathbf{s}$ is the steering vector and $a(t)$ is the complex amplitude of the desired signal, and $\mathbf{y}(t)$ is due to the interferences and noise alone. The desired signal has power $p = E[|a|^2]$.

Assuming $\mathbf{y}(t)$ is second-order stationary with zero-mean, the SINR (Signal-to-Interference-plus-Noise Ratio) of the desired signal in the weight-and-sum $\mathbf{w}^H\mathbf{x}$ is

$$\text{SINR} = p \left|\mathbf{s}^H \mathbf{w}\right|^2 / \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{10}$$

where $\mathbf{R} = E[\mathbf{y}\mathbf{y}^H]$.

If the weight vector $\mathbf{w}$ is chosen so as to null *all* the interferences in $\mathbf{y}$, then the SNR (Signal-to-Noise Ratio) of the desired signal in the weight-and-sum $\mathbf{w}^H\mathbf{x}$ is

$$\text{SNR} = p \left|\mathbf{s}^H \mathbf{w}\right|^2 / \left(\mathcal{N}\|\mathbf{w}\|^2\right) \tag{11}$$

where $\mathcal{N}$ is the variance of the zero-mean i.i.d. noise in each element of $\mathbf{y}$.

If the weight vector $\mathbf{w}$ is chosen so as to null a *subset* of the interferences in $\mathbf{y}$, then the SINR will be given by (10), but with the difference that $\mathbf{R}$ is now only due to the non-nulled interferences and noise.

3. This is a lower bound which assumes that the steering vectors are linearly independent, which is typically the case if $L < M$.

3 The Proposed GPS Anti-Jamming System

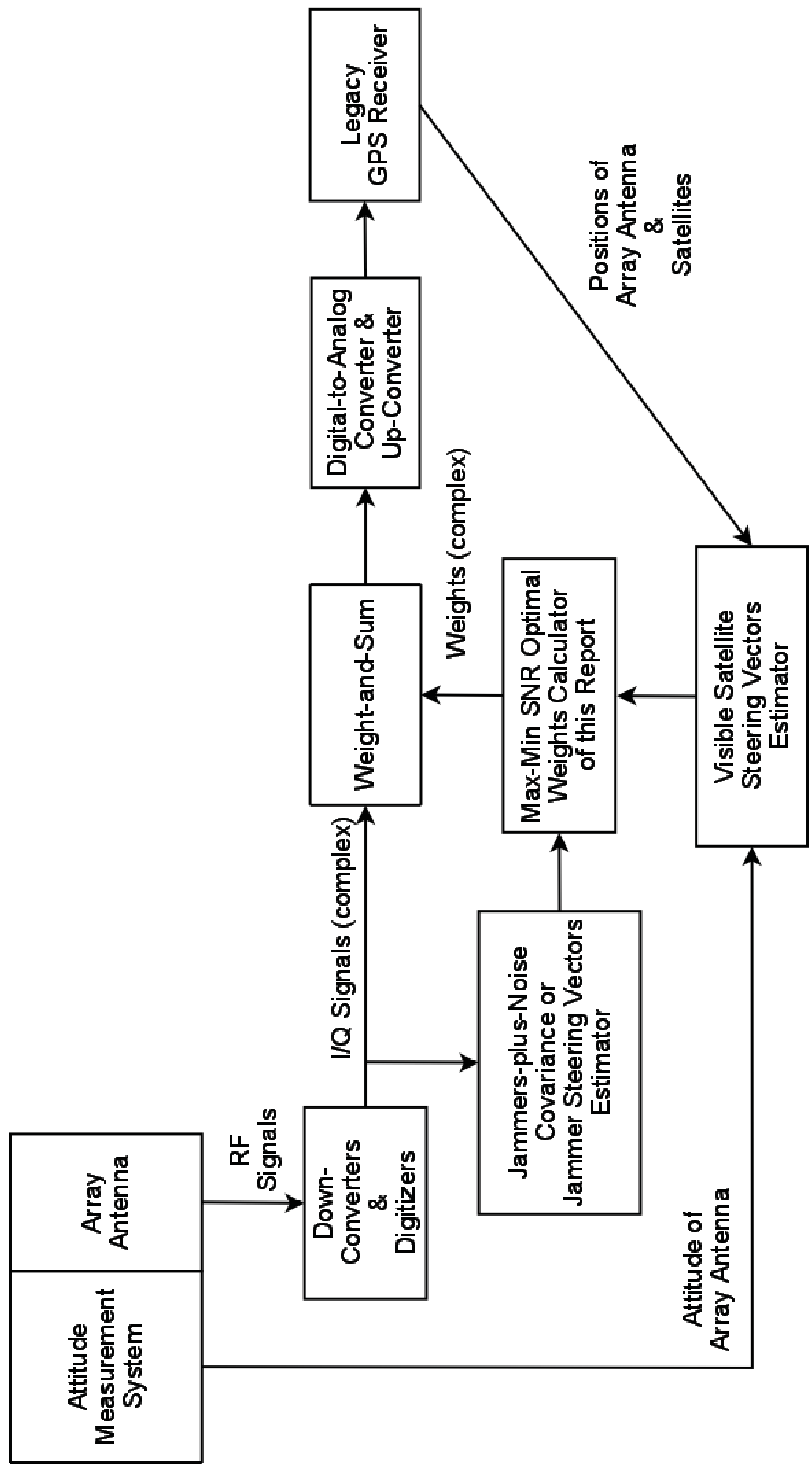

Figure 1: *The Proposed GPS Anti-Jamming System*

4 Assumptions used in this Report

The optimal methods of this report assume certain knowledge about satellites and jammers in relation to the array antenna. These assumptions are stated below together with justification. Note that knowledge of Directions of Arrival (DOAs), mentioned in the introduction, is replaced with knowledge of steering vectors.

Assumption 1 (Used in all methods) *The steering vectors of the visible satellites are known. The received power levels of satellites are equal.*

First, the DOAs of the satellites can be calculated from the measurement of the attitude of the array antenna and feedback from the GPS receiver about positions of the array and the satellites. Then the steering vectors can be obtained using the array antenna's manifold, i.e. mapping from DOA to steering vector. GPS has been designed so that received power levels of satellites are equal.

Assumption 2 (Used in 'Max-Min SINR') *The array output vector due to interference from jammers and noise has zero mean and known positive definite covariance matrix* $\mathbf{R}$.

Because the received satellite signal powers are negligible compared to the noise power, $\mathbf{R}$ can be estimated from the data received by the array antenna. Due to noise, $\mathbf{R}$ will be positive definite.

Assumption 3 (Used in 'Max-Min SINR') *The interference-plus-noise seen in the reception of any satellite's signal is the same for all satellites.*

This is because the received satellite signal powers are negligible compared to the noise power.

Assumption 4 (Used in 'Max-Min SNR with Nulling') *The steering vectors of all the jammers are known.*

These can be estimated, from the data received by the array antenna, by using any direction finding method, e.g. MUSIC, Maximum-Likelihood [5], assuming that the jammer power levels are high enough.

Assumption 5 (Used in 'Max-Min SINR with Partial Nulling') *The steering vectors of a subset of jammers are known. The array output vector due to noise and interference from the other jammers has zero mean and known positive definite covariance matrix* $\mathbf{R}$.

After estimating the steering vectors and power levels of a subset of jammers, their contribution to the interference can be subtracted to obtain $\mathbf{R}$ of the other jammers.

5 'Max-Min SINR' Optimization Problem

In this section, we propose to choose the weight vector $\mathbf{w}$ of the spatial combining array antenna so as to maximize the minimum SINR (Signal-to-Interference-plus-Noise Ratio) of the visible satellites. Denoting the satellite index by i, the optimization problem we wish to solve is

$$\max_{\mathbf{w}} \min_{i} \mathrm{SINR}_i. \tag{12}$$

We do this under Assumptions 1, 2, and 3 stated in Section 4.

Suppose there are K visible satellites, and their steering vectors are $\{\mathbf{s}_i : i = 1, 2, \ldots, K\}$. Then the SINR of the i^{th} satellite is (see (10))

$$\mathrm{SINR}_i = p_i \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{13}$$

where p_i is the satellite's signal power. Henceforth we will not show p_i as it can be assumed to be constant over i or absorbed into the steering vector $\mathbf{s}_i$.

So the optimization problem we wish to solve is

$$\max_{\mathbf{w}} \min_{i} \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \mathbf{w}^H \mathbf{R} \mathbf{w}. \tag{14}$$

We can transform the problem as follows. We can factorize $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{C}^H \mathbf{C} \tag{15}$$

for some square matrix $\mathbf{C}$, e.g. the Cholesky factor or the square root[6]. Since $\mathbf{R}$ is positive definite, $\mathbf{C}$ is invertible.

Define

$$\mathbf{u} = \mathbf{C}\mathbf{w} \tag{16}$$

$$\mathbf{v}_i = \mathbf{C}^{-H}\mathbf{s}_i \qquad \text{for } i = 1, 2, \ldots, K. \tag{17}$$

Then

$$\mathbf{s}_i^H \mathbf{w} = \mathbf{s}_i^H \mathbf{C}^{-1}\mathbf{C}\mathbf{w} = \mathbf{v}_i^H \mathbf{u} \tag{18}$$

$$\mathbf{w}^H \mathbf{R} \mathbf{w} = \mathbf{w}^H \mathbf{C}^H \mathbf{C} \mathbf{w} = \mathbf{u}^H \mathbf{u}. \tag{19}$$

Using the above, the optimization problem (14) can be written as

$$\max_{\mathbf{u}} \min_{i} \left|\mathbf{v}_i^H \mathbf{u}\right|^2 / \|\mathbf{u}\|^2. \tag{20}$$

Denote by $\mathbf{u}_{opt}$ the optimum solution of the above problem. Then the optimum solution $\mathbf{w}_{opt}$ of problem (14) is given by

$$\mathbf{w}_{opt} = \mathbf{C}^{-1}\mathbf{u}_{opt}. \tag{21}$$

The optimization problem (20) is treated from Section 8 onwards as the generic 'Max-Min SNR' problem.

5.1 An Implementation Consideration

To implement the above approach we need **C** rather than **R**. The Cholesky factor **C** can be directly estimated from the data by Givens rotations which can be efficiently implemented in hardware [7].

6 'Max-Min SNR with Nulling' Optimization Problem

In this section, we propose to choose the weight vector $\mathbf{w}$ of the spatial combining array antenna so as to maximize the minimum SNR (Signal-to-Noise Ratio) of the visible satellites, while nulling all the jammers. Denoting the satellite index by i, the optimization problem we wish to solve is

$$\max_{\mathbf{w}} \min_{i} \mathrm{SNR}_i. \tag{22}$$

We do this under Assumptions 1 and 4 stated in Section 4.

Suppose there are L jammers and their steering vectors are $\{\mathbf{q}_l : l = 1, 2, \ldots, L\}$. Then the weight vector $\mathbf{w}$ has to satisfy the constraints

$$\mathbf{q}_l^H \mathbf{w} = 0 \qquad \text{for } l = 1, 2, \ldots, L. \tag{23}$$

This is feasible if $L < M$ where M is the number of antenna elements.

Suppose there are K visible satellites, and their steering vectors are $\{\mathbf{s}_i : i = 1, 2, \ldots, K\}$. Assuming that the noise in the outputs of the antenna elements is i.i.d. (independent and identically distributed) with variance $\mathcal{N}$, the SNR of the i^{th} satellite is (see (11))

$$\mathrm{SNR}_i = (p_i/\mathcal{N}) \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \tag{24}$$

where p_i is the satellite's signal power. Henceforth we will not show $(p_i/\mathcal{N})$ as it can be assumed to be constant over i or absorbed into the steering vector $\mathbf{s}_i$.

So the optimization problem we wish to solve is

$$\max_{\mathbf{w}} \min_{i} \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \tag{25}$$

subject to the nulling constraints (23).

We can transform the problem as follows. Denote $\mathcal{Q} = \operatorname{span}\{\mathbf{q}_l : l = 1, 2, \ldots, L\}$. The orthogonal complement of $\mathcal{Q}$ is denoted by $\mathcal{Q}^\perp$. Then (23) implies that $\mathbf{w} \in \mathcal{Q}^\perp$. Let $\mathbf{s}_i = \mathbf{v}_i + \mathbf{u}_i$ be the orthogonal decomposition such that $\mathbf{v}_i \in \mathcal{Q}^\perp$ and $\mathbf{u}_i \in \mathcal{Q}$. Then $\mathbf{s}_i^H \mathbf{w} = \mathbf{v}_i^H \mathbf{w}$. Now we have the equivalent optimization problem

$$\max_{\mathbf{w}} \min_{i} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \tag{26}$$

subject to the constraints (23), or equivalently $\mathbf{w} \in \mathcal{Q}^\perp$. The latter constraint can be ignored because the unconstrained optimum $\mathbf{w}$ of (26) is in $\operatorname{span}\{\mathbf{v}_i : i = 1, 2, \ldots, K\} \subset \mathcal{Q}^\perp$, and hence (23) is automatically satisfied. This will be shown in Section 8.

The optimization problem (26) is treated from Section 8 onwards as the generic 'Max-Min SNR' problem.

6.1 An Implementation Consideration

Note that we do not necessarily need the jammer steering vectors $\{\mathbf{q}_l\}$. Rather, any basis for their span $\mathcal{Q}$ will be sufficient. Estimation of $\mathcal{Q}$ (in terms a basis) usually precedes the estimation of $\{\mathbf{q}_l\}$.

7 'Max-Min SINR with Partial Nulling' Optimization Problem

In this section, we propose a method which is the hybrid of the methods proposed in Sections 5 and 6. Specifically, we propose to choose the weight vector $\mathbf{w}$ of the spatial combining array antenna so as to maximize the minimum SINR (Signal-to-Interference-plus-Noise Ratio) of the visible satellites, while nulling a subset of the jammers.

We do this under Assumptions 1, 3, and 5 stated in Section 4.

Suppose the L jammers which we want to null have steering vectors $\{\mathbf{q}_l : l = 1, 2, \ldots, L\}$. Then the weight vector $\mathbf{w}$ has to satisfy the constraints

$$\mathbf{q}_l^H \mathbf{w} = 0 \qquad \text{for } l = 1, 2, \ldots, L. \tag{27}$$

This is feasible if $L < M$ where M is the number of antenna elements.

Suppose there are K visible satellites, and their steering vectors are $\{\mathbf{s}_i : i = 1, 2, \ldots, K\}$. Then the SINR of the i^{th} satellite, after nulling the L jammers, is (see last paragraph of Section 2.3)

$$\text{SINR}_i = p_i \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{28}$$

where p_i is the satellite's signal power. Henceforth we will not show p_i as it can be assumed to be constant over i or absorbed into the steering vector $\mathbf{s}_i$.

So the optimization problem we wish to solve is

$$\max_{\mathbf{w}} \min_{i} \left|\mathbf{s}_i^H \mathbf{w}\right|^2 / \mathbf{w}^H \mathbf{R} \mathbf{w} \tag{29}$$

subject to the nulling constraints (27).

We can transform the problem as follows. We can factorize $\mathbf{R}$ as

$$\mathbf{R} = \mathbf{C}^H \mathbf{C} \tag{30}$$

for some square matrix $\mathbf{C}$, e.g. the Cholesky factor or the square root[6]. Since $\mathbf{R}$ is positive definite, $\mathbf{C}$ is invertible.

Define

$$\begin{aligned} \mathbf{u} &= \mathbf{C}\mathbf{w} && (31)\\ \mathbf{p}_l &= \mathbf{C}^{-H}\mathbf{q}_l \qquad \text{for } l = 1, 2, \ldots, L && (32)\\ \mathbf{r}_i &= \mathbf{C}^{-H}\mathbf{s}_i \qquad \text{for } i = 1, 2, \ldots, K. && (33) \end{aligned}$$

Then

$$\begin{aligned} \mathbf{s}_i^H \mathbf{w} &= \mathbf{s}_i^H \mathbf{C}^{-1} \mathbf{C} \mathbf{w} = \mathbf{r}_i^H \mathbf{u} && (34)\\ \mathbf{q}_l^H \mathbf{w} &= \mathbf{q}_l^H \mathbf{C}^{-1} \mathbf{C} \mathbf{w} = \mathbf{p}_l^H \mathbf{u} && (35)\\ \mathbf{w}^H \mathbf{R} \mathbf{w} &= \mathbf{w}^H \mathbf{C}^H \mathbf{C} \mathbf{w} = \mathbf{u}^H \mathbf{u}. && (36) \end{aligned}$$

Using the above, the optimization problem (29) can be written as

$$\max_{\mathbf{u}} \min_{i} \left|\mathbf{r}_i^H \mathbf{u}\right|^2 / \|\mathbf{u}\|^2 \tag{37}$$

subject to the nulling constraints

$$\mathbf{p}_l^H \mathbf{u} = 0 \qquad \text{for } l = 1, 2, \ldots, L. \tag{38}$$

We can further transform the problem as follows. Suppose $\mathcal{P} = \text{span}\{\mathbf{p}_l : l = 1, 2, \ldots, L\}$. The orthogonal complement of $\mathcal{P}$ is denoted by $\mathcal{P}^\perp$. Then (38) implies that $\mathbf{u} \in \mathcal{P}^\perp$. Let $\mathbf{r}_i = \mathbf{v}_i + \mathbf{t}_i$ be the orthogonal decomposition such that $\mathbf{v}_i \in \mathcal{P}^\perp$ and $\mathbf{t}_i \in \mathcal{P}$. Then $\mathbf{r}_i^H \mathbf{u} = \mathbf{v}_i^H \mathbf{u}$. Now we have the equivalent optimization problem

$$\max_{\mathbf{u}} \min_{i} \left|\mathbf{v}_i^H \mathbf{u}\right|^2 / \|\mathbf{u}\|^2 \tag{39}$$

with no need for any constraints. As it will be shown in Section 8, the optimum $\mathbf{u}$ is in $\text{span}\{\mathbf{v}_i : i = 1, 2, \ldots, K\} \subset \mathcal{P}^\perp$, and hence (38) is automatically satisfied.

Denote by $\mathbf{u}_{opt}$ the optimum solution of the above problem. Then the optimum solution $\mathbf{w}_{opt}$ of problem (29) is given by

$$\mathbf{w}_{opt} = \mathbf{C}^{-1} \mathbf{u}_{opt}. \tag{40}$$

The optimization problem (39) is treated from Section 8 onwards as the generic 'Max-Min SNR' problem.

8 The Generic 'Max-Min SNR' Optimization Problem

In the three previous sections, we posed optimization problems and reduced them to the same generic form (see (20), (26), and (39)). The rest of the report deals with this generic problem

$$\max_{\mathbf{w}} \min_{i \in I} \left| \mathbf{v}_i^H \mathbf{w} \right|^2 / \|\mathbf{w}\|^2 \tag{41}$$

which we call the 'Max-Min SNR' optimization problem. Here $I = \{1, 2, \ldots, K\}$ is the set of all visible satellites.[4]

Denote $\mathcal{V} = \operatorname{span}\{\mathbf{v}_i : i \in I\}$. We now show that the optimum $\mathbf{w} \in \mathcal{V}$. This was claimed previously in relation to (26) and (39). For any $\mathbf{w} \notin \mathcal{V}$, let $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ be the orthogonal decomposition such that $\mathbf{w}_1 \in \mathcal{V}$ and $\mathbf{w}_2 \in \mathcal{V}^\perp$. Then $\mathbf{v}_i^H \mathbf{w} = \mathbf{v}_i^H \mathbf{w}_1$ and $\|\mathbf{w}\|^2 = \|\mathbf{w}_1\|^2 + \|\mathbf{w}_2\|^2 > \|\mathbf{w}_1\|^2$. Therefore

$$\left| \mathbf{v}_i^H \mathbf{w}_1 \right|^2 / \|\mathbf{w}_1\|^2 > \left| \mathbf{v}_i^H \mathbf{w} \right|^2 / \|\mathbf{w}\|^2 \qquad \forall i \in I \tag{42}$$

which shows that the optimum $\mathbf{w} \in \mathcal{V}$.

Note that in (41), due to the $\min_i$ operation, the objective function that is to be maximized is not a differentiable function of $\mathbf{w}$. The standard approach to this type of problem is to convert it to a constrained optimization problem as follows. Introducing an auxiliary scalar variable γ, the above problem can be stated equivalently as

$$\max_{\gamma, \mathbf{w}} \gamma \tag{43}$$

$$\text{subject to} \qquad \left| \mathbf{v}_i^H \mathbf{w} \right|^2 / \|\mathbf{w}\|^2 \geq \gamma \qquad \text{for } i = 1, 2, \ldots, K. \tag{44}$$

In the above problem, the objective and constraint functions are differentiable.

8.1 Karush-Kuhn-Tucker Conditions for Optimality

Here we state the Karush-Kuhn-Tucker (KKT) conditions for optimality in (43),(44). These are derived in Annex A.

If $\mathbf{w}$ is optimum, then there is a set of Lagrange multipliers $\{\lambda_i \geq 0\}$ such that the following KKT conditions are satified:

$$\sum_{i=1}^{K} \lambda_i = 1 \tag{45}$$

$$\left[\sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H - \gamma \mathbf{I} \right] \mathbf{w} = \mathbf{0} \tag{46}$$

$$\lambda_i \left(\gamma - \left| \mathbf{v}_i^H \mathbf{w} \right|^2 / \|\mathbf{w}\|^2 \right) = 0 \qquad \text{for } i = 1, 2, \ldots, K. \tag{47}$$

4. If $\mathbf{w}$ is optimum then so is $\alpha \mathbf{w}$ for any complex scalar α. If desired, the problem can be reformulated by removing $\|\mathbf{w}\|^2$ from the denominator and introducing the constraint $\|\mathbf{w}\| = 1$ or even $\|\mathbf{w}\| \leq 1$.

Eq. (46) shows that $(\gamma, \mathbf{w})$ is an eigenvalue-eigenvector pair of the matrix $\sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H$. Eq. (47) says that if the constraint (44) is satisfied with strict inequality ($>$), then the corresponding $\lambda_i = 0$

One way to solve the optimization problem is to find all triples $(\gamma, \mathbf{w}, \{\lambda_i\})$ that satisfy the KKT conditions and choose the one with the largest γ. This is easier said than done. Nevertheless, the KKT conditions will be useful in developing a strategy to solve the optimization problem.

8.2 The Active Set

Suppose $(\gamma, \mathbf{w})$ is optimal in (43), (44). Then the subset $\mathbf{I}_a$ of $I = \{1, 2, \ldots, K\}$ at which

$$\left|\mathbf{v}_i^H \mathbf{w}\right|^2 = \gamma \|\mathbf{w}\|^2 \tag{48}$$

is called the active set. We may also call the corresponding set of $\mathbf{v}_i$ the active set, i.e.,

$$\mathbf{v}_a = \{\mathbf{v}_i : i \in \mathbf{I}_a\}. \tag{49}$$

How the term is being used will be clear from the context.

For the vectors $\mathbf{v}_i$ in the complement of the active set, $\left|\mathbf{v}_i^H \mathbf{w}\right|^2 > \gamma \, ||\mathbf{w}| \, |^2$, i.e., (44) is automatically satified, and hence these vectors can be removed from the optimization problem without loss of optimality, provided we know the active set of course! This also means that the optimum $\mathbf{w} \in \text{span}(\mathbf{v}_a)$.

Suppose $\{\lambda_i \geq 0\}$ is the set of Lagrange multipliers corresponding to the optimal $(\gamma, \mathbf{w})$. Then (47) shows that

$$\lambda_i = 0 \qquad i \notin I_a \tag{50}$$

$$\sum_{i \in \mathbf{I}_a} \lambda_i = 1. \tag{51}$$

8.3 A Canonical Form for the Optimum $\mathbf{w}$

We next derive a canonical form for the optimum $\mathbf{w}$, assuming that we know the active set. In the active set,

$$\mathbf{v}_i^H \mathbf{w} = e^{\jmath\theta_i} \sqrt{\gamma} \|\mathbf{w}\| \tag{52}$$

for some real θ_i. We can write (46) as

$$\begin{aligned}
\gamma \mathbf{w} &= \sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H \mathbf{w} && (53) \\
&= \sum_{i \in \mathbf{I}_a} \lambda_i \mathbf{v}_i \mathbf{v}_i^H \mathbf{w} && (54) \\
&= \sum_{i \in \mathbf{I}_a} \lambda_i \mathbf{v}_i e^{\jmath\theta_i} \sqrt{\gamma} \|\mathbf{w}\| && (55) \\
&= \sqrt{\gamma} \|\mathbf{w}\| \sum_{i \in \mathbf{I}_a} \lambda_i e^{\jmath\theta_i} \mathbf{v}_i && (56)
\end{aligned}$$

and hence

$$\mathbf{w}/\|\mathbf{w}\| = (1/\sqrt{\gamma}) \sum_{i \in \mathbf{I}_a} \lambda_i e^{\jmath\theta_i} \mathbf{v}_i. \tag{57}$$

Thus the optimum $\mathbf{w}$ is a special type of complex linear combination of vectors in $\mathbf{V}_a$.

For simplicity, we assume henceforth that $\mathbf{I}_a = \{1, \dots, k\}$, and use the notation

$$\mathbf{V}_a = [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k]. \tag{58}$$

Then we can write (57) in matrix form as

$$\mathbf{w}/\|\mathbf{w}\| = (1/\sqrt{\gamma}) \mathbf{V}_a \begin{bmatrix} \lambda_1 e^{\jmath\theta_1} \\ \vdots \\ \lambda_k e^{\jmath\theta_k} \end{bmatrix}. \tag{59}$$

Henceforth, we will assume that a repesentation of the above form exists for the optimum $\mathbf{w}$ such that the columns of $\mathbf{V}_a$ are linearly independent (LI).

8.3.1 An Equation Relating γ, $\{\lambda_i\}$, and $\{\theta_i\}$

By premultiplying (57) by $\mathbf{v}_k^H$ for $k \in \mathbf{I}_a$, and using (52), we get

$$\gamma e^{\jmath\theta_k} = \sum_{i \in \mathbf{I}_a} \mathbf{v}_k^H \mathbf{v}_i \lambda_i e^{\jmath\theta_i}. \tag{60}$$

Letting k run through $\mathbf{I}_a$, we get the matrix equation

$$\gamma \begin{bmatrix} e^{\jmath\theta_1} \\ \vdots \\ e^{\jmath\theta_k} \end{bmatrix} = \mathbf{W}_a \begin{bmatrix} \lambda_1 e^{\jmath\theta_1} \\ \vdots \\ \lambda_k e^{\jmath\theta_k} \end{bmatrix} \tag{61}$$

where again for simplicity we have assumed that $\mathbf{I}_a = \{1, \dots, k\}$, and used the notation

$$\mathbf{W}_a = \mathbf{V}_a^H \mathbf{V}_a \tag{62}$$

where $\mathbf{V}_a$ is given by (58). Note that $\mathbf{W}_a$ is invertible, since the columns of $\mathbf{V}_a$ are assumed to be linearly independent.

Pre-multiplying (61), first by $\mathbf{W}_a^{-1}$, and then by $\mathrm{diag}\left(e^{-j\theta_1}, e^{-j\theta_2}, \dots, e^{-j\theta_k}\right)$, we get

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \end{pmatrix} / \gamma = \begin{pmatrix} e^{-j\theta_1} & 0 & \dots & & 0 \\ 0 & e^{-j\theta_2} & 0 & & \vdots \\ & 0 & \ddots & & \\ \vdots & & & \ddots & 0 \\ 0 & & \dots & 0 & e^{-j\theta_k} \end{pmatrix} \mathbf{W}_a^{-1} \begin{pmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_k} \end{pmatrix} \tag{63}$$

This equation shows that if $\{\theta_i\}$ at optimality is known, then $\{\lambda_i\}$ and γ can be determined by first evaluating the RHS and then normalizing it using (51).

Pre-multiplying (63) by the all-ones row vector, and using (51), we get

$$1/\gamma = \left(e^{-j\theta_1}, e^{-j\theta_2}, \ldots, e^{-j\theta_k}\right) \mathbf{W}_a^{-1} \begin{pmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_k} \end{pmatrix}. \tag{64}$$

This equation suggests that the optimum $\{\theta_i\}$ and the maximum γ can be found by minimizing the RHS, which is an unconstrained minimization problem.

8.4 An Unconstrained Optimization Problem for Finding $\{\theta_i\}$

Theorem 1 *Assuming that we know the LI active set, the optimum $\{\theta_i\}$, which defines the optimum $\mathbf{w}$ through (64), (63) and (57), can be found by solving the following unconstrained minimization problem:*

$$\min_{\{\theta_i\}} y(\theta_1, \ldots, \theta_k) \tag{65}$$

where

$$y(\theta_1, \ldots, \theta_k) = [e^{-j\theta_1}, \ldots, e^{-j\theta_k}]\mathbf{W}_a^{-1} \begin{bmatrix} e^{j\theta_1} \\ \vdots \\ e^{j\theta_k} \end{bmatrix}. \tag{66}$$

Proof:

The gradient vector of the above objective function is

$$\frac{\partial y}{\partial \theta} = 2\Im \left[\begin{pmatrix} e^{-j\theta_1} & 0 & \ldots & & 0 \\ 0 & e^{-j\theta_2} & 0 & & \vdots \\ & 0 & \ddots & & \\ \vdots & & & \ddots & 0 \\ 0 & & \ldots & 0 & e^{-j\theta_k} \end{pmatrix} \mathbf{W}_a^{-1} \begin{pmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_k} \end{pmatrix} \right]. \tag{67}$$

This is derived in Annex B and given by (B.12) where $\mathbf{A}$ stands for $\mathbf{W}_a^{-1}$. Note that the gradient is the imaginary part of the RHS of (63) multiplied by two. Therefore the evaluation of the RHS of (63) at a stationary point of y is a real-valued vector. This must be the case at the minimum.

Let $\{\theta_{i,opt}\}$, $\{\lambda_{i,opt}\}$ and γ_{opt} correspond to the optimum $\mathbf{w}_{opt}$ (see (57)). Then (63) and (67) show that $\{\theta_{i,opt}\}$ is a stationary point of y in (66). If $\{\theta_{i,opt}\}$ does not minimize y, then we can derive a contradiction as follows.

Let $\{\theta_i\}$ correspond to the minimum of y and compute γ by (64) and $\{\lambda_i\}$ by(63). Now (64) shows that $\gamma > \gamma_{opt}$. Since $\{\theta_i\}$ is a stationary point of y, (67) shows that $\{\lambda_i\}$ values are all real (but not necessarily non-negative). Use the $\{\theta_i\}$ and $\{\lambda_i\}$ to construct a $\mathbf{w}$ as follows (compare with (59)):

$$\mathbf{w} = \mathbf{V}_a \begin{bmatrix} \lambda_1 e^{\jmath\theta_1} \\ \vdots \\ \lambda_k e^{\jmath\theta_k} \end{bmatrix}. \tag{68}$$

Then we can show, using the equivalence between (63) and (61),

$$\mathbf{V}_a^H \mathbf{w} = \mathbf{W}_a \begin{bmatrix} \lambda_1 e^{\jmath\theta_1} \\ \vdots \\ \lambda_k e^{\jmath\theta_k} \end{bmatrix} = \gamma \begin{bmatrix} e^{\jmath\theta_1} \\ \vdots \\ e^{\jmath\theta_k} \end{bmatrix} \tag{69}$$

$$\|\mathbf{w}\|^2 = \left[\lambda_1 e^{-\jmath\theta_1}, \ldots, \lambda_k e^{-\jmath\theta_k}\right] \mathbf{W}_a \begin{bmatrix} \lambda_1 e^{\jmath\theta_1} \\ \vdots \\ \lambda_k e^{\jmath\theta_k} \end{bmatrix} = \gamma \sum_i \lambda_i = \gamma. \tag{70}$$

Using (69) and (70), we get

$$\left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 = \gamma \qquad \forall i \in \mathbf{I}_a. \tag{71}$$

Since $\gamma > \gamma_{opt}$, the $\mathbf{w}$ which we have constructed contradicts the optimality of $\mathbf{w}_{opt}$. Therefore, we conclude that $\{\theta_{i,opt}\}$ minimizes y of (66). □

To facilitate the minimization of y, expressions for its gradient vector and Hessian matrix are derived in Annex B where we have replaced $\mathbf{W}_a^{-1}$ with $\mathbf{A}$ for convenience.

Since we don't know the active set a priori, we must consider all linearly independent (LI) subsets of $\{\mathbf{v}_i : i \in I\}$.

Suppose that for an arbitrarily chosen LI subset of $\{\mathbf{v}_i\}$, we minimize y and use the $\{\theta_i\}$ to evaluate γ by (64) and $\{\lambda_i\}$ by (63). Two scenarios are possible at the minimum:

1. All λ_i values are non-negative. In this case, the chosen subset is the active set for the problem over that subset. This subset can potentially be the active set for the problem over the whole set.
2. Some λ_i values are negative. In this case, the chosen subset is not the active set for the problem over that subset. This subset cannot be the active set for the problem over the whole set.

8.5 The Special Cases of One and Two Satellites

8.5.1 One Satellite

We want to solve the problem

$$\max_{\mathbf{w}} \left|\mathbf{v}_1^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. \tag{72}$$

By Schwarz inequality, the optimum $\mathbf{w} = \mathbf{v}_1$ and the achieved maximum is $\|\mathbf{v}_1\|^2$.

8.5.2 Two Satellites

We want to solve the problem

$$\max_{\mathbf{w}} \min_{i\in\{1,2\}} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. \tag{73}$$

At the optimum, these three cases are possible:

1. Only $\mathbf{v}_1$ is active:
   Since the problem reduces to the one-satellite case above, the optimum $\mathbf{w} = \mathbf{v}_1$. Since $\mathbf{v}_2$ is inactive, we must have $\|\mathbf{v}_1\|^2 < |\mathbf{v}_2^H \mathbf{v}_1|$.
2. Only $\mathbf{v}_2$ is active:
   The optimum $\mathbf{w} = \mathbf{v}_2$. This happens only if $\|\mathbf{v}_2\|^2 < |\mathbf{v}_1^H \mathbf{v}_2|$.
3. Both $\mathbf{v}_1$ and $\mathbf{v}_2$ are active. This happens only if $|\mathbf{v}_1^H \mathbf{v}_2| \leq \min\left(\|\mathbf{v}_1\|^2, \|\mathbf{v}_2\|^2\right)$. This happens trivially when $\mathbf{v}_1 = \alpha \mathbf{v}_2$ for scalar α such that $|\alpha| = 1$. The more interesting linearly independent (LI) case is treated below.

Define

$$\begin{aligned} \mathbf{V} &= [\mathbf{v}_1, \mathbf{v}_2] & (74)\\ \mathbf{W} &= \mathbf{V}^H \mathbf{V} & (75)\\ &= \begin{bmatrix} \|\mathbf{v}_1\|^2 & \mathbf{v}_1^H \mathbf{v}_2 \\ \mathbf{v}_2^H \mathbf{v}_1 & \|\mathbf{v}_2\|^2 \end{bmatrix}. & (76) \end{aligned}$$

Assuming $\mathbf{v}_1$ and $\mathbf{v}_2$ are LI, $\mathbf{W}$ is invertible.

In light of Theorem 1, let us first minimize the function (66)

$$\begin{aligned} y &= [e^{-j\theta_1}, e^{-j\theta_2}] \mathbf{W}^{-1} \begin{bmatrix} e^{j\theta_1} \\ e^{j\theta_2} \end{bmatrix} & (77)\\ &= \frac{1}{|\mathbf{W}|} [e^{-j\theta_1}, e^{-j\theta_2}] \begin{bmatrix} \|\mathbf{v}_2\|^2 & -\mathbf{v}_1^H \mathbf{v}_2 \\ -\mathbf{v}_2^H \mathbf{v}_1 & \|\mathbf{v}_1\|^2 \end{bmatrix} \begin{bmatrix} e^{j\theta_1} \\ e^{j\theta_2} \end{bmatrix} & (78)\\ &= \frac{1}{|\mathbf{W}|} \left(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - \mathbf{v}_1^H \mathbf{v}_2 e^{-j(\theta_1-\theta_2)} - \mathbf{v}_2^H \mathbf{v}_1 e^{j(\theta_1-\theta_2)} \right). & (79) \end{aligned}$$

The minimum of y is achieved for $(\theta_1 - \theta_2) = \text{angle}\left(\mathbf{v}_1^H \mathbf{v}_2\right)$, and the minimum is

$$y_{min} = \frac{1}{|\mathbf{W}|} \left(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - 2|\mathbf{v}_1^H \mathbf{v}_2| \right). \tag{80}$$

The achieved maximum in (73) is $\gamma = 1/y_{min}$.

Next let us evaluate (63) for $(\theta_1 - \theta_2) = \text{angle}\left(\mathbf{v}_1^H \mathbf{v}_2\right)$.

$$\begin{aligned}
\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} / \gamma &= \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} \mathbf{W}^{-1} \begin{bmatrix} e^{j\theta_1} \\ e^{j\theta_2} \end{bmatrix} \qquad (81) \\
&= \frac{1}{|\mathbf{W}|} \begin{bmatrix} e^{-j\theta_1} & 0 \\ 0 & e^{-j\theta_2} \end{bmatrix} \begin{bmatrix} \|\mathbf{v}_2\|^2 & -\mathbf{v}_1^H \mathbf{v}_2 \\ -\mathbf{v}_2^H \mathbf{v}_1 & \|\mathbf{v}_1\|^2 \end{bmatrix} \begin{bmatrix} e^{j\theta_1} \\ e^{j\theta_2} \end{bmatrix} \\
&= \frac{1}{|\mathbf{W}|} \begin{bmatrix} \|\mathbf{v}_2\|^2 - \mathbf{v}_1^H \mathbf{v}_2 e^{-j(\theta_1 - \theta_2)} \\ \|\mathbf{v}_1\|^2 - \mathbf{v}_2^H \mathbf{v}_1 e^{j(\theta_1 - \theta_2)} \end{bmatrix} \qquad (82) \\
&= \frac{1}{|\mathbf{W}|} \begin{bmatrix} \|\mathbf{v}_2\|^2 - |\mathbf{v}_1^H \mathbf{v}_2| \\ \|\mathbf{v}_1\|^2 - |\mathbf{v}_2^H \mathbf{v}_1| \end{bmatrix}. \qquad (83)
\end{aligned}$$

Note that in case 3, we have $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. They cannot both be zero because of the LI assumption. Now we can construct the optimum $\mathbf{w}$ by (57).

In case 1, we have $\lambda_1 > 0$ and $\lambda_2 < 0$, because by Schwarz inequality $\|\mathbf{v}_1\|^2 < |\mathbf{v}_2^H \mathbf{v}_1| \implies |\mathbf{v}_2^H \mathbf{v}_1| < \|\mathbf{v}_2\|^2$. Similarly in case 2, we have $\lambda_1 < 0$ and $\lambda_2 > 0$.

8.6 A Strategy for Solving the Max-Min SNR Problem

8.6.1 On the Solution of the Problem over a Subset of Satellites

Recall that $I = \{1, 2, \ldots, K\}$ is the set of all visible satellites and define $\mathbf{V}_I = \{\mathbf{v}_i : i \in I\}$. For a subset J of the visible satellites, i.e. $J \subset I$, define

$$\mathbf{V}_J = \{\mathbf{v}_i : i \in J\}. \qquad (84)$$

We use the notation $|J|$ for the number of elements in J.

Denote by P_J the following optimization problem over the subset J of satellites:

$$\max_{\mathbf{w}} \min_{i \in J} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. \qquad (85)$$

Regarding the solution of P_J, denote by γ_J the achieved maximum and by $\mathbf{w}_J$ the $\mathbf{w}$ which achieves it. We next state a simple lemma which will be used to develop a solution strategy for P_I.

Lemma 1 *1.* $\gamma_J \geq \gamma_I$

2. If

$$\left|\mathbf{v}_i^H \mathbf{w}_J\right|^2 / \|\mathbf{w}_J\|^2 \geq \gamma_J \qquad \forall i \notin J, \qquad (86)$$

then $\gamma_J = \gamma_I$, *and* $\mathbf{w}_J$ *solves* P_I.

Proof : For any $\mathbf{w}$, since $J \subset I$,

$$\min_{i \in J} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \geq \min_{i \in I} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. \qquad (87)$$

Taking $\max_{\mathbf{w}}$ on both sides gives part 1.

Since γ_J and $\mathbf{w}_J$ solve P_J, we have

$$\left|\mathbf{v}_i^H \mathbf{w}_J\right|^2 / \|\mathbf{w}_J\|^2 \geq \gamma_J \qquad \forall i \in J \tag{88}$$

Combining this with (86), we have

$$\left|\mathbf{v}_i^H \mathbf{w}_J\right|^2 / \|\mathbf{w}_J\|^2 \geq \gamma_J \qquad \forall i \in I \tag{89}$$

which shows $(\gamma_J, \mathbf{w}_J)$ is feasible for P_I, and hence $\gamma_I \geq \gamma_J$. Combining with part 1, we get part 2. $\square$.

8.6.2 Strategy for Solving Problem over All Satellites

Consider all linearly independent (LI) subsets $\mathbf{V}_J$ of $\mathbf{V}_I$, in the order of increasing (non-decreasing) $|J|$, until the solution $\mathbf{w}_J$ of P_J also serves as the solution of P_I, i.e. (86) is satisfied.

The problem P_J can be solved in closed-form for $|J| = 1$ and $|J| = 2$, as shown in Section 8.5. For $|J| \geq 3$, we can solve P_J by first solving (66) numerically, using the gradient vector and Hessian matrix of y derived in Annex B, and then using Theorem 1.

When considering an LI subset $\mathbf{V}_J$, identify all its subsets which have exactly one element less. Suppose these correspond to $J_k \subset J$ for $k = 1, 2, \ldots, |J|$. Denote by $\mathbf{u}_k$ the vector that is in $\mathbf{V}_J$ but not in $\mathbf{V}_{J_k}$

We assume that we have solved P_{J_k} for $k = 1, 2, \ldots, |J|$, and the solutions are $(\gamma_{J_k}, \mathbf{w}_{J_k})$.

Pass through $\{J_k\}$ as follows. If

$$|\mathbf{u}_k^H \mathbf{w}_{J_k}|^2 / \|\mathbf{w}_{J_k}\|^2 \geq \gamma_{J_k} \tag{90}$$

then the solution of P_{J_k} is also the solution of P_J. Otherwise, set a flag and move to the next J_k. Repeat this until a solution of P_J has been inferred or all J_k have been considered. If the solution of P_J has not been inferred, i.e., all flags have been set, then solve P_J via Theorem 1, assuming $\mathbf{V}_J$ is the active set.

Whenever, a solution $\mathbf{w}_J$ of P_J has been found, check whether (86) is satisfied. If so then $\mathbf{w}_J$ also serves as the solution of P_I, and we can stop.

8.6.3 Number of Subsets to Consider

For an array antenna with M elements, the vectors $\{\mathbf{v}_i\}$ are of length M. Since an LI subset $\mathbf{V}_J$ can have at most $L = \min(M, K)$ vectors, the number of subsets that have to be considered is at most

$$N = {}^K C_1 + {}^K C_2 + \ldots + {}^K C_L \tag{91}$$

where ${}^K C_n = \frac{K!}{n!(K-n)!}$.

9 'Maximum Weighted-Average SNR' Optimization Problem

In this section, we present alternative methods of choosing the array antenna's weights, which are of practical interest in their own right. Moreover, as we shall see in the next section, the discussion presented here has a theoretical use in developing another solution strategy for the 'Max-Min SNR' problem.

Recall that there are K satellites and let $\{\mu_i \geq 0\}$ be a set of numbers, which we call weights,[5] such that

$$\sum_{i=1}^{K} \mu_i = 1. \tag{92}$$

In Section 5, we proposed to maximize the minimum SINR. A mathematically simpler alternative is to maximize the weighted-average SINR :

$$\max_{\mathbf{w}} \sum_{i=1}^{K} \mu_i \mathrm{SINR}_i. \tag{93}$$

Similarly, a simpler alternative to the approach of Section 6 is to maximize the weighted-average SNR :

$$\max_{\mathbf{w}} \sum_{i=1}^{K} \mu_i \mathrm{SNR}_i \tag{94}$$

subject to the nulling constraints (23). A similar alternative exists also for the approach of Section 7. In all these alternatives, the performance of the optimized system will depend on the choice of the weights $\{\mu_i\}$.

These alternatives can all be reduced to the following generic form which we call the 'Maximum Weighted-Average SNR' problem (compare with (41)):

$$\max_{\mathbf{w}} \sum_{i=1}^{K} \mu_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. \tag{95}$$

The objective function in (95), without the normalizing $\|\mathbf{w}\|^2$, can be written as

$$\begin{aligned} \sum_{i=1}^{K} \mu_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 &= \sum_{i=1}^{K} \mu_i \mathbf{w}^H \mathbf{v}_i \mathbf{v}_i^H \mathbf{w} && (96) \\ &= \mathbf{w}^H \left(\sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H \right) \mathbf{w} && (97) \\ &= \mathbf{w}^H \mathbf{U} \mathbf{w} && (98) \end{aligned}$$

5. These should be distinguished from the weights applied to the array antenna outputs.

where

$$\mathbf{U} = \sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H. \tag{99}$$

Using this, the optimization problem (95) can be written as

$$\max_{\mathbf{w}} \mathbf{w}^H \mathbf{U} \mathbf{w} / \|\mathbf{w}\|^2. \tag{100}$$

Therefore, the solution to the optimization problem (95) is the largest eigenvalue of **U** and the corresponding eigenvector [8] (Theorem 4.2.2, Rayleigh-Ritz).

We will use this result in the next section to derive an upper bound on 'Max-Min SNR' and a solution strategy for the 'Max-Min SNR' problem.

10 An Upper Bound and a Solution Strategy for ‘Max-Min SNR’

Here we will use the solution of the ‘Maximum Weighted-Average SNR’ problem, derived in the previous section, to first derive an upper bound on the maximal objective value of the ‘Max-Min SNR’ problem and then derive a solution strategy for the ‘Max-Min SNR’ problem, subject to a condition.

To derive an upper bound, recall the fact that, given a set of real numbers, their weighted average is greater than or equal to their minimum. Thus we can write the following inequalities using the numbers $\{\mu_i \geq 0\}$ satisfying (92) defined in the previous section:

$$\min_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \leq \sum_{i=1}^{K} \mu_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \qquad \forall \mathbf{w} \tag{101}$$

$$\max_{\mathbf{w}} \min_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \leq \max_{\mathbf{w}} \sum_{i=1}^{K} \mu_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \tag{102}$$

$$= \text{largest eigenvalue of} \left(\sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H \right). \tag{103}$$

The last line is from the solution of the ‘Maximum Weighted-Average SNR’ problem. The RHS of (103) is a class of upper bounds, parameterized by $\{\mu_i\}$, on the Max-Min SNR of (41). The least upper bound of this class can be obtained by minimizing the RHS of (103) w.r.t. $\{\mu_i\}$.

Thus the ‘Max-Min SNR’ can be upper bounded as follows

$$\max_{\mathbf{w}} \min_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \leq \min_{\{\mu_i \geq 0\}} \text{largest eigenvalue of} \left(\sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H \right) \tag{104}$$

subject to the constraint (92). The minimization problem on the RHS of (104) is also the Lagrangian dual of the problem (43)-(44) of Section 8. See (A.3) of Annex A.

The following theorem says that the solution of the problem on the RHS of (104) directly yields the solution of the problem on the LHS subject to a condition. The proof is given in Annex C.

Theorem 2 *Suppose that for the optimal $\{\mu_i\}$ in the RHS of (104) above, the largest eigenvalue ρ of $\mathbf{U} = \sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H$ is simple (of multiplicity 1). Then*

1. *Equality holds in (104)*
2. *The eigenvector $\mathbf{w}$ corresponding to the largest eigenvalue of $\mathbf{U}$ is optimal for the ‘Max-Min SNR’ problem in the LHS*
3. *Identifying $\gamma = \rho$ and $\{\lambda_i = \mu_i\}$, $(\gamma, \mathbf{w}, \{\lambda_i\})$ together satisfy the KKT conditions (45), (46), and (47) for the ‘Max-Min SNR’ problem.*

10.1 On the Convexity of the RHS of (104)

The optimization problem on the RHS of (104) is a convex problem. This can be inferred from the general fact that Lagrangian dual problems are always convex optimization problems; see [9], Proposition 5.1.2.

For a direct proof, denote by $g(.)$ the largest eigenvalue function of a Hermitian matrix. If $\mathbf{X}$ and $\mathbf{Y}$ are Hermitian matrices of the same size, then it is known that

$$g(\mathbf{X}+\mathbf{Y}) \leq g(\mathbf{X}) + g(\mathbf{Y}). \tag{105}$$

This follows from the characterization of the largest eigenvalue as the maximum of the Rayleigh-Ritz ratio or Rayleigh quotient; see Theorem 4.2.2 of [8]. This result is also a special case of a general result on the eigenvalues of a sum of two Hermitian matrices; see Theorem 4.3.1 of [8] or Theorem 8.1.5 of [6].

For $\mu = (\mu_1, \mu_2, \ldots, \mu_K)$, denote by $f(\mu)$ the largest eigenvalue of $\mathbf{U} = \sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H$, i.e.,

$$f(\mu) = g\left(\sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H\right). \tag{106}$$

Let $\tilde{\mu}$ and $\hat{\mu}$ be two realizations of the vector μ, and let α be a scalar such that $0 < \alpha < 1$. Then, using (105), we can show that

$$f(\alpha\tilde{\mu} + (1-\alpha)\hat{\mu}) \leq \alpha f(\tilde{\mu}) + (1-\alpha) f(\hat{\mu}), \tag{107}$$

i.e., $f(.)$ is convex. Since the constraints on μ are also convex, the optimization problem is convex.

Efficient numerical procedures are known to exist for solving convex optimization problems [10], and therefore, the problem on the RHS of (104) can be solved.

10.2 A Solution Strategy for 'Max-Min SNR'

In view of the preceding discussion and Theorem 2, the 'Max-Min SNR' problem can potentially be solved as follows:

1. Solve the convex problem on the RHS of (104)
2. Check whether the largest eigenvalue of $\mathbf{U} = \sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H$, at optimality, is simple
3. If the largest eigenvalue is simple, then the corresponding eigenvector is a solution of the 'Max-Min SNR' problem.

If the largest eigenvalue is not simple, then the corresponding eigenvector is not unique. However, the subspace spanned by the eigenvector is unique. That subspace may contain the solution to the 'Max-Min SNR' problem.

Note that the above strategy can be applied even on a subset of satellites, particularly on any subset considered in Section 8.6.

11 Conclusion and Suggestions for Further Work

After briefly reviewing some array antenna signal processing theory, particularly the use of 'weight-and-sum' operation on the outputs of an array antenna to shape the reception pattern, we considered the problem of optimally choosing the weights to protect a legacy GPS receiver from jammers.

Assuming knowledge of the steering vectors (SVs) of the visible satellites, we proposed a class of optimal approaches that aim to achieve high array gain towards the satellites while mitigating or nulling the jammers. Within this class, we proposed three variations:

1. Maximize the minimum SINR of the satellites, assuming knowledge of the total covariance matrix of the jammers and noise. Thus the jammers are mitigated but not perfectly nulled.
2. Maximize the minimum SNR of the satellites, while (perfectly) nulling the jammers, assuming knowledge of the SVs of the jammers.
3. This is a hybrid of the first and second. In this, we (perfectly) null a subset of jammers, based on knowledge of their SVs, and maximize the minimum SINR of the satellites, based on knowledge of the covariance matrix of the other jammers and noise.

We showed that the mathematical statements of the above three optimization problems can be reduced to the same generic form which we called 'Max-Min SNR' problem. We derived mathematical results, including Karush-Kuhn-Tucker conditions and a canonical form of the optimal solution, to facilitate the solution of the 'Max-Min SNR' problem. We then developed a method of solving the problem when the active set (satellites which will have minimum SNR) is known a priori. Finally, we developed a solution strategy which is combinatorial in nature and finds the active set and the solution.

We then posed and solved the 'Max Weighted-Average SNR' problem and showed its connection to the 'Max-Min SNR' problem. Using this, we obtained an upper-bound on 'Max-Min SNR', given as the solution of the related 'Minimize the Largest Eigenvalue' (MLE) problem. We showed, under a condition, that the solution of the MLE problem directly gives the solution of the 'Max-Min SNR' problem. We also showed that the MLE problem is a convex optimization problem and hence can be efficiently solved. Based on these, we proposed a second solution strategy for the 'Max-Min SNR' problem.

Caution must be exercised in applying the above proposed approaches when a satellite and a jammer have nearly the same DOA. In this situation, the said satellite may weigh down the SINR of the other satellites, and hence that satellite should best be identified and excluded. The theoretical framework provided in this report would help to identify such satellites. However, if there is sufficient DOA-separation between the satellites and jammers, the proposed approaches would allow more accurate pseudo-range measurements which in turn would provide more accurate measurements of GPS user position and velocity.

The overall contribution of this report is a theoretical framework for optimally choosing the weights of an array antenna for protecting a legacy GPS receiver via the weight-and-sum operation.

11.1 Further Work

1. The proposed solution strategies must be implemented and tested.
2. A proof of Theorem 2 must be attempted without assuming that the largest eigenvalue is simple.
3. An efficient method should be developed for adapting to changes of problem parameters such as number and DOA of satellites and jammers and jammers-plus-noise covariance.
4. The sensitivity of the solution to problem parameters should be studied.

References

[1] P. Misra and P. Enge, Global Positioning System: Signals, Measurements, and Performance, 2nd edition, Ganga-Jamuna Press, 2006.

[2] B. Forssell and T. B. Olsen, Jamming GPS: Susceptibility of Some Civil GPS Receivers, GPS world, Vol. 14, No. 1, pp. 54-58, January 2003.

[3] F. Vizard, Safeguarding GPS, Scientific American, April 14, 2003.

[4] J. Bird, Private Communication, Oct 2011.

[5] H. L. Van Trees, Optimum Array Processing - Part IV of Detection, Estimation and Modulation Theory, Wiley-Interscience, 2002.

[6] G. H. Golub and C. F. Van Loan, Matrix Computations, 3rd ed., The Johns Hopkins University Press, 1996.

[7] C. M. Rader, VLSI Systolic Arrays for Adaptive Nulling, IEEE Signal Processing Magazine, pp. 29-49, July 1996.

[8] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, 1990.

[9] D. P. Bertsekas, Nonlinear Programming, 2nd ed., Athena Scientific, 1999.

[10] S. Boyd and L. Vandenberghe, Convex Optimization, Cambridge University Press, 2004.

[11] D. G. Luenberger, Optimization by Vector Space Methods, John-wiley & Sons Inc., 1969.

[12] J. R. Magnus, On Differentiating Eigenvalues and Eigenvectors, Econometric Theory, Vol. 1, 1985, pp. 179-191.

[13] J. R. Magnus and H. Neudecker, Matrix Differential Calculus, John Wiley & Sons, Revised Edition, 1999.

Annex A: Karush-Kuhn-Tucker Conditions for (43), (44)

Here we derive the first-order Karush-Kuhn-Tucker (KKT) conditions for the optimization problem (43), (44) of Section 8. These are necessary conditions for optimality that must be satisfied by the optimum $(\gamma, \mathbf{w})$. We apply Theorem 1 of [11], Section 9.4, page 249, together with the definition of regularity on page 248, to derive the KKT conditions. [6]

We rewrite the constraints (44) in standard form as

$$\gamma - \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \leq 0 \tag{A.1}$$

and show that the regularity assumption of the above cited theorem is satisfied. First of all, for any nonzero $\mathbf{w}$ we can choose a γ so that the constraints (A.1) for $i = 1, 2, \ldots, K$ are satisfied. Starting from a feasible point $(\gamma, \mathbf{w})$, there exists a direction in which we can move, even slightly, so that the new point will satisfy the constraints (A.1) with strict inequality (< 0). For example, we can just reduce γ while leaving $\mathbf{w}$ the same. Therefore, the regularity assumption of the Theorem of [11] (page 249) is satified, and we can proceed to derive the KKT conditions.

Treating the problem as the minimization of $(-\gamma)$ and associating the Lagrange multiplier $\lambda_i \geq 0$ with the constraint (A.1), the Lagrangian for the minimization problem is

$$\begin{aligned} \mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\}) &= -\gamma + \sum_{i=1}^{K} \lambda_i \left(\gamma - \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2\right) &\text{(A.2)} \\ &= \gamma \left(-1 + \sum_{i=1}^{K} \lambda_i\right) - \sum_{i=1}^{K} \lambda_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2. &\text{(A.3)} \end{aligned}$$

If $\mathbf{w}$ is optimum, then there is a set $\{\lambda_i \geq 0\}$ such that the following complementary slackness condition and stationarity condition are satified.

The complementary slackness condition is

$$\lambda_i \left(\gamma - \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2\right) = 0 \qquad \text{for } i = 1, 2, \ldots, K. \tag{A.4}$$

The stationarity condition consists of

$$\frac{\partial}{\partial \gamma} \mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\}) = 0 \qquad \text{and} \qquad \frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\}) = \mathbf{0}. \tag{A.5}$$

Differentiating $\mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\})$ w.r.t. γ and setting it to zero, we get

$$\sum_{i=1}^{K} \lambda_i = 1. \tag{A.6}$$

6. This theorem, although abstract, involves a regularity assumption which is easy to verify. An easy-to-understand statement of the KKT conditions is given in [9] (Section 3.3.1, proposition 3.3.1). However, it involves equality constraints as well, and its regularity assumption seems difficult to verify.

To differentiate $\mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\})$ w.r.t. $\mathbf{w}$, we first note that $\|\mathbf{w}\|^2$ and $\left|\mathbf{v}_i^H \mathbf{w}\right|^2$ are quadratic forms in $\mathbf{w}$, and write

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 &= \left(\|\mathbf{w}\|^2 (2\mathbf{v}_i \mathbf{v}_i^H \mathbf{w}) - \left|\mathbf{v}_i^H \mathbf{w}\right|^2 (2\mathbf{w}) \right) / \|\mathbf{w}\|^4 && \text{(A.7)} \\ &= \left(2/\|\mathbf{w}\|^2\right) \left[\mathbf{v}_i \mathbf{v}_i^H - \left(\left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \right) \mathbf{I} \right] \mathbf{w}. && \text{(A.8)} \end{aligned}$$

Using this, we differentiate $\mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\})$ w.r.t. $\mathbf{w}$ as follows.

$$\begin{aligned} \frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\}) &= -\sum_{i=1}^{K} \lambda_i \frac{\partial}{\partial \mathbf{w}} \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 && \text{(A.9)} \\ &= -\frac{2}{\|\mathbf{w}\|^2} \left[\sum_{i=1}^{K} \lambda_i \left[\mathbf{v}_i \mathbf{v}_i^H - \left(\left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \right) \mathbf{I} \right] \right] \mathbf{w} && \text{(A.10)} \\ &= -\frac{2}{\|\mathbf{w}\|^2} \left[\sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H - \left(\sum_{i=1}^{K} \lambda_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \right) \mathbf{I} \right] \mathbf{w}. && \text{(A.11)} \end{aligned}$$

Setting $\frac{\partial}{\partial \mathbf{w}} \mathcal{L}(\gamma, \mathbf{w}, \{\lambda_i\}) = 0$, we get

$$\left[\sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H - \left(\sum_{i=1}^{K} \lambda_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 \right) \mathbf{I} \right] \mathbf{w} = \mathbf{0}. \tag{A.12}$$

This can be simplified as follows. Summing (A.4) over i and rearranging,

$$\sum_{i=1}^{K} \lambda_i \left|\mathbf{v}_i^H \mathbf{w}\right|^2 / \|\mathbf{w}\|^2 = \gamma \sum_{i=1}^{K} \lambda_i = \gamma, \tag{A.13}$$

where we have also used (A.6). Therefore, (A.12) can be writen as

$$\left[\sum_{i=1}^{K} \lambda_i \mathbf{v}_i \mathbf{v}_i^H - \gamma \mathbf{I} \right] \mathbf{w} = \mathbf{0}. \tag{A.14}$$

In summary, the KKT conditions are (A.4), (A.6), and (A.14).

Annex B: The Gradient and Hessian of (66)

Here we derive expressions for the gradient vector and the Hessian matrix of the objective function of (66). For convenience, we replace $\mathbf{W}_a^{-1}$ with $\mathbf{A}$ and write

$$y = [e^{-j\theta_1}, \ldots, e^{-j\theta_K}]\mathbf{A} \begin{bmatrix} e^{j\theta_1} \\ \vdots \\ e^{j\theta_K} \end{bmatrix} \tag{B.1}$$

where $\mathbf{A} = [a_{k,m}]$ is a Hermitian matrix. The expressions to be derived will be useful in the minimization of y.

B.1 Gradient

Denoting $\theta = (\theta_1, \theta_2, \ldots, \theta_K)^T$, the gradient vector is defined as

$$\frac{\partial y}{\partial \theta} = \left(\frac{\partial y}{\partial \theta_1}, \frac{\partial y}{\partial \theta_2}, \ldots, \frac{\partial y}{\partial \theta_K} \right)^T. \tag{B.2}$$

An expression for this can be derived easily via the chain rule as follows.

Define the complex scalar and vector variables

$$\begin{aligned} x_k &= e^{j\theta_k} \qquad \text{for } k = 1, 2, \ldots, K & \text{(B.3)} \\ \mathbf{x} &= (x_1, x_2, \ldots, x_K)^T. & \text{(B.4)} \end{aligned}$$

Let $\mathbf{x} = \mathbf{x}_r + j\mathbf{x}_i$ be the real-imaginary decomposition of $\mathbf{x}$. Then we have the standard results

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial \theta} &= \frac{\partial \mathbf{x}_r}{\partial \theta} + j\frac{\partial \mathbf{x}_i}{\partial \theta} = j\text{diag}\left(e^{j\theta_1}, e^{j\theta_2}, \ldots, e^{j\theta_K}\right) & \text{(B.5)} \\ \frac{\partial y}{\partial \mathbf{x}} &= \frac{\partial y}{\partial \mathbf{x}_r} + j\frac{\partial y}{\partial \mathbf{x}_i} = 2\mathbf{A}\mathbf{x} & \text{(B.6)} \end{aligned}$$

By the chain rule of differentiation[7]

$$\begin{aligned} \frac{\partial y}{\partial \theta} &= \frac{\partial \mathbf{x}_r}{\partial \theta}\frac{\partial y}{\partial \mathbf{x}_r} + \frac{\partial \mathbf{x}_i}{\partial \theta}\frac{\partial y}{\partial \mathbf{x}_i} & \text{(B.7)} \\ &= \Re\left[\left(\frac{\partial \mathbf{x}_r}{\partial \theta} - j\frac{\partial \mathbf{x}_i}{\partial \theta} \right) \left(\frac{\partial y}{\partial \mathbf{x}_r} + j\frac{\partial y}{\partial \mathbf{x}_i} \right) \right] & \text{(B.8)} \\ &= \Re\left[\left(\frac{\partial \mathbf{x}}{\partial \theta} \right)^* \left(\frac{\partial y}{\partial \mathbf{x}} \right) \right] & \text{(B.9)} \\ &= \Re\left[-j\text{diag}\left(e^{-j\theta_1}, e^{-j\theta_2}, \ldots, e^{-j\theta_K}\right)(2\mathbf{A}\mathbf{x}) \right] & \text{(B.10)} \\ &= 2\Im\left[\text{diag}\left(e^{-j\theta_1}, e^{-j\theta_2}, \ldots, e^{-j\theta_K}\right)(\mathbf{A}\mathbf{x}) \right]. & \text{(B.11)} \end{aligned}$$

7. The $(k,l)^{th}$ element of $\frac{\partial \mathbf{x}}{\partial \theta}$ is $\frac{\partial x_l}{\partial \theta_k}$.

Substituting for $\mathbf{x}$ from (B.4),(B.3), we get the expression

$$\frac{\partial y}{\partial \theta} = 2\Im\left[\begin{pmatrix} e^{-j\theta_1} & 0 & \dots & & 0 \\ 0 & e^{-j\theta_2} & 0 & & \vdots \\ & 0 & \ddots & & \\ \vdots & & & \ddots & 0 \\ 0 & & \dots & 0 & e^{-j\theta_K} \end{pmatrix} \mathbf{A} \begin{pmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_K} \end{pmatrix}\right]. \tag{B.12}$$

The k^{th} element of the gradient vector has the following form which will be useful in deriving the Hessian; Recall that $\mathbf{A} = [a_{k,m}]$:

$$\frac{\partial y}{\partial \theta_k} = 2\Im\left[e^{-j\theta_k}\sum_{m=1}^{K} a_{k,m}e^{j\theta_m}\right] \tag{B.13}$$

$$= 2\Im\left[a_{k,k} + e^{-j\theta_k}\sum_{\substack{m=1 \\ m\neq k}}^{K} a_{k,m}e^{j\theta_m}\right] \tag{B.14}$$

$$= 2\Im\left[e^{-j\theta_k}\sum_{\substack{m=1 \\ m\neq k}}^{K} a_{k,m}e^{j\theta_m}\right] \tag{B.15}$$

since $a_{k,k}$ is real.

For later use, define for $k = 1, 2, \dots, K$

$$\mu_k = e^{-j\theta_k}\sum_{m=1}^{K} a_{k,m}e^{j\theta_m} \tag{B.16}$$

and note that (see (B.13))

$$\frac{\partial y}{\partial \theta_k} = 2\Im\left[\mu_k\right]. \tag{B.17}$$

In matrix form,

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_K \end{pmatrix} = \begin{pmatrix} e^{-j\theta_1} & 0 & \dots & & 0 \\ 0 & e^{-j\theta_2} & 0 & & \vdots \\ & 0 & \ddots & & \\ \vdots & & & \ddots & 0 \\ 0 & & \dots & 0 & e^{-j\theta_K} \end{pmatrix} \mathbf{A} \begin{pmatrix} e^{j\theta_1} \\ e^{j\theta_2} \\ \vdots \\ e^{j\theta_K} \end{pmatrix} \tag{B.18}$$

B.1.1 Testing for Stationary Point

A point $\theta = (\theta_1, \theta_2, \dots, \theta_K)^T$ at which $\frac{\partial y}{\partial \theta} = \mathbf{0}$ is called a stationary point of y. By (B.17), at a stationary point, all μ_k values are real.

At a stationary point

$$y = \sum_{k=1}^{K} \mu_k. \tag{B.19}$$

This is seen by premultiplying (B.18) by the all-ones row vector.

B.2 Hessian

The Hessian is the matrix of second partial derivatives of y with respect to the elements of θ. First we find the second partial derivatives and then arrange them into a matrix.

$$\begin{aligned}
\frac{\partial^2 y}{\partial \theta_k^2} &= 2\Im\left[-je^{-j\theta_k}\sum_{\substack{m=1\\m\neq k}}^{K} a_{k,m}e^{j\theta_m}\right] && \text{(B.20)}\\
&= 2\Re\left[-e^{-j\theta_k}\sum_{\substack{m=1\\m\neq k}}^{K} a_{k,m}e^{j\theta_m}\right] && \text{(B.21)}\\
&= 2\Re\left[a_{k,k} - e^{-j\theta_k}\sum_{m=1}^{K} a_{k,m}e^{j\theta_m}\right] && \text{(B.22)}\\
&= 2\Re\left[a_{k,k} - \mu_k\right] && \text{(B.23)}
\end{aligned}$$

where we have used (B.16).

For $k \neq m$,

$$\begin{aligned}
\frac{\partial^2 y}{\partial \theta_m \partial \theta_k} &= 2\Im\left[e^{-j\theta_k}a_{k,m}je^{j\theta_m}\right] && \text{(B.24)}\\
&= 2\Re\left[e^{-j\theta_k}a_{k,m}e^{j\theta_m}\right]. && \text{(B.25)}
\end{aligned}$$

Denote

$$\begin{aligned}
\Omega &= \operatorname{diag}\left(e^{j\theta_1}, e^{j\theta_2}, \ldots, e^{j\theta_K}\right) && \text{(B.26)}\\
\mathcal{M} &= \operatorname{diag}\left(\mu_1, \mu_2, \ldots, \mu_K\right) && \text{(B.27)}
\end{aligned}$$

Then the Hessian matrix has the following form

$$2\Re\left[\Omega^* \mathbf{A}\Omega - \mathcal{M}\right]. \tag{B.28}$$

B.2.1 Testing for Local Minimum

A point $\theta = (\theta_1, \theta_2, \ldots, \theta_K)^T$ which is stationary and at which the Hessian matrix is positive definite is a local minimum of y.

This page intentionally left blank.

Annex C: Proof of Theorem 2

The key steps of the proof below are: 1) derive the KKT conditions for the minimization problem on the RHS of (104), 2) show that its solution gives a feasible point for the problem (43)-(44). The claims of the theorem will follow easily.

As in Annex A, we apply Theorem 1 of [11], Section 9.4, page 249 to derive the KKT conditions. See also the footnote on page 29.

Since the reasoning that led to (104) will still be valid if the constraint (92) is replaced with the inequality $\sum_{i=1}^{K} \mu_i \geq 1$, we adopt the latter. Recall the other constraints $\{\mu_i \geq 0\}$. Starting from a feasible point $\mu = (\mu_1, \mu_2, \ldots, \mu_K)$, we can move in the direction $(1, 1, \ldots, 1)$, even slightly, so that the new point will satisfy all the constraints with strict inequality ($>$). Therefore, the regularity assumption of the Theorem of [11] (page 249) is satified, and we can proceed to derive the KKT conditions.

Associate the Lagrange multiplier $\eta \geq 0$ with the constraint

$$1 - \sum_{i=1}^{K} \mu_i \leq 0, \tag{C.1}$$

and associate the Lagrange multipliers $\{\eta_i \geq 0\}$ with the constraints $\{(-\mu_i) \leq 0\}$. Then the Lagrangian for the minimization problem is

$$\mathcal{L}(\mu, \eta, \{\eta_i\}) = \text{largest eigenvalue of } \mathbf{U} + \eta \left(1 - \sum_{i=1}^{K} \mu_i\right) + \sum_{i=1}^{K} \eta_i(-\mu_i). \tag{C.2}$$

Although we don't have an expression for the largest eigenvalue of $\mathbf{U} = \sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H$, we can easily obtain an expression for its gradient w.r.t. $\mu = (\mu_1, \mu_2, \ldots, \mu_K)$ provided the largest eigenvalue is simple (of multiplicity 1). We will do this by applying the following theorem (Theorem 6.3.12 of [8]). See also [12] or [13], Chapter 8.

Theorem. Let $A(t) \in M_n$ (n-by-n complex matrices) be differentiable at $t = 0$. Assume that λ is an algebraically simple eigenvalue of $A(0)$ and that $\lambda(t)$ is an eigenvalue of $A(t)$, for small t, such that $\lambda(0) = \lambda$. Let x be a right λ eigenvector of $A(0)$ and let y be a left λ eigenvector of $A(0)$. Then

$$\lambda^{'}(0) = \frac{y^* A^{'}(0) x}{y^* x}. \tag{C.3}$$

□

Assume that the largest eigenvalue of $\mathbf{U}$ is simple and denote it by ρ. Denote the corresponding normalized eigenvector by $\mathbf{w}$, i.e. $\|\mathbf{w}\| = 1$. Then by the above theorem the partial derivative of the largest eigenvalue w.r.t. μ_i is

$$\mathbf{w}^H \mathbf{v}_i \mathbf{v}_i^H \mathbf{w} = \left|\mathbf{v}_i^H \mathbf{w}\right|^2, \tag{C.4}$$

and therefore the gradient of the largest eigenvalue is

$$\left(\left|\mathbf{v}_1^H\mathbf{w}\right|^2, \left|\mathbf{v}_2^H\mathbf{w}\right|^2, \ldots, \left|\mathbf{v}_K^H\mathbf{w}\right|^2\right)^T. \tag{C.5}$$

The stationarity condition $\partial\mathcal{L}\left(\mu, \eta, \{\eta_i\}\right)/\partial\mu = \mathbf{0}$ is

$$\left(\left|\mathbf{v}_1^H\mathbf{w}\right|^2, \left|\mathbf{v}_2^H\mathbf{w}\right|^2, \ldots, \left|\mathbf{v}_K^H\mathbf{w}\right|^2\right)^T - \eta(1, 1, \ldots, 1)^T - \sum_{i=1}^{K} \eta_i \mathbf{e}_i^T = \mathbf{0} \tag{C.6}$$

where the column vector $\mathbf{e}_i$ contains zeros everywhere except for a one in the i^{th} position. This simplifies to

$$\left(|\left|\mathbf{v}_1^H\mathbf{w}\right|^2, \left|\mathbf{v}_2^H\mathbf{w}\right|^2, \ldots, \left|\mathbf{v}_K^H\mathbf{w}\right|^2\right) = (\eta + \eta_1, \eta + \eta_2, \ldots, \eta + \eta_K). \tag{C.7}$$

The complementary slackness conditions are

$$\eta\left(1 - \sum_{i=1}^{K} \mu_i\right) = 0 \tag{C.8}$$

$$\eta_i \mu_i = 0 \qquad \text{for } i = 1, 2, \ldots, K. \tag{C.9}$$

Having done step 1), we proceed to step 2).

Since $\mathbf{w}$ is the normalized eigenvector corresponding to eigenvalue ρ of $\mathbf{U}$,

$$\rho = \sum_{i=1}^{K} \mu_i \mathbf{w}^H \mathbf{v}_i \mathbf{v}_i^H \mathbf{w} \tag{C.10}$$

$$= \sum_{i=1}^{K} \mu_i \left|\mathbf{v}_i^H\mathbf{w}\right|^2. \tag{C.11}$$

Substituting the stationarity and complementary slackness conditions into (C.11),

$$\rho = \eta \sum_{i=1}^{K} \mu_i + \sum_{i=1}^{K} \mu_i \eta_i \tag{C.12}$$

$$= \eta. \tag{C.13}$$

Since $\eta > 0$, we conclude from (C.8)

$$\sum_{i=1}^{K} \mu_i = 1, \tag{C.14}$$

and hence not all μ_i can be zero. Denote by $\mathcal{I}$ the subset of $\{1, 2, \ldots, K\}$ where $\mu_i > 0$. For $i \in \mathcal{I}$,

$$\begin{aligned} \eta_i &= 0 \qquad \text{(from } (C.9)) && \text{(C.15)} \\ \left|\mathbf{v}_i^H \mathbf{w}\right|^2 &= \eta. \qquad \text{(from } (C.7) && \text{(C.16)} \\ &= \rho. && \text{(C.17)} \end{aligned}$$

For $i \notin \mathcal{I}$, since $\eta_i \geq 0$,

$$\left|\mathbf{v}_i^H \mathbf{w}\right|^2 \geq \rho. \tag{C.18}$$

Thus $(\rho, \mathbf{w})$, which is the optimal point of the RHS of (104), is also a feasible point $(\gamma, \mathbf{w})$, with $\gamma = \rho$, for the LHS of (104), stated equivalently via (43)-(44). This proves claims 1 and 2 of the theorem.

To prove claim 3, identify $\gamma = \rho$ and $\{\lambda_i = \mu_i\}$. Then first note (C.14) is (45). Since $(\rho, \mathbf{w})$ is an eigenvalue-eigenvector pair of $\mathbf{U}$,

$$\left[\sum_{i=1}^{K} \mu_i \mathbf{v}_i \mathbf{v}_i^H - \rho \mathbf{I}\right] \mathbf{w} = \mathbf{0}. \tag{C.19}$$

which is (46). Since $\mu_i = 0$ for $i \notin \mathcal{I}$ and $\left|\mathbf{v}_i^H \mathbf{w}\right|^2 = \rho$ for $i \in \mathcal{I}$,

$$\mu_i \left(\rho - \left|\mathbf{v}_i^H \mathbf{w}\right|^2\right) = 0 \qquad \forall i \tag{C.20}$$

which is (47) after normalizing to $\|\mathbf{w}\| = 1$.

The proof of Theorem 2 is now complete.

This page intentionally left blank.

DOCUMENT CONTROL DATA

(Security classification of title, body of abstract and indexing annotation must be entered when document is classified)

1. ORIGINATOR (The name and address of the organization preparing the document. Organizations for whom the document was prepared, e.g. Centre sponsoring a contractor's report, or tasking agency, are entered in section 8.) Defence R&D Canada – Ottawa 3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4	2. SECURITY CLASSIFICATION (Overall security classification of the document including special warning terms if applicable.) UNCLASSIFIED (NON-CONTROLLED GOODS) DMC A REVIEW: GCEC June 2010

3. TITLE (The complete document title as indicated on the title page. Its classification should be indicated by the appropriate abbreviation (S, C or U) in parentheses after the title.)

Max-Min SNR: an optimum approach to array antenna signal processing for GPS anti-jamming

4. AUTHORS (Last name, followed by initials – ranks, titles, etc. not to be used.)

Yasotharan, A.

5. DATE OF PUBLICATION (Month and year of publication of document.) December 2011	6a. NO. OF PAGES (Total containing information. Include Annexes, Appendices, etc.) 56	6b. NO. OF REFS (Total cited in document.) 13

7. DESCRIPTIVE NOTES (The category of the document, e.g. technical report, technical note or memorandum. If appropriate, enter the type of report, e.g. interim, progress, summary, annual or final. Give the inclusive dates when a specific reporting period is covered.)

Technical Memorandum

8. SPONSORING ACTIVITY (The name of the department project office or laboratory sponsoring the research and development – include address.)

Defence R&D Canada – Ottawa
3701 Carling Avenue, Ottawa, Ontario, Canada K1A 0Z4

9a. PROJECT NO. (The applicable research and development project number under which the document was written. Please specify whether project or grant.) 15en01	9b. GRANT OR CONTRACT NO. (If appropriate, the applicable number under which the document was written.)
10a. ORIGINATOR'S DOCUMENT NUMBER (The official document number by which the document is identified by the originating activity. This number must be unique to this document.) DRDC Ottawa TM 2011-202	10b. OTHER DOCUMENT NO(s). (Any other numbers which may be assigned this document either by the originator or by the sponsor.)

11. DOCUMENT AVAILABILITY (Any limitations on further dissemination of the document, other than those imposed by security classification.)

(X) Unlimited distribution
(   ) Defence departments and defence contractors; further distribution only as approved
(   ) Defence departments and Canadian defence contractors; further distribution only as approved
(   ) Government departments and agencies; further distribution only as approved
(   ) Defence departments; further distribution only as approved
(   ) Other (please specify):

12. DOCUMENT ANNOUNCEMENT (Any limitation to the bibliographic announcement of this document. This will normally correspond to the Document Availability (11). However, where further distribution (beyond the audience specified in (11)) is possible, a wider announcement audience may be selected.)

Unlimited

13. ABSTRACT (A brief and factual summary of the document. It may also appear elsewhere in the body of the document itself. It is highly desirable that the abstract of classified documents be unclassified. Each paragraph of the abstract shall begin with an indication of the security classification of the information in the paragraph (unless the document itself is unclassified) represented as (S), (C), (R), or (U). It is not necessary to include here abstracts in both official languages unless the text is bilingual.)

Global Positioning System (GPS) receivers are extremely vulnerable to jamming. One mitigation method is to use a CRPA (Controlled Reception Pattern Antenna) which is an array antenna whose signal outputs are weighted-and-summed and fed into the GPS receiver. Ideally, the weights must be chosen so as to null the jammers and receive with high strength all satellite signals. This is a challenging problem if the GPS receiver has one common input port through which all satellite signals are received. In conventional CRPA, the weights are calculated without considering where the satellites or jammers are located. In this approach, there is limited control over the reception pattern of the array antenna.

In this report, we propose a class of optimal approaches that assume knowledge of the directions of arrival (DOA) of the visible satellites. Within this, three variations are presented: 1) maximize satellites' SINR (Signal-to-Interference-plus-Noise Ratio) assuming that jammers-plus-noise covariance is known, 2) maximize satellites' SNR (Signal-to-Noise Ratio) while nulling the jammers, assuming that jammer DOAs are known, 3) a hybrid of these. The maximization is in the 'max-min SINR' and 'max-min SNR' senses, where the 'min' is over the satellites. We show that all three variations can be reduced to the same generic form where a 'max-min SNR' problem is solved as if there were no jammers. We show a connection between the 'max-min SNR' and 'max weighted-average SNR' problems. We derive several results, including Karush-Kuhn-Tucker conditions, to facilitate the solution of the generic problem, and propose solution strategies.

14. KEYWORDS, DESCRIPTORS or IDENTIFIERS (Technically meaningful terms or short phrases that characterize a document and could be helpful in cataloguing the document. They should be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location may also be included. If possible keywords should be selected from a published thesaurus. e.g. Thesaurus of Engineering and Scientific Terms (TEST) and that thesaurus identified. If it is not possible to select indexing terms which are Unclassified, the classification of each should be indicated as with the title.)

GPS, Global Positioning System
NAVWAR, Navigation Warfare
anti-jamming, interference-nulling
array antenna, CRPA, Controlled Reception Pattern Antenna
optimum array weights
Max-Min SINR, Max-Min SNR, Max Weighted-Average SNR
eigenvalue minimization, convex optimization

Defence R&D Canada

Canada's leader in Defence
and National Security
Science and Technology

R & D pour la défense Canada

Chef de file au Canada en matière
de science et de technologie pour
la défense et la sécurité nationale

DEFENCE R&D DÉFENSE

www.drdc-rddc.gc.ca