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Maximum likelihood emitter location estimation using hybrid sensor measurements: Theoretical derivations

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Abstract

Emitter location estimation (geolocation) is often implemented by processing measurements of a single signal parameter, such as angle of arrival (AOA), time of arrival (TOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA) or received signal strength (RSS). In recent years, approaches for utilizing hybrid sensor measurements for improved geolocation accuracy have attracted much interest. This technical memorandum derives a hybrid maximum likelihood emitter location estimator that utilizes geolocation data from a mixture of AOA, RSS and two-channel phase interferometer (ambiguous AOA) sensors. To provide the basis for subsequent derivations, we first formulate the maximum likelihood emitter location estimator for each individual type of sensor data. The hybrid algorithm is then obtained as a corollary under the assumption of statistical independence for the three types of sensor data. The Fisher information matrix and the Cramer-Rao lower bound (CRLB) for the average miss distance are also derived for each of the emitter location estimators. These results provide very useful theoretical benchmarks for the performance analysis of geolocation systems and for the estimation of geolocation performance metrics, such as the CEP (circular error probable) and the EEP (elliptical error probable).

Résumé

L'estimation de la position d'émetteurs (géolocalisation) est souvent exécutée en traitant les mesures d'un seul paramètre de signal, comme l'angle d'incidence, le temps d'arrivée, la différence de temps d'arrivée, la différence de fréquence d'arrivée ou la puissance du signal reçu. Ces dernières années, les méthodes permettant d'utiliser des mesures hybrides de capteur pour accroître l'exactitude de la géolocalisation ont suscité beaucoup d'intérêt.

Dans le présent mémoire technique, on dérive un estimateur hybride à vraisemblance maximale qui repose sur une combinaison de données de géolocalisation, dont l'angle d'incidence, la puissance du signal reçu et les capteurs d'interférométrie en phase à deux voies (angle d'incidence ambiguë). En vue d'établir une base pour des dérivations ultérieures, on a d'abord traité l'estimateur à vraisemblance maximale de chaque type de données de capteurs. L'algorithme hybride est ensuite obtenu sous forme corollaire selon l'hypothèse d'une indépendance statistique des trois types de données de capteurs. On dérive également la matrice de données de Fisher et la limite inférieure de Cramer-Rao de la distance de passage moyenne pour chaque estimateur de position d'émetteurs. Ces résultats constituent des étalons théoriques très utiles pour analyser le rendement des systèmes de géolocalisation et en estimer les indicateurs, comme l'erreur circulaire probable et l'erreur elliptique probable.

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Executive summary

Maximum likelihood emitter location estimation using hybrid sensor measurements: Theoretical derivations

Sichun Wang, Brad R. Jackson, Robert Inkol; DRDC Ottawa TM 2011-196; Defence R&D Canada – Ottawa; December 2011.

Background

Accurate emitter location estimation (geolocation) has long been a challenging problem due to the existence of many forms of signal impairments such as multipath propagation and channel fading. To improve the reliability of geolocation estimates, one approach is to utilize hybrid sensor measurements obtained from a mixture of different sensors spatially dispersed in the area of interest (AOI). In the open literature, some investigations have been reported on the development of hybrid geolocation algorithms for localization in wireless sensor networks. This technical memorandum develops a hybrid maximum likelihood emitter location estimator that uses data from a mixture of AOA, RSS and two-channel phase interferometer (ambiguous AOA) sensors. A theoretical framework for performance analysis for the proposed algorithm is established.

Principal results

Simple expressions for the likelihood functions for maximum likelihood emitter location estimators that use data from AOA, RSS, two-channel phase interferometer (ambiguous AOA) sensors or a mixture of these sensors are obtained. These results are formulated under the log-normal path loss model for RSS measurements and under the Gaussian and von Mises models for the AOA and phase measurements. The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are derived in easily computable formulas for all the estimators developed in the technical memorandum.

Significance of results

The simple formulas derived for the likelihood function, the Fisher information matrix and the Cramer-Rao lower bound for the average miss distance provide the necessary theoretical tools essential for algorithm implementation and performance analysis. The Fisher information matrices are also useful for determining estimates for performance metrics such as the CEP (circular error probable) and the EEP (elliptical error probable).

Future work

Systematic computer simulations and tests with off-the-air measurement data are planned to obtain a comprehensive understanding of various aspects of the derived algorithms including geolocation accuracy and implementation complexity.

Sommaire

Maximum likelihood emitter location estimation using hybrid sensor measurements: Theoretical derivations

Sichun Wang, Brad R. Jackson, Robert Inkol; DRDC Ottawa TM 2011-196; R & D pour la défense Canada – Ottawa; décembre 2011.

Contexte

L'estimation exacte de la position d'émetteurs (géolocalisation) est depuis longtemps un véritable casse tête, en raison des nombreux types d'affaiblissement des signaux, comme la propagation par trajets multiples et l'affaiblissement des canaux. L'une des façons d'améliorer la fiabilité de l'estimation de la position d'émetteurs consiste à utiliser des mesures hybrides de capteurs prises par une combinaison de différents capteurs dispersés dans la zone d'intérêt. Dans la littérature non classifiée, on a cité certaines études portant sur la conception d'algorithmes de géolocalisation hybrides à des fins de positionnement au sein de réseaux de capteurs sans fil. Le présent mémoire technique porte sur la conception d'un estimateur hybride à vraisemblance maximale qui repose sur une combinaison de données, y compris l'angle d'incidence, la puissance du signal reçu et les capteurs d'interférométrie en phase à deux voies (angle d'incidence ambiguë). On y établit également un cadre théorique pour analyser le rendement de l'algorithme proposé.

Principaux résultats

On a obtenu des expressions simples pour les fonctions de vraisemblance des estimateurs à vraisemblance maximale qui traitent l'angle d'arrivé, la puissance du signal reçu et les capteurs d'interférométrie en phase à deux voies (angle d'arrivé ambiguë) ou une combinaison de ces capteurs. Les mesures de puissance du signal reçu sont traitées dans le modèle d'affaiblissement sur le trajet log normal et les mesures de l'angle d'incidence et de la phase, dans les modèles de gaussien et de von Mises. Pour chaque estimateur mentionné dans le présent mémoire technique, on a dérivé en formules de calcul simples la matrice de données de Fisher connexe et la limite inférieure de Cramer-Rao de la distance de passage moyenne.

Importance des résultats

Les formules simples dérivées pour la fonction de vraisemblance, la matrice de données de Fisher et la limite inférieure de Cramer-Rao de la distance de passage moyenne

constituent des outils théoriques essentiels à la mise en œuvre d'un algorithme et à une analyse du rendement. Les matrices de données de Fisher sont également utiles pour déterminer les estimations des indicateurs de rendement, comme l'erreur circulaire probable et l'erreur elliptique probable.

Recherches futures

On prévoit exécuter des essais et des simulations informatiques systématiques avec les données de mesures hors ondes, afin de comprendre les divers aspects des algorithmes dérivés, notamment l'exactitude de la géolocalisation et la complexité de sa mise en œuvre.

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Acknoledgments

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1 Introduction

Emitter location estimates are usually obtained by processing measurements of one of the following signal parameters: angle of arrival (AOA), time of arrival (TOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA) or received signal strength (RSS). Various aspects of geolocation problems have been investigated under different assumptions [1], [2]. Traditionally the literature seems to have largely focused on algorithms which utilize only one type of sensor data. However, according to estimation theory, emitter geolocation accuracy can be expected to improve if additional data from other types of sensors is available and can be used [3], i.e., properly designed hybrid approaches that make effective use of available information should provide better geolocation accuracy. Some investigations of hybrid geolocation techniques have been reported [4]-[7]. However, many aspects of the characterization and practical implementation of hybrid geolocation algorithms still present major difficulties, due to the large number of variables involved. In particular, the impact of the emitter-sensor geometry involves various intricacies and requires further studies [8], [9].

The aim of this technical memorandum is to formulate a hybrid maximum likelihood (ML) emitter location estimator based on a mixture of the three different types of measurement data obtained from ground-based RSS and AOA (DF) and two-channel phase interferometer (ambiguous AOA) sensors. As a first step in the development, the ML emitter location estimator for each individual sensor type (i.e., RSS, AOA or two-channel phase interferometer (ambiguous AOA)) is formulated. Following the computation of the likelihood function for each individual type of sensor data, the likelihood function for the hybrid ML emitter location estimator is obtained as a relatively simple corollary under the assumption of statistical independence of the three different types of sensor data. In addition, the associated Fisher information matrix and Cramer-Rao lower bound for the average miss distance are derived for each ML geolocation algorithm. These theoretical developments provide the necessary mathematical tools for comparing the maximum likelihood emitter location estimators considered in this technical memorandum.

This technical memorandum is organized as follows. Section 2 introduces basic assumptions and notation. Section 3 derives the ML estimator of the emitter coordinates (x, y) based on RSS measurements from n RSS sensors. Sections 4 and 5 respectively derive the ML estimator of (x, y) based on AOA measurements from p AOA sensors and the ML estimator of (x, y) based on phase measurements from q two-channel phase interferometers. In these two sections, both the von Mises and the Gaussian probability densities are used to model the AOA errors and the phase (ambiguous AOA) errors and the associated likelihood functions for both densities are presented. Section 6 formulates a hybrid ML emitter location estimator using data from a mix of RSS, AOA and two-channel phase interferometer (ambiguous AOA) sensors. This involves minimization of an objective function defined as an appropriately weighted sum of objective functions associated with each single type of sensor data, with the positive weights determined by the signal model parameters. Section 7 concludes this technical memorandum.

2 Assumptions and notation

The theoretical derivations in this technical memorandum are based on the following assumptions:

1. The distances between the emitter and the sensors are assumed to be relatively small so that the curvature of the surface of the earth can be completely neglected;
2. A two-dimensional (2D) Cartesian coordinate system is set up and that the xy plane of the given 2D coordinate system coincides with the plane of the earth's surface;
3. A single emitter, whose position is to be estimated, is located on the xy plane (the plane of the earth) at a position defined by (x, y);
4. Three types of ground-based sensor data are available from RSS, AOA and two-channel phase interferometer (ambiguous AOA) sensors;
5. There are n ground-based RSS sensors located at the positions $R_k = (x_k, y_k)$, $1 \leq k \leq n$, with the RSS measured at R_k denoted by Ω_k (see Figure. 1);
6. The RSS measurement Ω_k is assumed to follow the path loss model [10]-[13]:

$$\Omega_k = c - 10\gamma \log_{10}(d_k) + \xi_k, \quad 1 \leq k \leq n \tag{1}$$

   where c is a constant for all $1 \leq k \leq n$, γ is the path loss exponent also assumed to be constant for all $1 \leq k \leq n$, d_k is the Euclidean distance from the emitter to the ground-based RSS sensor at R_k computed by:

$$d_k = \sqrt{(x - x_k)^2 + (y - y_k)^2} \tag{2}$$

   and ξ_k, $1 \leq k \leq n$, are independent zero-mean Gaussian random variables with variances $\sigma_k^2 > 0$.

7. There are p ground-based AOA sensors located at positions $A_l = (u_l, v_l)$, $1 \leq l \leq p$, with the true angle of arrival at A_l defined as the angle between the vector pointing from A_l to the emitter and the unit vector in the positive direction of the x-axis (see Figure. 2). The true angle of arrival at A_l shall be denoted by θ_l and its measured value shall be denoted by ϕ_l. Both θ_l and ϕ_l take values from the interval $(-\pi, \pi]$.
8. The AOA measurements ϕ_l, $1 \leq l \leq p$, are statistically independent;
9. The AOA measurement ϕ_l is modeled by two different probability distributions. The first is the von Mises distribution [14] with its probability density denoted by V_l and computed by:

$$V_l(\phi_l) = \frac{e^{\kappa_l \cos(\phi_l - \theta_l)}}{2\pi I_0(\kappa_l)}, \quad -\pi \leq \phi_l \leq \pi, \quad 1 \leq l \leq p \tag{3}$$

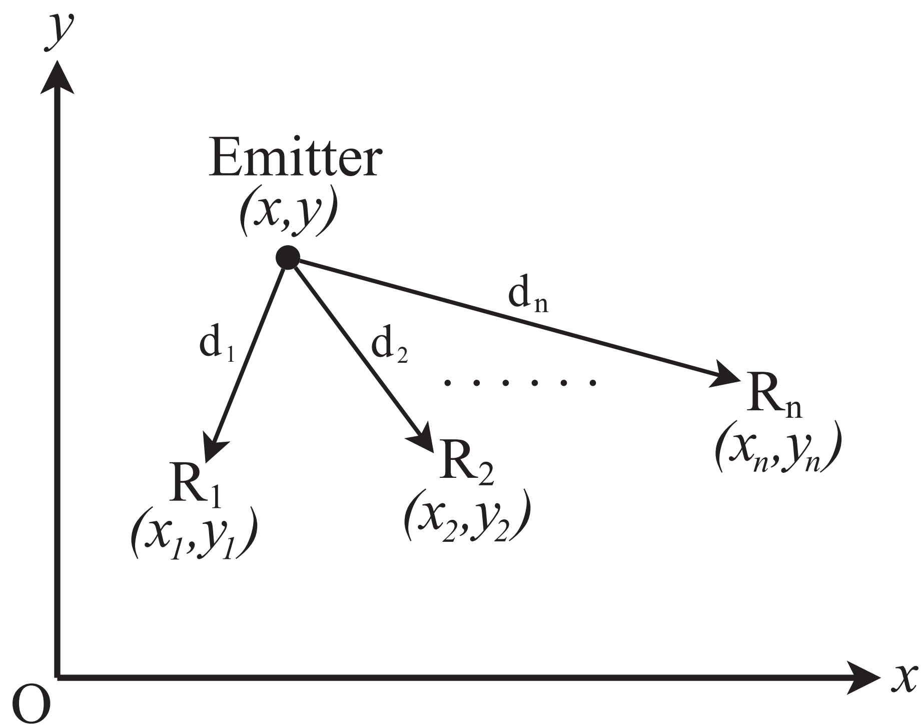

Figure 1: *Positions of n ground-based RSS sensors.*

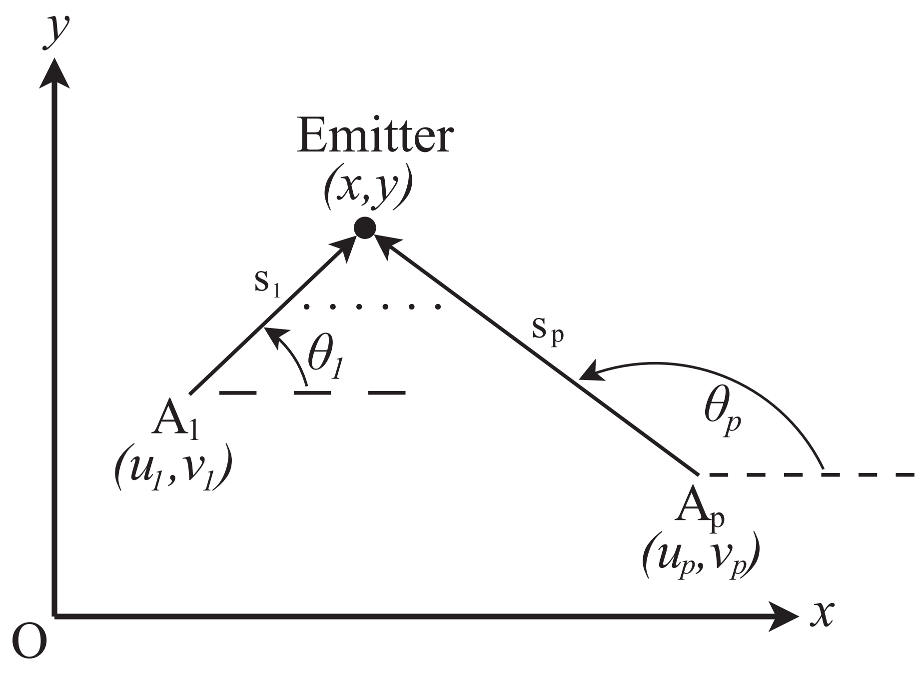

Figure 2: *Positions of p ground-based AOA sensors.*

where $\kappa_l > 0$ is a positive dimensionless constant that characterizes the accuracy of the AOA measurement ϕ_l, I_0 is the modified Bessel function of the first kind with order 0, θ_l is the true angle of arrival of the emitter defined by:

$$\theta_l = \mathrm{atan2}(y - v_l, x - u_l) \tag{4}$$

with

$$\mathrm{atan2}(y, x) = \begin{cases} \arctan\left(\frac{y}{x}\right) & x \geq 0, y \geq 0 \\ \pi + \arctan\left(\frac{y}{x}\right) & x \leq 0, y \geq 0 \\ \arctan\left(\frac{y}{x}\right) & x \geq 0, y \leq 0 \\ -\pi + \arctan\left(\frac{y}{x}\right) & x \leq 0, y \leq 0 \end{cases} \tag{5}$$

The second is the Gaussian distribution with its probability density denoted by G_l and computed by:

$$G_l(\phi_l) = \frac{1}{\sqrt{2\pi}\tau_l} \exp\left\{-\frac{1}{2\tau_l^2}(\phi_l - \theta_l)^2\right\}, \quad 1 \leq l \leq p \tag{6}$$

where $\tau_l > 0$ is the standard deviation of ϕ_l in radians.

10. The Euclidean distance from the AOA sensor at A_l to the emitter at (x, y) is denoted by s_l with

$$s_l = \sqrt{(x - u_l)^2 + (y - v_l)^2}, \quad 1 \leq l \leq p \tag{7}$$

11. There are q ground-based two-channel phase interferometers located on the xy plane, with the center of the l-th interferometer located at position $I_l = (w_l, z_l)$, $1 \leq l \leq q$, and the spacing between the two antenna elements being all identical and equal to d_0 for the q two-channel phase interferometers (see Figure. 3). Let the angle between the positive direction of the l-th interferometer and the positive direction of the x-axis be denoted by β_l; let the angle between the vector pointing from the center of the l-th interferometer, I_l, to the emitter and the positive direction of the x-axis be denoted by α_l and let the phase measurement made by the lth interferometer be denoted by ψ_l, with α_l, β_l and ψ_l all assuming values in the interval $(-\pi, \pi]$. It can be verified that

$$\alpha_l = \mathrm{atan2}(y - z_l, x - w_l), \quad 1 \leq l \leq q \tag{8}$$

12. The sensor headings of the q two-channel phase interferometers defined by β_l, $1 \leq l \leq q$, are assumed to be known;
13. The antenna spacing d_0 and the wavelength λ_0 of the received signal are assumed to be known;
14. The phase measurements ψ_l, $1 \leq l \leq q$, are statistically independent;

15. The phase measurement of the l-th interferometer, ψ_l, is modeled by two different probability distributions. The first is the von Mises distribution [15], with its probability density denoted by $\overline{V}_l$ and computed by:

$$\overline{V}_l(\psi_l) = \frac{e^{\overline{\kappa}_l \cos\left(\psi_l - \frac{2\pi d_0}{\lambda_0}\cos(\alpha_l - \beta_l)\right)}}{2\pi I_0(\overline{\kappa}_l)}, \quad -\pi \le \psi_l \le \pi, \quad 1 \le l \le q \tag{9}$$

where $\overline{\kappa}_l > 0$ is a positive dimensionless constant, d_0 is the distance between the two antenna elements and $\lambda_0 > 0$ is the wavelength of the transmitted signal from the emitter. The second is the Gaussian distribution with its probability density denoted by $\overline{G}_l$ and computed by:

$$\overline{G}_l(\psi_l) = \frac{1}{\sqrt{2\pi}\overline{\tau}_l} \exp\left\{-\frac{1}{2\overline{\tau}_l^2}\left(\psi_l - \frac{2\pi d_0}{\lambda_0}\cos(\alpha_l - \beta_l)\right)^2\right\}, \quad 1 \le l \le q \tag{10}$$

where $\overline{\tau}_l > 0$ is the standard deviation of ψ_l in radians.

16. The Euclidean distance from the center of the l-th two-channel phase interferometer at I_l to the emitter at (x, y) is denoted by t_l with

$$t_l = \sqrt{(x - w_l)^2 + (y - z_l)^2}, \quad 1 \le l \le q \tag{11}$$

17. The two-channel phase interferometer cannot produce an unambiguous AOA estimate. In the simplest and most favorable case, if d_0 is less than $\lambda_0/2$, a single ambiguity exists since there are two angles relative to the sensor baseline, i.e., θ and $-\theta$, with θ defined by $\theta = \alpha_l - \beta_l$ in Figure 3, which correspond to the same phase measurement and, without additional information, it is impossible to determine which of the two angles θ and $-\theta$ is meaningful. Furthermore, if d_0 exceeds $\lambda_0/2$, additional ambiguities exist [17]. In this technical memorandum, the assumption is made that there will always be sufficient geolocation information available from the other sensors to obtain an unambiguous emitter location estimate.

In the following sections, the ML emitter location estimators based on the RSS measurements, Ω_k, $1 \le k \le n$, the AOA measurements, ϕ_l, $1 \le l \le p$, and the phase measurements, ψ_l, $1 \le l \le q$, are derived. The associated Fisher information matrix (c.f. [3]) and the Cramer-Rao lower bound for the miss distance are also computed for each ML estimator. A hybrid emitter location estimator based on the combination of the RSS, AOA and two-channel phase interferometer (ambiguous AOA) data is then derived as a corollary under the assumption of statistical independence of the RSS, AOA and phase measurements. These theoretical developments provide very useful mathematical tools for comparing the performance of the proposed maximum likelihood emitter location estimators for various combinations of sensors.

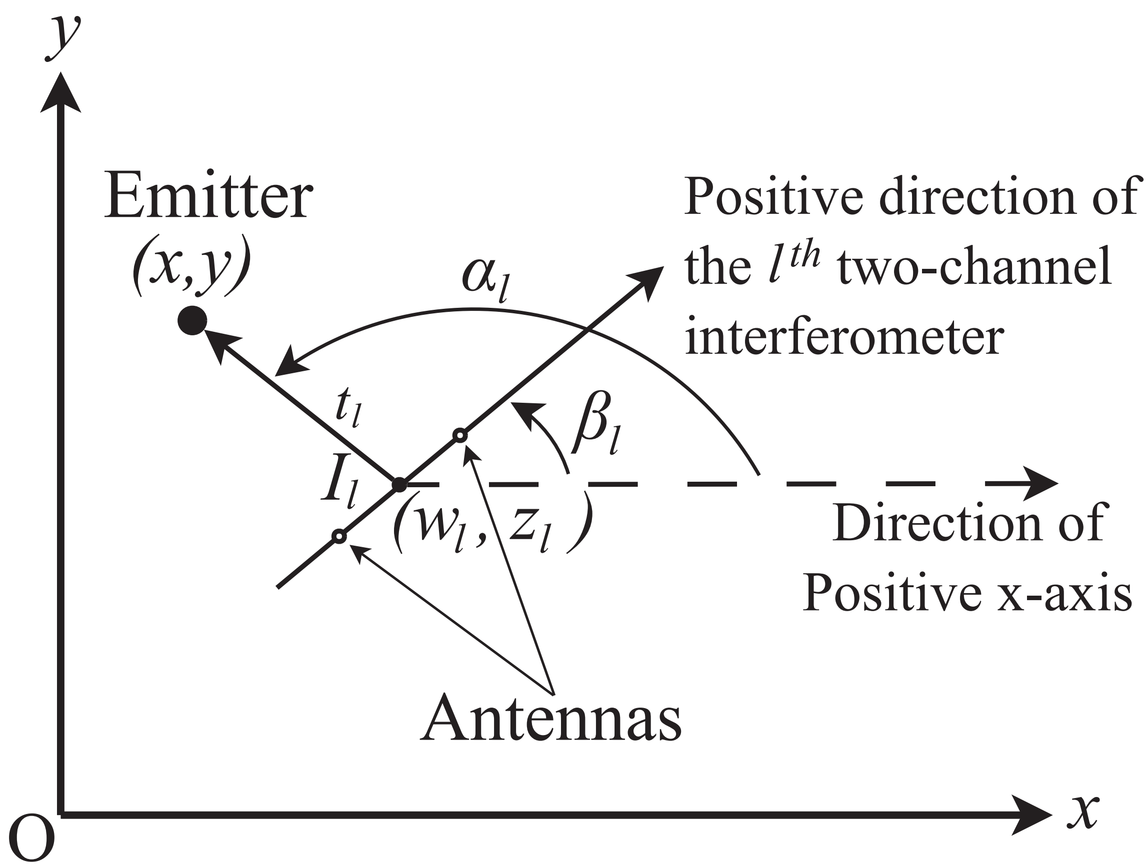

Figure 3: *Position of a typical ground-based two-channel phase interferometer. There are altogether q such two-channel phase interferometers.*

3 Maximum likelihood geolocation: Ground-based RSS sensors

Based on the path loss model (1), the maximum likelihood emitter location estimator using ground-based RSS measurements was previously described in [16]. Here the algorithm is reformulated in a simpler and more compact form. The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are then computed for unbiased emitter location estimators.

Define the $(n-1)$-dimensional random vector Ω as follows:

$$\Omega = [\Omega_{12}, \Omega_{13}, \ldots, \Omega_{1n}]^T \tag{12}$$

where $\Omega_{kl} = \Omega_k - \Omega_l, 1 \leq k, l \leq n$, and T denotes matrix transposition. The motivation for introducing the random vector Ω is rather simple: By considering the RSS differences, $\Omega_1 - \Omega_k$, $2 \leq k \leq n$, a maximum likelihood estimator of the emitter location (x, y) can be derived without being burdened by the nuisance parameter c in (1), which is irrelevant to geolocation. In fact, Ω has a Gaussian distribution and its covariance matrix of dimensions $(n-1) \times (n-1)$, denoted by $\mathbf{\Sigma}$, can be easily computed. We shall only provide detailed derivations for the case where the standard deviations σ_k of ξ_k, $1 \leq k \leq n$, are identical to a common constant $\sigma > 0$. The results for the general case shall be stated here without proofs since they are very similar to the derivations for the special case of equal variances.

Applying the assumption that $\sigma_1 = \sigma_2 = \ldots = \sigma_n = \sigma$, it can be verified that $\mathbf{\Sigma}$ is computed by [16]:

$$\mathbf{\Sigma} = \sigma^2 \begin{bmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 2 & \cdots & 1 \\ 1 & 1 & \cdots & 2 & 1 \\ 1 & 1 & \cdots & 1 & 2 \end{bmatrix} \tag{13}$$

$\mathbf{\Sigma}$ is non-singular and its inverse $\mathbf{\Sigma}^{-1}$ can be shown to be computed by:

$$\mathbf{\Sigma}^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} & -\frac{1}{n} \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{n-1}{n} \end{bmatrix} = \frac{1}{\sigma^2} \left\{ \mathbf{I} - \frac{1}{n} \mathbf{1}\mathbf{1}^T \right\} \tag{14}$$

where $\mathbf{I}$ is the identity matrix of dimensions $(n-1) \times (n-1)$ and $\mathbf{1}$ is the column vector of length $(n-1)$ with all its entries equal to 1. It can be verified that, for

$1 \leq k \leq n$,

$$E\left(\Omega_{1k}\right) = 10\gamma \log_{10}\left(\frac{d_k}{d_1}\right) = 5\gamma \log_{10}\left(\frac{(x-x_k)^2+(y-y_k)^2}{(x-x_1)^2+(y-y_1)^2}\right) \tag{15}$$

The probability density function of Ω, denoted by f_Ω here, is then computed by:

$$f_\Omega = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left\{-\frac{1}{2}(\Omega - E(\Omega))^T \mathbf{\Sigma}^{-1}(\Omega - E(\Omega))\right\} \tag{16}$$

with

$$\begin{aligned}
& (\Omega - E(\Omega))^T \mathbf{\Sigma}^{-1}(\Omega - E(\Omega)) \\
&= \frac{1}{\sigma^2}(\Omega - E(\Omega))^T \left\{\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right\}(\Omega - E(\Omega)) \\
&= \frac{1}{\sigma^2}\left\{(\Omega - E(\Omega))^T(\Omega - E(\Omega)) - \frac{1}{n}\left|\mathbf{1}^T(\Omega - E(\Omega))\right|^2\right\} \\
&= \frac{1}{\sigma^2}\left\{\left|\Omega - E(\Omega)\right|^2 - \frac{1}{n}\left|\mathbf{1}^T(\Omega - E(\Omega))\right|^2\right\} \\
&= \frac{1}{\sigma^2}\left\{\sum_{k=1}^{n}\left[\Omega_{1k} - 10\gamma\log_{10}\left(\frac{d_k}{d_1}\right)\right]^2 - \frac{1}{n}\left|\sum_{k=1}^{n}\left(\Omega_{1k} - 10\gamma\log_{10}\left(\frac{d_k}{d_1}\right)\right)\right|^2\right\} \\
&= \frac{1}{\sigma^2}\left\{\sum_{k=1}^{n}\left[\Omega_{1k} - 10\gamma\log_{10}\left(\frac{d_k}{d_1}\right)\right]^2\right. \\
&\qquad\qquad \left. -n\left|\Omega_1 + 10\gamma\log_{10}(d_1) - \left[\overline{\Omega} + 10\gamma\overline{\log_{10}(d)}\right]\right|^2\right\} \\
&= \frac{1}{\sigma^2}\left\{\sum_{k=1}^{n}\left(\widetilde{\Omega}_k - \widetilde{\Omega}_1\right)^2 - n\widetilde{\Omega}_1^2\right\}
\end{aligned} \tag{17}$$

where

$$\overline{\Omega} = \frac{1}{n}\sum_{k=1}^{n}\Omega_k \tag{18}$$

$$\overline{\log_{10}(d)} = \frac{1}{n}\sum_{k=1}^{n}\log_{10}(d_k) = \frac{1}{n}\sum_{k=1}^{n}\log_{10}\left(\sqrt{(x-x_k)^2+(y-y_k)^2}\right) \tag{19}$$

$$\widetilde{\Omega}_k = \Omega_k + 10\gamma_g\log_{10}(d_k) - \left[\overline{\Omega} + 10\gamma_g\overline{\log_{10}(d)}\right] \tag{20}$$

Since

$$\sum_{k=1}^{n} \overline{\Omega}_k = 0 \tag{21}$$

it follows from (17) that

$$\begin{aligned}(\Omega - E(\Omega))^T \mathbf{\Sigma}^{-1} (\Omega - E(\Omega)) &= \frac{1}{\sigma^2} \left\{ \sum_{k=1}^{n} \left(\widetilde{\Omega}_k - \widetilde{\Omega}_1 \right)^2 - n\widetilde{\Omega}_1^2 \right\} \\ = \frac{1}{\sigma^2} \left\{ \sum_{k=1}^{n} \widetilde{\Omega}_k^2 - 2\widetilde{\Omega}_1 \sum_{k=1}^{n} \widetilde{\Omega}_k + n\widetilde{\Omega}_1^2 - n\widetilde{\Omega}_1^2 \right\} &= \frac{1}{\sigma^2} \sum_{k=1}^{n} \widetilde{\Omega}_k^2\end{aligned} \tag{22}$$

and the probability density function f_Ω is simplified to become:

$$f_\Omega = \frac{1}{(2\pi)^{n/2} \sqrt{\det \mathbf{\Sigma}}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{k=1}^{n} \widetilde{\Omega}_k^2 \right\} \tag{23}$$

The maximum likelihood (ML) estimator of (x, y) based on Ω, denoted by $(\hat{x}_\Omega, \hat{y}_\Omega)$, is then obtained by maximizing the log-likelihood function, $\log(f_\Omega)$. Since $-\frac{n}{2}\log(2\pi) - \frac{1}{2}\log(\det \mathbf{\Sigma})$ and σ^2 are all constants which do not depend on (x, y), the ML estimator $(\hat{x}_\Omega, \hat{y}_\Omega)$ can also be equivalently obtained by minimizing the objective function L_Ω defined by:

$$L_\Omega = \sum_{k=1}^{n} \widetilde{\Omega}_k^2 \tag{24}$$

i.e.,

$$(\hat{x}_\Omega, \hat{y}_\Omega) = \operatorname{argmin} \sum_{k=1}^{n} \widetilde{\Omega}_k^2 \tag{25}$$

where $\widetilde{\Omega}_k$ is defined by (20).

We next use formula (3.31) in [3] (Eq. (A.8) in Annex A) to compute the Fisher information matrix and the Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators based on the random vector Ω.

For $1 \leq k \leq n$, let φ_k and μ_k be defined, respectively, by:

$$\varphi_k = \operatorname{atan2}(y - y_k, x - x_k) \tag{26}$$

and

$$\mu_k = E(\Omega_{1k}) = 10\gamma \log_{10}(d_k/d_1) = 5\gamma \log_{10} \left[\frac{(x - x_k)^2 + (y - y_k)^2}{(x - x_1)^2 + (y - y_1)^2} \right] \tag{27}$$

It follows that

$$\frac{\partial \mu_k}{\partial x} = \frac{10\gamma}{\ln(10)} \left[\frac{\cos\varphi_k}{d_k} - \frac{\cos\varphi_1}{d_1} \right] \tag{28}$$

and

$$\frac{\partial \mu_k}{\partial y} = \frac{10\gamma}{\ln(10)} \left[\frac{\sin\varphi_k}{d_k} - \frac{\sin\varphi_1}{d_1} \right] \tag{29}$$

Define the length-n sequences τ and ρ by:

$$\tau_k = \frac{x - x_k}{d_k^2} = \frac{\cos\varphi_k}{d_k}, \quad \rho_k = \frac{y - y_k}{d_k^2} = \frac{\sin\varphi_k}{d_k}, \quad 1 \le k \le n \tag{30}$$

and denote the averages of τ, ρ, τ^2, ρ^2 and $\tau\rho$, by $\bar{\tau}$, $\bar{\rho}$, $\overline{\tau^2}$, $\overline{\rho^2}$ and $\overline{(\tau\rho)}$, respectively, i.e.,

$$\begin{cases} \bar{\tau} = \frac{\sum_{k=1}^n \tau_k}{n}, & \bar{\rho} = \frac{\sum_{k=1}^n \rho_k}{n} \\ \overline{\tau^2} = \frac{\sum_{k=1}^n \tau_k^2}{n}, & \overline{\rho^2} = \frac{\sum_{k=1}^n \rho_k^2}{n}, \qquad \overline{(\tau\rho)} = \frac{\sum_{k=1}^n \tau_k \rho_k}{n} \end{cases} \tag{31}$$

Let $C(\gamma, \sigma) = \frac{10\gamma}{\ln(10)\sigma}$ and let the 2×2 Fisher information matrix of Ω be denoted by:

$$\mathbf{J}_\Omega = \begin{bmatrix} I_{11} & I_{12} \\ I_{12} & I_{22} \end{bmatrix} \tag{32}$$

Then from formula (3.31) in [3] (Eq. (A.8) in Annex A) it follows that

$$\begin{aligned} I_{11} &= \frac{1}{n\sigma^2} \sum_{k=2}^n \left[n\frac{\partial \mu_k}{\partial x} - \sum_{l=2}^n \frac{\partial \mu_l}{\partial x} \right] \frac{\partial \mu_k}{\partial x} \\ &= \frac{1}{n\sigma^2} \left[n \sum_{k=2}^n \left(\frac{\partial \mu_k}{\partial x} \right)^2 - \left[\sum_{k=2}^n \frac{\partial \mu_k}{\partial x} \right]^2 \right] = \frac{1}{\sigma^2} \left\{ \sum_{k=2}^n \left(\frac{\partial \mu_k}{\partial x} \right)^2 - \frac{1}{n} \left[\sum_{k=2}^n \frac{\partial \mu_k}{\partial x} \right]^2 \right\} \\ &= C(\gamma,\sigma)^2 \left\{ \sum_{k=2}^n \left[\frac{\cos\varphi_k}{d_k} - \frac{\cos\varphi_1}{d_1} \right]^2 - \frac{1}{n} \left[\sum_{k=2}^n \frac{\cos\varphi_k}{d_k} - \frac{(n-1)\cos\varphi_1}{d_1} \right]^2 \right\} \\ &= C(\gamma,\sigma)^2 \left\{ \sum_{k=2}^n [\tau_k - \tau_1]^2 - \frac{1}{n} \left[\sum_{k=2}^n \tau_k - (n-1)\tau_1 \right]^2 \right\} \\ &= C(\gamma,\sigma)^2 \left\{ \sum_{k=1}^n [\tau_k - \tau_1]^2 - \frac{1}{n} \left[\sum_{k=1}^n \tau_k - n\tau_1 \right]^2 \right\} \\ &= nC(\gamma,\sigma)^2 \left\{ \sum_{k=1}^n [\tau_k - \tau_1]^2 - n [\bar{\tau} - \tau_1]^2 \right\} = nC(\gamma,\sigma)^2 \left\{ \overline{\tau^2} - (\bar{\tau})^2 \right\} \end{aligned} \tag{33}$$

$$
\begin{aligned}
I_{12} &= \frac{1}{n\sigma^2}\sum_{k=2}^{n}\left[n\frac{\partial\mu_k}{\partial x}-\sum_{l=2}^{n}\frac{\partial\mu_l}{\partial x}\right]\frac{\partial\mu_k}{\partial y} \\
&= \frac{1}{\sigma^2}\left\{\sum_{k=2}^{n}\frac{\partial\mu_k}{\partial x}\frac{\partial\mu_k}{\partial y}-\frac{1}{n}\left[\sum_{k=2}^{n}\frac{\partial\mu_k}{\partial x}\right]\left[\sum_{k=2}^{n}\frac{\partial\mu_k}{\partial y}\right]\right\} \\
&= C(\gamma,\sigma)^2\left\{\sum_{k=2}^{n}[\tau_k-\tau_1][\rho_k-\rho_1]-\frac{1}{n}\left[\sum_{k=2}^{n}\tau_k-(n-1)\tau_1\right]\left[\sum_{k=2}^{n}\rho_k-(n-1)\rho_1\right]\right\} \\
&= C(\gamma,\sigma)^2\left\{\sum_{k=1}^{n}[\tau_k-\tau_1][\rho_k-\rho_1]-\frac{1}{n}\left[\sum_{k=1}^{n}\tau_k-n\tau_1\right]\left[\sum_{k=1}^{n}\rho_k-n\rho_1\right]\right\} \\
&= C(\gamma,\sigma)^2\left\{\sum_{k=1}^{n}[\tau_k-\tau_1][\rho_k-\rho_1]-n\left[\bar{\tau}-\tau_1\right]\left[\bar{\rho}-\rho_1\right]\right\} = nC(\gamma,\sigma)^2\left\{\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right\} \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
I_{22} &= \frac{1}{n\sigma^2}\sum_{k=2}^{n}\left[n\frac{\partial\mu_k}{\partial y}-\sum_{l=2}^{n}\frac{\partial\mu_l}{\partial y}\right]\frac{\partial\mu_k}{\partial y} = \frac{1}{\sigma^2}\left\{\sum_{k=2}^{n}\left(\frac{\partial\mu_k}{\partial y}\right)^2-\frac{1}{n}\left[\sum_{k=2}^{n}\frac{\partial\mu_k}{\partial y}\right]^2\right\} \\
&= C(\gamma,\sigma)^2\left\{\sum_{k=1}^{n}\left[\rho_k-\rho_1\right]^2-n\left[\bar{\rho}-\rho_1\right]^2\right\} = nC(\gamma,\sigma)^2\left\{\overline{\rho^2}-\left(\bar{\rho}\right)^2\right\} \qquad (35)
\end{aligned}
$$

Substituting the formulas (33)-(35) in (32), the Fisher information matrix $\mathbf{J}_\Omega$ can be rewritten as:

$$
\mathbf{J}_\Omega = nC(\gamma,\sigma)^2\begin{bmatrix}\overline{\tau^2}-\left(\bar{\tau}\right)^2 & \overline{(\tau\rho)}-\bar{\tau}\bar{\rho} \\ \overline{(\tau\rho)}-\bar{\tau}\bar{\rho} & \overline{\rho^2}-\left(\bar{\rho}\right)^2\end{bmatrix} \qquad (36)
$$

and its inverse is computed by:

$$
\begin{aligned}
\left(\mathbf{J}_\Omega\right)^{-1} &= \frac{\frac{1}{nC(\gamma,\sigma)^2}\begin{bmatrix}\overline{\rho^2}-\left(\bar{\rho}\right)^2 & -\left[\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right] \\ -\left[\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right] & \overline{\tau^2}-\left(\bar{\tau}\right)^2\end{bmatrix}}{\det\begin{bmatrix}\overline{\tau^2}-\left(\bar{\tau}\right)^2 & \overline{(\tau\rho)}-\bar{\tau}\bar{\rho} \\ \overline{(\tau\rho)}-\bar{\tau}\bar{\rho} & \overline{\rho^2}-\left(\bar{\rho}\right)^2\end{bmatrix}} \\
&= \frac{\frac{1}{nC(\gamma,\sigma)^2}\begin{bmatrix}\overline{\rho^2}-\left(\bar{\rho}\right)^2 & -\left[\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right] \\ -\left[\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right] & \overline{\tau^2}-\left(\bar{\tau}\right)^2\end{bmatrix}}{\left[\overline{\tau^2}-\left(\bar{\tau}\right)^2\right]\left[\overline{\rho^2}-\left(\bar{\rho}\right)^2\right]-\left[\overline{(\tau\rho)}-\bar{\tau}\bar{\rho}\right]^2} \qquad (37)
\end{aligned}
$$

The Cramer-Rao lower bound for the average miss distance, denoted by $CRLB_\Omega$, is then defined and computed by:

$$CRLB_\Omega = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Omega)^{-1}\right]} \tag{38}$$

where **tr** denotes the trace of a square matrix, i.e., the sum of the diagonal elements.

The preceding results are derived under the assumption that the variances σ_k^2 of the zero-mean Gaussian random variables ξ_k, $1 \le k \le n$, are all equal to σ^2. Using the Sherman-Morrison formula from linear algebra [3], these results can all be generalized to the case where the variances σ_k^2 of ξ_k, $1 \le k \le n$, are not necessarily all equal. Since the derivations are very similar, we shall next present the general results without giving detailed proofs.

Define the weight vector, $\mathbf{w}$, by:

$$\begin{aligned}\mathbf{w} &= [w_1, w_2, \ldots, w_n]^T \\ &= \left(\sum_{l=1}^{n} \sigma_l^{-2}\right)^{-1} \left[\sigma_1^{-2}, \sigma_2^{-2}, \ldots, \sigma_k^{-2}, \ldots, \sigma_n^{-2}\right]^T\end{aligned} \tag{39}$$

Let

$$\overline{\Omega} = \sum_{l=1}^{n} w_l \Omega_l \tag{40}$$

$$\overline{\log_{10}(d)} = \sum_{l=1}^{n} w_l \log_{10}(d_l) \tag{41}$$

$$\overline{c} = \overline{\Omega} + 10\gamma\overline{\log_{10}(d)} \tag{42}$$

and, for $1 \le k \le n$, define

$$\widetilde{\Omega}_k = \Omega_k - (\overline{c} - 10\gamma\log_{10}(d_k)) = \Omega_k + 10\gamma\log_{10}(d_k) - \bar{c} \tag{43}$$

Then the probability density function f_Ω can be shown to be computed by:

$$f_\Omega = C(\sigma_1, \ldots, \sigma_n)\exp\left\{-\frac{1}{2}\sum_{k=1}^{n}\sigma_k^{-2}\widetilde{\Omega}_k^2\right\} \tag{44}$$

where $C(\sigma_1, \ldots, \sigma_n)$ is a positive constant independent of the emitter location (x, y) and the Fisher information matrix $\mathbf{J}_\Omega$ can be shown to be computed by:

$$\mathbf{J}_\Omega = \left[\sum_{l=1}^{n}\sigma_l^{-2}\right]\left[\frac{10\gamma}{\ln(10)}\right]^2\begin{bmatrix}\overline{\tau^2} - (\overline{\tau})^2 & \overline{\tau\rho} - \overline{\tau}\;\overline{\rho} \\ \overline{\tau\rho} - \overline{\tau}\;\overline{\rho} & \overline{\rho^2} - (\overline{\rho})^2\end{bmatrix} \tag{45}$$

where

$$\begin{cases} \overline{\tau} = \sum_{l=1}^{n} w_l \tau_l, & \overline{\rho} = \sum_{l=1}^{n} w_l \rho_l \\ \overline{\tau^2} = \sum_{l=1}^{n} w_l \tau_l^2, & \overline{\rho^2} = \sum_{l=1}^{n} w_l \rho_l^2 \\ \overline{(\tau\rho)} = \sum_{l=1}^{n} w_l \tau_l \rho_l \end{cases} \tag{46}$$

The Cramer-Rao lower bound for the average miss distance is still computed by (38) but with the Fisher information matrix given by (45).

4 Maximum likelihood geolocation: Ground-based AOA sensors

In this section, the maximum likelihood geolocation algorithm using the p ground-based AOA measurements ϕ_l, $1 \leq l \leq p$, is formulated, taking the von Mises distribution (3) as the statistical model for ϕ_l. The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are then presented for unbiased emitter location estimators. For ease of comparisons, the maximum likelihood geolocation algorithm is also formulated here using the Gaussian distribution (6) as the statistical model for ϕ_l, $1 \leq l \leq p$. The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are also presented for unbiased emitter location estimators for the Gaussian model. We shall only provide detailed derivations for the simpler case where the AOA measurements ϕ_l have identical variances as the general case of unequal variances is very similar.

Define the p-dimensional random vector Φ as follows:

$$\Phi = [\phi_1, \phi_2, \cdots, \phi_p]^T \tag{47}$$

Assume the AOA measurements ϕ_l follow the von Mises density (3) with the parameters κ_l, $1 \leq l \leq p$, all identical and equal to the common positive constant $\kappa > 0$. Since the random variables ϕ_k, $1 \leq k \leq p$, are independent and identically distributed, applying formula (3), the probability density function of Φ, denoted by f_Φ, can be written as:

$$f_\Phi = V_1(\phi_1)V_2(\phi_2)\cdots V_p(\phi_p) = \prod_{k=1}^{p} \frac{e^{\kappa \cos(\phi_k - \theta_k)}}{2\pi I_0(\kappa)} = \frac{e^{\kappa \sum_{k=1}^{p} \cos(\phi_k - \theta_k)}}{(2\pi I_0(\kappa))^p} \tag{48}$$

where θ_k is defined by (4). The log-likelihood function $\log(f_\Phi)$ is then computed by:

$$\begin{aligned}\log(f_\Phi) &= -p\log(2\pi I_0(\kappa)) + \kappa \sum_{l=1}^{p} \cos(\phi_l - \theta_l) \\ &= -p\log(2\pi I_0(\kappa)) + p\kappa - 2\sum_{l=1}^{p} \left\{ \sin\left[\frac{\phi_l - \theta_l}{2}\right]\right\}^2 \end{aligned} \tag{49}$$

The maximum likelihood emitter location estimate using the AOA measurements ϕ_l, $1 \leq l \leq p$, denoted by $(\hat{x}_\Phi, \hat{y}_\Phi)$, is then obtained by maximizing $\log(f_\Phi)$, or equivalently, by minimizing the objective function L_Φ defined by:

$$L_\Phi = \sum_{l=1}^{p} \left\{ \sin\left[\frac{\phi_l - \theta_l}{2}\right]\right\}^2 \tag{50}$$

since $-p\log(2\pi I_0(\kappa))$ and $p\kappa$ are constants independent of the emitter location (x, y). Hence we can write

$$(\hat{x}_\Phi, \hat{y}_\Phi) = \text{argmax} \sum_{l=1}^{p} \cos(\phi_l - \theta_l) = \text{argmin} \sum_{l=1}^{p} \left\{ \sin\left[\frac{\phi_l - \theta_l}{2}\right] \right\}^2 \tag{51}$$

If the Gaussian distributions (6) are used to model the AOA measurements ϕ_l, $1 \leq l \leq p$, with the standard deviations τ_l all identical and equal to the common positive constant $\tau > 0$, the maximum likelihood emitter location estimate, also denoted by $(\hat{x}_\Phi, \hat{y}_\Phi)$, can be shown to be obtained by minimizing the objective function $\widetilde{L}_\Phi$ defined by:

$$\widetilde{L}_\Phi = \sum_{l=1}^{p} (\phi_l - \theta_l)^2 \tag{52}$$

i.e.,

$$(\hat{x}_\Phi, \hat{y}_\Phi) = \text{argmin } \widetilde{L}_\Phi = \text{argmin} \sum_{l=1}^{p} (\phi_l - \theta_l)^2 \tag{53}$$

The objective functions in (51) and (53) look very similar but there are subtle differences. As AOA measurements are always defined in the interval $(-\pi, \pi]$, in theory, the objective function in (53) may not be able to handle difficulties arising from the fact that $-\pi$ and π actually represent the same angle of arrival, although such difficulties may occur only infrequently in practice.

We next compute the Fisher information matrix of Φ and the Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators based on the random vector Φ.

First assume the von Mises densities (3) are used to model the AOA measurements ϕ_l, $1 \leq l \leq p$, with $\kappa_1 = \kappa_2 = \ldots = \kappa$.

We can write

$$\begin{aligned}
\frac{\partial \log(f_\Phi)}{\partial x} &= \sum_{l=1}^{p} \kappa \frac{\partial}{\partial x} \cos(\phi_l - \theta_l) = \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{\partial \theta_l}{\partial x} \\
&= \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{-(y - v_l)}{(x - u_l)^2 + (y - v_l)^2} \\
&= \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{(-\sin\theta_l)}{s_l}
\end{aligned} \tag{54}$$

and

$$\frac{\partial \log(f_\Phi)}{\partial y} = \sum_{l=1}^{p} \kappa \frac{\partial}{\partial y} \cos(\phi_l - \theta_l) = \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{\partial \theta_l}{\partial y}$$

$$= \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{(x - u_l)}{(x - u_l)^2 + (y - v_l)^2}$$

$$= \sum_{l=1}^{p} \kappa \left[\sin(\phi_l - \theta_l)\right] \frac{\cos \theta_l}{s_l} \tag{55}$$

Hence

$$E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial x}\right]^2\right\} = \sum_{l=1}^{p} \kappa^2 \left[\frac{\sin \theta_l}{s_l}\right]^2 E\left\{\left[\sin(\phi_l - \theta_l)\right]^2\right\} \tag{56}$$

where

$$E\left\{\left[\sin(\phi_l - \theta_l)\right]^2\right\} = \frac{1}{2\pi I_0(\kappa)} \int_{-\pi}^{\pi} \left[\sin(\phi_l - \theta_l)\right]^2 \exp\left\{\kappa \cos(\phi_l - \theta_l)\right\} d\phi_l$$

$$= \frac{1}{2\pi I_0(\kappa)} \int_{-\pi}^{\pi} e^{\kappa \cos t} \sin^2 t dt = \frac{1}{\pi I_0(\kappa)} \int_{0}^{\pi} e^{\kappa \cos t} \sin^2 t \ dt = \frac{g(\kappa)}{I_0(\kappa)} \tag{57}$$

where

$$g(x) = \frac{1}{\pi} \int_{0}^{\pi} e^{x \cos t} \sin^2 t \ dt \tag{58}$$

We thus have

$$E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial x}\right]^2\right\} = \frac{\kappa^2 g(\kappa)}{I_0(\kappa)} \sum_{l=1}^{p} \left[\frac{\sin \theta_l}{s_l}\right]^2 \tag{59}$$

Similarly

$$E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial y}\right]^2\right\} = \frac{\kappa^2 g(\kappa)}{I_0(\kappa)} \sum_{l=1}^{p} \left[\frac{\cos \theta_l}{s_l}\right]^2 \tag{60}$$

and

$$E\left\{\frac{\partial \log(f_\Phi)}{\partial x} \frac{\partial \log(f_\Phi)}{\partial y}\right\} = -\frac{\kappa^2 g(\kappa)}{I_0(\kappa)} \sum_{l=1}^{p} \frac{\cos \theta_l \sin \theta_l}{s_l^2} \tag{61}$$

The Fisher information matrix of Φ, denoted by $\mathbf{J}_\Phi$, is then computed [3] by:

$$\begin{aligned}
\mathbf{J}_\Phi &= \begin{bmatrix} E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial x}\right]^2\right\} & E\left\{\frac{\partial \log(f_\Phi)}{\partial x}\frac{\partial \log(f_\Phi)}{\partial y}\right\} \\ E\left\{\frac{\partial \log(f_\Phi)}{\partial x}\frac{\partial \log(f_\Phi)}{\partial y}\right\} & E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial y}\right]^2\right\} \end{bmatrix} \\
&= \frac{\kappa^2 g(\kappa)}{I_0(\kappa)} \begin{bmatrix} \sum_{l=1}^p \left[\frac{\sin\theta_l}{s_l}\right]^2 & -\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ -\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^p \left[\frac{\cos\theta_l}{s_l}\right]^2 \end{bmatrix}
\end{aligned} \tag{62}$$

and its inverse is computed by:

$$\begin{aligned}
(\mathbf{J}_\Phi)^{-1} &= \frac{I_0(\kappa)}{\kappa^2 g(\kappa)} \frac{\begin{bmatrix} \sum_{l=1}^p \left[\frac{\cos\theta_l}{s_l}\right]^2 & \sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ \sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^p \left[\frac{\sin\theta_l}{s_l}\right]^2 \end{bmatrix}}{\det \begin{bmatrix} \sum_{l=1}^p \left[\frac{\sin\theta_l}{s_l}\right]^2 & -\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ -\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^p \left[\frac{\cos\theta_l}{s_l}\right]^2 \end{bmatrix}} \\
&= \frac{I_0(\kappa)}{\kappa^2 g(\kappa)} \frac{\begin{bmatrix} \sum_{l=1}^p \left[\frac{\cos\theta_l}{s_l}\right]^2 & \sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ \sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^p \left[\frac{\sin\theta_l}{s_l}\right]^2 \end{bmatrix}}{\left\{\sum_{l=1}^p \left[\frac{\cos\theta_l}{s_l}\right]^2\right\}\left\{\sum_{l=1}^p \left[\frac{\sin\theta_l}{s_l}\right]^2\right\} - \left\{\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2}\right\}^2}
\end{aligned} \tag{63}$$

The Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators, denoted by $CRLB_\Phi$, is then computed by:

$$\begin{aligned}
CRLB_\Phi &= \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Phi)^{-1}\right]} \\
&= \frac{\sqrt{\frac{I_0(\kappa)}{g(\kappa)}}}{\kappa} \sqrt{\frac{\sum_{l=1}^p \frac{1}{s_l^2}}{\left[\sum_{l=1}^p \frac{\cos^2\theta_l}{s_l^2}\right]\left[\sum_{l=1}^p \frac{\sin^2\theta_l}{s_l^2}\right] - \left[\sum_{l=1}^p \frac{\cos\theta_l \sin\theta_l}{s_l^2}\right]^2}}
\end{aligned} \tag{64}$$

Next assume that the Gaussian densities (6) are used to model the AOA measurements, ϕ_l, $1 \le l \le p$, with $\tau_l = \tau_2 = \ldots = \tau > 0$.

We have

$$E(\phi_l) = \theta_l = \operatorname{atan2}(y - v_l, x - u_l) \tag{65}$$

Since ϕ_l are independent and identically distributed, the covariance matrix of Φ is computed by $\tau^2\mathbf{I}$, where $\mathbf{I}$ is the $p \times p$ identity matrix. We also have

$$\frac{\partial E(\phi_l)}{\partial x} = \frac{\partial}{\partial x}\arctan\left(\frac{y - v_l}{x - u_l}\right) = -\frac{y - v_l}{(x - u_l)^2 + (y - v_l)^2} = -\frac{\sin\theta_l}{s_l} \tag{66}$$

$$\frac{\partial E(\phi_l)}{\partial y} = \frac{\partial}{\partial y} \arctan\left(\frac{y - v_l}{x - u_l}\right) = \frac{x - u_l}{(x - u_l)^2 + (y - v_l)^2} = \frac{\cos\theta_l}{s_l} \tag{67}$$

Applying the formula (3.31) in [3] (Eq. (A.8) in Annex A), we can write the Fisher information matrix of Φ, $\mathbf{J}_\Phi$, as follows:

$$\mathbf{J}_\Phi = \frac{1}{\tau^2} \begin{bmatrix} \sum_{l=1}^{p} \left[\frac{\sin\theta_l}{s_l}\right]^2 & -\sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ -\sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^{p} \left[\frac{\cos\theta_l}{s_l}\right]^2 \end{bmatrix} \tag{68}$$

The inverse matrix of $\mathbf{J}_\Phi$ is then computed by

$$(\mathbf{J}_\Phi)^{-1} = \frac{\tau^2 \begin{bmatrix} \sum_{l=1}^{p} \left[\frac{\cos\theta_l}{s_l}\right]^2 & \sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ \sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^{p} \left[\frac{\sin\theta_l}{s_l}\right]^2 \end{bmatrix}}{\left\{\sum_{l=1}^{p} \left[\frac{\cos\theta_l}{s_l}\right]^2\right\} \left\{\sum_{l=1}^{p} \left[\frac{\sin\theta_l}{s_l}\right]^2\right\} - \left\{\sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2}\right\}^2} \tag{69}$$

and the Cramer-Rao lower bound $CRLB_\Phi$ is computed by

$$\begin{aligned} CRLB_\Phi &= \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Phi)^{-1}\right]} \\ &= \tau \sqrt{\frac{\sum_{l=1}^{p} \frac{1}{s_l^2}}{\left[\sum_{l=1}^{p} \frac{\cos^2\theta_l}{s_l^2}\right] \left[\sum_{l=1}^{p} \frac{\sin^2\theta_l}{s_l^2}\right] - \left[\sum_{l=1}^{p} \frac{\cos\theta_l \sin\theta_l}{s_l^2}\right]^2}} \end{aligned} \tag{70}$$

It can be immediately observed that the Fisher information matrices and the Cramer-Rao lower bounds derived under the von Mises distribution (3) and the Gaussian distribution (6) are identical, except for the different scaling constants in (62), (64), (68) and (70).

The preceding results are derived under the assumption that the AOA measurements ϕ_k, $1 \leq k \leq p$, have the same variances. These results can be easily extended to the general case where the parameters κ_l or the standard deviations τ_l are not necessarily identical. We shall only state here the relevant results without detailed proofs.

First assume that the von Mises densities (3) are used to model the AOA measurements ϕ_l, $1 \leq l \leq p$. The probability density function f_Φ can then be written as:

$$f_\Phi = C(\kappa_1, \ldots, \kappa_p) \exp\left\{\sum_{l=1}^{p} \kappa_l \cos(\phi_l - \theta_l)\right\} \tag{71}$$

where $C(\kappa_1, \ldots, \kappa_p)$ is a positive constant independent of the emitter location (x, y). The objective function L_Φ defined by (50) is now transformed to become

$$L_\Phi = \sum_{l=1}^{p} a_l \left\{ \sin \left[\frac{\phi_l - \theta_l}{2} \right] \right\}^2 \tag{72}$$

where

$$a_l = \frac{\kappa_l}{\sum_{m=1}^{p} \kappa_m} \tag{73}$$

The Fisher information matrix of Φ now takes the form:

$$\begin{aligned} \mathbf{J}_\Phi &= \begin{bmatrix} E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial x}\right]^2\right\} & E\left\{\frac{\partial \log(f_\Phi)}{\partial x}\frac{\partial \log(f_\Phi)}{\partial y}\right\} \\ E\left\{\frac{\partial \log(f_\Phi)}{\partial x}\frac{\partial \log(f_\Phi)}{\partial y}\right\} & E\left\{\left[\frac{\partial \log(f_\Phi)}{\partial y}\right]^2\right\} \end{bmatrix} \\ &= \begin{bmatrix} \sum_{l=1}^{p} \frac{\kappa_l^2 g(\kappa_l)}{I_0(\kappa_l)} \left[\frac{\sin\theta_l}{s_l}\right]^2 & -\sum_{l=1}^{p} \frac{\kappa_l^2 g(\kappa_l)}{I_0(\kappa_l)} \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ -\sum_{l=1}^{p} \frac{\kappa_l^2 g(\kappa_l)}{I_0(\kappa_l)} \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^{p} \frac{\kappa_l^2 g(\kappa_l)}{I_0(\kappa_l)} \left[\frac{\cos\theta_l}{s_l}\right]^2 \end{bmatrix} \end{aligned} \tag{74}$$

and the Cramer-Rao lower bound for the average miss distance can then be computed as follows:

$$CRLB_\Phi = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Phi)^{-1}\right]} \tag{75}$$

where $\mathbf{J}_\Phi$ is given by (74).

Next assume that the Gaussian densities (6) are used to model the AOA measurements ϕ_l, $1 \le l \le p$. The probability density function f_Φ can then be written as:

$$f_\Phi = C(\tau_1, \ldots, \tau_p) \exp\left\{ -\frac{1}{2} \sum_{l=1}^{p} \sigma_l^{-2} (\phi_l - \theta_l)^2 \right\} \tag{76}$$

where $C(\tau_1, \ldots, \tau_p)$ is a positive constant independent of the emitter location (x, y). The objective function $\widetilde{L}_\Phi$ defined by (52) now becomes

$$\widetilde{L}_\Phi = \sum_{l=1}^{p} b_l \left(\phi_l - \theta_l\right)^2 \tag{77}$$

with

$$b_l = \frac{\tau_l^{-2}}{\sum_{m=1}^{p} \tau_m^{-2}} \tag{78}$$

The Fisher information matrix of Φ now takes the form:

$$\mathbf{J}_\Phi = \begin{bmatrix} \sum_{l=1}^{p} \tau_l^{-2} \left[\frac{\sin\theta_l}{s_l}\right]^2 & -\sum_{l=1}^{p} \tau_l^{-2} \frac{\cos\theta_l \sin\theta_l}{s_l^2} \\ -\sum_{l=1}^{p} \tau_l^{-2} \frac{\cos\theta_l \sin\theta_l}{s_l^2} & \sum_{l=1}^{p} \tau_l^{-2} \left[\frac{\cos\theta_l}{s_l}\right]^2 \end{bmatrix} \tag{79}$$

and the Cramer-Rao lower bound for the average miss distance can then be computed by (75) with $\mathbf{J}_\Phi$ computed by (79).

5 Maximum likelihood geolocation: Two-channel phase interferometers

This section formulates the maximum likelihood geolocation algorithm using phase measurements, ψ_l, $1 \leq l \leq q$, obtained from q ground-based two-channel phase interferometers where the statistics of ψ_l, $1 \leq l \leq q$, conform to the von Mises distribution (9). The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are then presented for unbiased emitter location estimators. For ease of comparisons, the maximum likelihood geolocation algorithm is also formulated here using the Gaussian distribution (10) as the statistical model for ψ_l, $1 \leq l \leq q$. The associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance are also presented for unbiased emitter location estimators for the Gaussian model. We shall only provide detailed derivations for the simpler case where the phase measurements ψ_l have identical variances as the general case of unequal variances is very similar.

Define the q-dimensional random vector Ψ as follows:

$$\Psi = [\psi_1, \psi_2, \cdots, \psi_q]^T \tag{80}$$

First assume that the random variables ψ_l, $1 \leq l \leq q$, follow the von Mises distributions (9) and the positive parameters $\overline{\kappa}_l$ are all identical and equal to the common positive constant $\overline{\kappa} > 0$. Since the random variables ψ_k, $1 \leq k \leq q$, are independent and identically distributed, applying the probability density (9), the probability density function of Ψ, denoted by f_Ψ, is computed by:

$$\begin{aligned} f_\Psi &= \overline{V}_1(\psi_1)\overline{V}_2(\psi_2)\cdots\overline{V}_q(\psi_q) = \prod_{l=1}^{q} \frac{e^{\overline{\kappa}\cos\left(\psi_l - \frac{2\pi d_0}{\lambda_0}\cos(\alpha_l - \beta_l)\right)}}{2\pi I_0(\overline{\kappa})} \\ &= \frac{e^{\overline{\kappa}\sum_{l=1}^{q}\cos(\psi_l - c_0\cos(\alpha_l-\beta_l))}}{(2\pi I_0(\overline{\kappa}))^q}, \quad c_0 = \frac{2\pi d_0}{\lambda_0} \end{aligned} \tag{81}$$

The log-likelihood function $\log(f_\Psi)$ is then computed by:

$$\log(f_\Psi) = -q\log(2\pi I_0(\overline{\kappa})) + \overline{\kappa}\sum_{l=1}^{q}\cos\left(\psi_l - c_0\cos\left(\alpha_l - \beta_l\right)\right) \tag{82}$$

The maximum likelihood emitter location estimator using the phase measurements ψ_l, $1 \leq l \leq q$, denoted by $(\hat{x}_\Psi, \hat{y}_\Psi)$, is then computed by maximizing $\log(f_\Psi)$, or equivalently, by minimizing the objective function L_Ψ defined by

$$L_\Psi = \sum_{l=1}^{q}\left\{\sin\left[\frac{\psi_l - c_0\cos\left(\alpha_l - \beta_l\right)}{2}\right]\right\}^2 \tag{83}$$

since $-q\log(2\pi I_0(\overline{\kappa}))$ and $\overline{\kappa}$ are constants independent of the emitter location (x,y) and

$$\sum_{l=1}^{q}\cos\left(\psi_l - c_0\cos\left(\alpha_l-\beta_l\right)\right) = q - 2\sum_{l=1}^{q}\left\{\sin\left[\frac{\psi_l - c_0\cos\left(\alpha_l-\beta_l\right)}{2}\right]\right\}^2 \tag{84}$$

If the phase measurements ψ_l, $1 \le l \le q$, follow the Gaussian distributions (10) with the standard deviations $\overline{\tau}_l$ all identical to $\overline{\tau} > 0$, the maximum likelihood emitter location estimator, also denoted by $(\hat{x}_\Psi, \hat{y}_\Psi)$, can be shown to be obtained by the following minimization:

$$(\hat{x}_\Psi, \hat{y}_\Psi) = \text{argmin } L_\Psi \tag{85}$$

where

$$L_\Psi = \sum_{l=1}^{q}\left[\psi_l - c_0\cos\left(\alpha_l-\beta_l\right)\right]^2 \tag{86}$$

We next compute the Fisher information matrix of Ψ and the Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators.

First assume that the random variables ψ_l, $1 \le l \le q$, follow the von Mises distributions (9) and the positive parameters $\overline{\kappa}_l$ are all identical and equal to $\overline{\kappa} > 0$. From (82) and (8) it follows that

$$\begin{aligned}
\frac{\partial\log(f_\Psi)}{\partial x} &= \sum_{l=1}^{q}\overline{\kappa}\frac{\partial}{\partial x}\cos\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right) \\
&= -c_0\sum_{l=1}^{q}\overline{\kappa}\left[\sin\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right)\right]\left[\sin\left(\alpha_l-\beta_l\right)\right]\frac{\partial\alpha_l}{\partial x} \\
&= -c_0\sum_{l=1}^{q}\overline{\kappa}\left[\sin\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right)\right]\left[\sin\left(\alpha_l-\beta_l\right)\right]\frac{[-(y-z_l)]}{(x-w_l)^2+(y-z_l)^2} \\
&= c_0\,\overline{\kappa}\sum_{l=1}^{q}\left[\sin\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right)\right]\frac{\left[\sin\left(\alpha_l-\beta_l\right)\right]\sin\alpha_l}{t_l}
\end{aligned} \tag{87}$$

and

$$\begin{aligned}
\frac{\partial\log(f_\Psi)}{\partial y} &= \sum_{l=1}^{q}\overline{\kappa}\frac{\partial}{\partial y}\cos\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right) \\
&= -c_0\sum_{l=1}^{q}\overline{\kappa}\left[\sin\left(\psi_l - c_0\cos(\alpha_l-\beta_l)\right)\right]\left[\sin\left(\alpha_l-\beta_l\right)\right]\frac{\partial\alpha_l}{\partial y}
\end{aligned}$$

$$= -c_0 \sum_{l=1}^{q} \overline{\kappa} \left[\sin\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right] \left[\sin\left(\alpha_l - \beta_l\right)\right] \frac{x - w_l}{(x - w_l)^2 + (y - z_l)^2}$$

$$= -c_0 \; \overline{\kappa} \sum_{l=1}^{q} \left[\sin\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right] \frac{\left[\sin\left(\alpha_l - \beta_l\right)\right] \cos \alpha_l}{t_l} \tag{88}$$

Hence

$$E\left\{\left[\frac{\partial \log(f_\Psi)}{\partial x}\right]^2\right\} =$$

$$(c_0 \; \overline{\kappa})^2 \sum_{l=1}^{q} \left[\frac{\left[\sin\left(\alpha_l - \beta_l\right)\right] \sin \alpha_l}{t_l}\right]^2 E\left\{\left[\sin\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right]^2\right\} \tag{89}$$

where

$$E\left\{\left[\sin\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right]^2\right\} = \frac{1}{2\pi I_0(\overline{\kappa})} \times$$

$$\int_{-\pi}^{\pi} \left[\sin\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right]^2 \exp\left\{\overline{\kappa} \cos\left(\psi_l - c_0 \cos(\alpha_l - \beta_l)\right)\right\} d\psi_l$$

$$= \frac{1}{2\pi I_0(\overline{\kappa})} \int_{-\pi}^{\pi} e^{\overline{\kappa} \cos t} \sin^2 t dt = \frac{1}{\pi I_0(\overline{\kappa})} \int_{0}^{\pi} e^{\overline{\kappa} \cos t} \sin^2 t \; dt = \frac{g(\overline{\kappa})}{I_0(\overline{\kappa})} \tag{90}$$

where

$$g(x) = \frac{1}{\pi} \int_{0}^{\pi} e^{x \cos t} \sin^2 t \; dt$$

Hence

$$E\left\{\left[\frac{\partial \log(f_\Psi)}{\partial x}\right]^2\right\} = (c_0 \; \overline{\kappa})^2 \; \frac{g(\overline{\kappa})}{I_0(\overline{\kappa})} \sum_{l=1}^{q} \left[\frac{\left[\sin\left(\alpha_l - \beta_l\right)\right] \sin \alpha_l}{t_l}\right]^2 \tag{91}$$

Similarly,

$$E\left\{\left[\frac{\partial \log(f_\Psi)}{\partial y}\right]^2\right\} = (c_0 \; \overline{\kappa})^2 \; \frac{g(\overline{\kappa})}{I_0(\overline{\kappa})} \sum_{l=1}^{q} \left[\frac{\left[\sin\left(\alpha_l - \beta_l\right)\right] \cos \alpha_l}{t_l}\right]^2 \tag{92}$$

and

$$E\left\{\frac{\partial \log(f_\Psi)}{\partial x} \frac{\partial \log(f_\Psi)}{\partial y}\right\} = -\left(c_0 \; \overline{\kappa}\right)^2 \; \frac{g(\overline{\kappa})}{I_0(\overline{\kappa})} \sum_{l=1}^{q} \frac{\left[\sin\left(\alpha_l - \beta_l\right)\right]^2 \cos \alpha_l \sin \alpha_l}{t_l^2} \tag{93}$$

The Fisher information matrix of Ψ, denoted by $\mathbf{J}_\Psi$, is then computed by:

$$\mathbf{J}_\Psi = \begin{bmatrix} E\left\{\left[\frac{\partial \log(f_\Psi)}{\partial x}\right]^2\right\} & E\left\{\frac{\partial \log(f_\Psi)}{\partial x} \frac{\partial \log(f_\Psi)}{\partial y}\right\} \\ E\left\{\frac{\partial \log(f_\Psi)}{\partial x} \frac{\partial \log(f_\Psi)}{\partial y}\right\} & E\left\{\left[\frac{\partial \log(f_\Psi)}{\partial y}\right]^2\right\} \end{bmatrix}$$

$$= \frac{(c_0\ \overline{\kappa})^2 g(\overline{\kappa})}{I_0(\overline{\kappa})} \begin{bmatrix} \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 & -\sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 \end{bmatrix} \quad (94)$$

and its inverse is computed by

$$(\mathbf{J}_\Psi)^{-1} = \frac{I_0(\overline{\kappa})}{(c_0\ \overline{\kappa})^2 g(\overline{\kappa})} \times$$

$$\frac{\begin{bmatrix} \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 & \sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ \sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 \end{bmatrix}}{\det \begin{bmatrix} \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 & -\sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^{q} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^{q} \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 \end{bmatrix}} \quad (95)$$

The Cramer-Rao lower bound for the average miss distance, denoted by $CRLB_\Psi$, is then computed by

$$CRLB_\Psi = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Psi)^{-1}\right]} = \frac{\sqrt{\frac{I_0(\overline{\kappa})}{g(\overline{\kappa})}}}{c_0\ \overline{\kappa}} \times$$

$$\left\{ \frac{\sum_{l=1}^{q} \frac{(\sin(\alpha_l-\beta_l))^2}{t_l^2}}{\left[\sum_{l=1}^{q} \frac{(\sin(\alpha_l-\beta_l))^2\cos^2\alpha_l}{t_l^2}\right]\left[\sum_{l=1}^{q} \frac{(\sin(\alpha_l-\beta_l))^2\sin^2\alpha_l}{t_l^2}\right] - \left[\sum_{l=1}^{q} \frac{(\sin(\alpha_l-\beta_l))^2\cos\alpha_l\sin\alpha_l}{t_l^2}\right]^2} \right\}^{\frac{1}{2}} \quad (96)$$

Next assume that the phase measurements ψ_l, $1 \leq l \leq q$, follow the Gaussian distribution (10) with the standard deviations $\overline{\tau}_l$ all identical and equal to $\overline{\tau} > 0$.

We can write

$$E(\psi_l) = c_0 \cos(\alpha_l - \beta_l) \quad (97)$$

Since ψ_l are independent and identically distributed, the covariance matrix of Ψ is computed by $\overline{\tau}^2\mathbf{I}$, where $\mathbf{I}$ is the $q \times q$ identity matrix. We also have

$$\frac{\partial E(\psi_k)}{\partial x} = -c_0 \sin(\alpha_l - \beta_l)\frac{\partial \alpha_l}{\partial x} = c_0 \sin(\alpha_l - \beta_l)\frac{\sin\alpha_l}{t_l} \quad (98)$$

$$\frac{\partial E(\psi_l)}{\partial y} = -c_0 \sin(\alpha_l - \beta_l)\frac{\partial \alpha_l}{\partial y} = -c_0 \sin(\alpha_l - \beta_l)\frac{\cos\alpha_l}{t_l} \quad (99)$$

Applying the formula (3.31) in [3], or the formula (A.8) in Annex A, we can write the Fisher information matrix of Ψ, $\mathbf{J}_\Psi$, as follows:

$$\mathbf{J}_\Psi = \frac{c_0^2}{\overline{\tau}^2} \begin{bmatrix} \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 & -\sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 \end{bmatrix} \tag{100}$$

The inverse matrix of $\mathbf{J}_\Psi$ is then computed by:

$$(\mathbf{J}_\Psi)^{-1} = \frac{\frac{\overline{\tau}^2}{c_0^2}\begin{bmatrix} \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 & \sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ \sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 \end{bmatrix}}{\det\begin{bmatrix} \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 & -\sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^q \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^q \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 \end{bmatrix}} \tag{101}$$

and the Cramer-Rao lower bound $CRLB_\Psi$ is computed by

$$CRLB_\Psi = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Psi)^{-1}\right]} = \frac{\overline{\tau}}{c_0} \times \left\{\frac{\sum_{l=1}^q \frac{(\sin(\alpha_l-\beta_l))^2}{t_l^2}}{\left[\sum_{l=1}^q \frac{(\sin(\alpha_l-\beta_l))^2\cos^2\alpha_l}{t_l^2}\right]\left[\sum_{l=1}^q \frac{(\sin(\alpha_l-\beta_l))^2\sin^2\alpha_l}{t_l^2}\right] - \left[\sum_{l=1}^q \frac{(\sin(\alpha_l-\beta_l))^2\cos\alpha_l\sin\alpha_l}{t_l^2}\right]^2}\right\}^{\frac{1}{2}} \tag{102}$$

Again it can be observed that the Fisher information matrices and the Cramer-Rao lower bounds derived under the von Mises distribution (9) and the Gaussian distribution (10) are identical, except for the different scaling constants in (94), (96), (100) and (102).

The preceding results are derived under the assumption that the phase measurements ψ_k, $1 \le k \le q$, have the same variances. These results can be easily extended to the general case where the parameters $\overline{\kappa}_l$ or the standard deviations $\overline{\tau}_l$ are not necessarily identical. We shall only state here the relevant results without detailed proofs.

First assume that the von Mises densities (9) are used to model the phase measurements ψ_l, $1 \le l \le q$. The probability density function f_Ψ can then be written as:

$$f_\Psi = C(\overline{\kappa}_1, \ldots, \overline{\kappa}_q)\ \exp\left\{\sum_{l=1}^q \overline{\kappa}_l \cos\left(\psi_l - c_0\cos\left(\alpha_l - \beta_l\right)\right)\right\} \tag{103}$$

where $C(\overline{\kappa}_1, \ldots, \overline{\kappa}_q)$ is a positive constant independent of the emitter location (x, y). The objective function L_Ψ defined by (83) is now transformed to become

$$L_\Psi = \sum_{l=1}^{q} g_l \left\{ \sin\left[\frac{\psi_l - c_0 \cos(\alpha_l - \beta_l)}{2} \right] \right\}^2 \tag{104}$$

where

$$g_l = \frac{\overline{\kappa}_l}{\sum_{m=1}^{q} \overline{\kappa}_m} \tag{105}$$

and the Fisher information matrix of Ψ now takes the form:

$$\begin{aligned}
\mathbf{J}_\Psi &= \begin{bmatrix} E\left\{ \left[\frac{\partial \log(f_\Psi)}{\partial x} \right]^2 \right\} & E\left\{ \frac{\partial \log(f_\Psi)}{\partial x} \frac{\partial \log(f_\Psi)}{\partial y} \right\} \\ E\left\{ \frac{\partial \log(f_\Psi)}{\partial x} \frac{\partial \log(f_\Psi)}{\partial y} \right\} & E\left\{ \left[\frac{\partial \log(f_\Psi)}{\partial y} \right]^2 \right\} \end{bmatrix} \\
&= c_0^2 \begin{bmatrix} \sum_{l=1}^{q} \frac{\overline{\kappa}_l^2 g(\overline{\kappa}_l)}{I_0(\overline{\kappa}_l)} \left[\frac{[\sin(\alpha_l - \beta_l)] \sin\alpha_l}{t_l} \right]^2 & -\sum_{l=1}^{q} \frac{\overline{\kappa}_l^2 g(\overline{\kappa}_l)}{I_0(\overline{\kappa}_l)} \frac{[\sin(\alpha_l - \beta_l)]^2 \cos\alpha_l \sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^{q} \frac{\overline{\kappa}_l^2 g(\overline{\kappa}_l)}{I_0(\overline{\kappa}_l)} \frac{[\sin(\alpha_l - \beta_l)]^2 \cos\alpha_l \sin\alpha_l}{t_l^2} & \sum_{l=1}^{q} \frac{\overline{\kappa}_l^2 g(\overline{\kappa}_l)}{I_0(\overline{\kappa}_l)} \left[\frac{[\sin(\alpha_l - \beta_l)] \cos\alpha_l}{t_l} \right]^2 \end{bmatrix}
\end{aligned} \tag{106}$$

and the Cramer-Rao lower bound for the average miss distance can then be computed by:

$$CRLB_\Psi = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Psi)^{-1} \right]} \tag{107}$$

where $\mathbf{J}_\Psi$ is computed by (106).

Next assume that the Gaussian model (10) is used to model the phase measurement ψ_l, $1 \leq l \leq q$. The probability density function f_Ψ can then be written as:

$$f_\Psi = C(\overline{\tau}_1, \ldots, \overline{\tau}_q) \exp\left\{ -\frac{1}{2} \sum_{l=1}^{q} \overline{\tau}_l^{-2} \left(\psi_l - c_0 \cos(\alpha_l - \beta_l) \right)^2 \right\} \tag{108}$$

where $C(\overline{\tau}_1, \ldots, \overline{\tau}_q)$ is a positive constant independent of the emitter location (x, y). The objective function $\widetilde{L}_\Phi$ defined by (86) now becomes

$$L_\Psi = \sum_{l=1}^{q} h_l \left[\psi_l - c_0 \cos(\alpha_l - \beta_l) \right]^2 \tag{109}$$

with

$$h_l = \frac{\overline{\tau}_l^{-2}}{\sum_{m=1}^{q} \overline{\tau}_m^{-2}} \tag{110}$$

The Fisher information matrix of Ψ now takes the form:

$$\mathbf{J}_\Psi = c_0^2 \begin{bmatrix} \sum_{l=1}^{q} \frac{1}{\overline{\tau}_l^2} \left[\frac{[\sin(\alpha_l-\beta_l)]\sin\alpha_l}{t_l}\right]^2 & -\sum_{l=1}^{q} \frac{1}{\overline{\tau}_l^2} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} \\ -\sum_{l=1}^{q} \frac{1}{\overline{\tau}_l^2} \frac{[\sin(\alpha_l-\beta_l)]^2\cos\alpha_l\sin\alpha_l}{t_l^2} & \sum_{l=1}^{q} \frac{1}{\overline{\tau}_l^2} \left[\frac{[\sin(\alpha_l-\beta_l)]\cos\alpha_l}{t_l}\right]^2 \end{bmatrix} \quad (111)$$

and the Cramer-Rao lower bound for the average miss distance can then be computed by (107) with $\mathbf{J}_\Psi$ computed by (111).

6 Maximum likelihood geolocation using a combination of different ground-based sensors

If data measurements from the three types of sensors discussed in the previous sections are utilized in combination for emitter geolocation, a maximum likelihood emitter location estimator can be obtained by maximizing the joint probability density function of the three random vectors Ω, Φ and Ψ, which are defined respectively by (12), (47) and (80) and assumed to be statistically independent. For simplicity, we shall only present the relevant formulas for the simple case where the measurements for each type of sensor have the same variance and the AOA and phase measurements are respectively modeled by the von Mises densities (3) and (9), with $\sigma_1 = \ldots = \sigma_n = \sigma > 0$, $\kappa_1 = \kappa_2 = \ldots = \kappa_p = \kappa > 0$ and $\overline{\kappa}_1 = \overline{\kappa}_2 = \ldots = \overline{\kappa}_q = \overline{\kappa} > 0$. All the other cases can be dealt with in a similar manner and are thus omitted here.

Specifically, let

$$\Delta = [\Omega^T, \Phi^T, \Psi^T]^T \tag{112}$$

and denote its probability density function by f_Δ. Since Ω, Φ, Ψ are independently distributed random vectors, it follows that $f_\Delta = f_\Omega f_\Phi f_\Psi$ and the log-likelihood function $\log(f_\Delta)$ is then computed by

$$\log(f_\Delta) = \log(f_\Omega) + \log(f_\Phi) + \log(f_\Psi) \tag{113}$$

Combining (23), (48) and (81) yields the following expression for $\log(f_\Delta)$:

$$\begin{aligned}\log(f_\Delta) &= \log(f_\Omega) + \log(f_\Phi) + \log(f_\Psi)\\ &= C - \frac{1}{2\sigma^2}\sum_{k=1}^{n}\widetilde{\Omega}_k^2 - 2\kappa\sum_{l=1}^{p}\left\{\sin\left[\frac{\phi_l - \theta_l}{2}\right]\right\}^2\\ &\quad -2\overline{\kappa}\sum_{l=1}^{q}\left\{\sin\left[\frac{\psi_l - c_0\cos\left(\alpha_l - \beta_l\right)}{2}\right]\right\}^2\end{aligned} \tag{114}$$

where C is a constant independent of the emitter location (x, y). The function $\log(f_\Delta)$ can be further rewritten as

$$\log(f_\Delta) = C - \left[\frac{1}{2\sigma^2} + 2\kappa + 2\overline{\kappa}\right] L_\Delta \tag{115}$$

where

$$L_\Delta = \mu_\Omega L_\Omega + \mu_\Phi L_\Phi + \mu_\Psi L_\Psi \tag{116}$$

with L_Ω, L_Φ and L_Ψ defined respectively by (24), (50) and (83) and the positive weights μ_Ω, μ_Λ, μ_Φ and μ_Ψ are computed by:

$$\begin{cases} \mu_\Omega &= \frac{\frac{1}{2\sigma^2}}{\frac{1}{2\sigma^2}+2\kappa+2\overline{\kappa}} > 0 \\ \mu_\Phi &= \frac{2\kappa}{\frac{1}{2\sigma^2}+2\kappa+2\overline{\kappa}} > 0 \\ \mu_\Psi &= \frac{2\overline{\kappa}}{\frac{1}{2\sigma^2}+2\kappa+2\overline{\kappa}} > 0 \end{cases} \tag{117}$$

with

$$\mu_\Omega + \mu_\Phi + \mu_\Psi = 1$$

The maximum likelihood emitter location estimate using all the three types of sensors, denoted by $(\hat{x}_\Delta, \hat{y}_\Delta)$, is then obtained by maximizing $\log(f_\Delta)$, or equivalently, by minimizing the objective function L_Δ defined by (116), i.e.,

$$\begin{aligned} (\hat{x}_\Delta, \hat{y}_\Delta) &= \text{argmin } L_\Delta \\ &= \arg\min \left\{ \mu_\Omega \sum_{k=1}^{n} \widetilde{\Omega}_k^2 + \mu_\Phi \sum_{l=1}^{p} \left\{ \sin\left[\frac{\phi_l - \theta_l}{2}\right]\right\}^2 \right. \\ &\qquad \left. + \mu_\Psi \sum_{l=1}^{q} \left\{ \sin\left[\frac{\psi_l - c_0\cos(\alpha_l - \beta_l)}{2}\right]\right\}^2 \right\} \end{aligned} \tag{118}$$

where $c_0 = \frac{2\pi d_0}{\lambda_0}$ (see Eq. (9) of this technical memorandum).

The statistical independence of Ω, Φ and Ψ also implies that the Fisher information matrix of Δ, denoted by $\mathbf{J}_\Delta$, is equal to the sum of the Fisher information matrices, $\mathbf{J}_\Omega$, $\mathbf{J}_\Phi$ and $\mathbf{J}_\Psi$, i.e.,

$$\mathbf{J}_\Delta = \mathbf{J}_\Omega + \mathbf{J}_\Phi + \mathbf{J}_\Psi \tag{119}$$

where $\mathbf{J}_\Omega$ is computed by (36) and $\mathbf{J}_\Phi$ and $\mathbf{J}_\Psi$ are computed respectively by (62) and (94) and the Cramer-Rao lower bound for the average miss distance, denoted by $CRLB_\Delta$, is then defined and computed by:

$$CRLB_\Delta = \sqrt{\mathbf{tr}\left[(\mathbf{J}_\Delta)^{-1}\right]} \tag{120}$$

where $\mathbf{J}_\Delta$ is computed by (119).

7 Conclusions

This technical memorandum investigates a maximum likelihood approach for estimating the position of an emitter in a two dimensional space using data from a combination of RSS, AOA and two-channel phase interferometer (ambiguous AOA) sensors. Both the von Mises and the Gaussian distributions were used to model the AOA and two-channel interferometer phase measurements. First, the maximum likelihood emitter location estimator and the associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance were derived for each type of sensor data. The hybrid ML emitter location estimator was then obtained as a corollary under the assumption of statistical independence for the different types of sensor data. In addition, the associated Fisher information matrix and the Cramer-Rao lower bound for the average miss distance for unbiased emitter location estimators were derived for hybrid data.

The simple formulas derived for the likelihood function, the Fisher information matrix and the Cramer-Rao lower bound for the average miss distance provide the necessary theoretical tools essential for algorithm implementation and performance analysis. The Fisher information matrices are particularly useful for estimating geolocation performance metrics such as the CEP (circular error probable) and the EEP (elliptical error probable) for the proposed hybrid maximum likelihood geolocation algorithm.

Results from more systematic computer simulations and tests with off-the-air measurement data will be provided in subsequent technical memoranda.

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Annex A: Formula for computing the Fisher information matrix for a Gaussian distribution

This section introduces a very useful formula for computing the Fisher information matrix for a Gaussian distributed random vector [3].

Suppose that the N-dimensional random vector $\mathbf{X}$ has a Gaussian distribution:

$$\mathbf{X} \sim \mathcal{N}(\mu(\theta), \mathbf{C}(\theta)) \tag{A.1}$$

where $\theta = [\theta_1, \theta_2, \cdots, \theta_M]^T$ is an M-dimensional parameter vector and $\mu(\theta)$ and $\mathbf{C}(\theta)$ are, respectively, the mean and the covariance of $\mathbf{X}$, which are denoted respectively by

$$\mu(\theta) = [\mu_1(\theta), \cdots, \mu_N(\theta)]^T \tag{A.2}$$

$$\mathbf{C}(\theta) = [C_{kl}(\theta)]_{1 \le k,l \le N} \tag{A.3}$$

Let the $M \times M$ Fisher information matrix of $\mathbf{X}$ be denoted by

$$\mathbf{J} = [I_{kl}]_{1 \le k,l \le M} \tag{A.4}$$

Let

$$\frac{\partial \mu}{\partial \theta_m} = \left[\frac{\partial \mu_1}{\partial \theta_m}, \cdots, \frac{\partial \mu_N}{\partial \theta_m} \right]^T \tag{A.5}$$

and

$$\frac{\partial \mathbf{C}(\theta)}{\partial \theta_m} = \left[\frac{\partial C_{kl}(\theta)}{\partial \theta_m} \right]_{1 \le k,l \le N} \tag{A.6}$$

Then (see Eqn. 3.31 of [3])

$$I_{mn} = \left(\frac{\partial \mu}{\partial \theta_m} \right)^T \mathbf{C}(\theta)^{-1} \frac{\partial \mu}{\partial \theta_n} + \frac{1}{2} \mathbf{tr} \left[\mathbf{C}(\theta)^{-1} \frac{\partial \mathbf{C}(\theta)}{\partial \theta_m} \mathbf{C}(\theta)^{-1} \frac{\partial \mathbf{C}(\theta)}{\partial \theta_n} \right] \tag{A.7}$$

If $\mathbf{C}(\theta) = \mathbf{C}$ does not depend on θ, then I_{mn} assumes the simple form:

$$I_{mn} = \left(\frac{\partial \mu}{\partial \theta_m} \right)^T \mathbf{C}^{-1} \frac{\partial \mu}{\partial \theta_n} \tag{A.8}$$

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Emitter location estimation (geolocation) is often implemented by processing measurements of a single signal parameter, such as angle of arrival (AOA), time of arrival (TOA), time difference of arrival (TDOA), frequency difference of arrival (FDOA) or received signal strength (RSS). In recent years, approaches for utilizing hybrid sensor measurements for improved geolocation accuracy have attracted much interest. This technical memorandum derives a hybrid maximum likelihood emitter location estimator that utilizes geolocation data from a mixture of AOA, RSS and two-channel phase interferometer (ambiguous AOA) sensors. To provide the basis for subsequent derivations, we first formulate the maximum likelihood emitter location estimator for each individual type of sensor data. The hybrid algorithm is then obtained as a corollary under the assumption of statistical independence for the three types of sensor data. The Fisher information matrix and the Cramer-Rao lower bound (CRLB) for the average miss distance are also derived for each of the emitter location estimators. These results provide very useful theoretical benchmarks for the performance analysis of geolocation systems and for the estimation of geolocation performance metrics, such as the CEP (circular error probable) and the EEP (elliptical error probable).

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