Sensitivity analysis of additive weighted scoring methods

How to fool your friends (again)

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Materiel Group Operational Research / AST
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Abstract

This paper generalizes a sensitivity analysis approach originally suggested by Brereton [Brereton, R.C. (1977), Weighting Factors and How to Fool Your Friends, (Staff Note ORD SN 1977/06) Operational Research Division, Ottawa, Canada] for additive weighted scoring methods. Using high-dimensional computational geometry, three different sensitivity measure classes are proposed: volume, distance and representativity. The methodology can be used to analyze the subjective selection of criteria weights and provide decision makers with quantitative evidence to evaluate the robustness of the output ranking to small variations in the proposed weights. In contrast to Brereton's approach, the proposed methodology can be used to analyze the sensitivity of a ranking for *any* subset of options. Furthermore the methodology is easy to apply using existing software available to analysts.

The Acquisition Support Team (AST) applied the methodology on an options analysis problem faced by Project Management Office Canadian Surface Combatant (PMO CSC). AST demonstrated that the overall ranking of options presented by PMO CSC is stable against moderate variations in the weights and has not produced any evidence for fine-tuning.

Résumé

Cet article propose une généralisation de la méthode d'analyse de sensibilité que Brereton a mis de l'avant il y a plusieurs années [Brereton, R.C. (1977), Weighting Factors and How to Fool Your Friends, note d'état major ORD SN 1977/06, Division de la recherche opérationnelle, Ottawa, Canada] pour la notation pondérée. À l'aide de la géométrie algorithmique en grandes dimensions, on définit trois types de mesures de sensibilité : volume, distance et représentativité. La méthode peut servir à analyser le choix subjectif des poids des critères et à fournir aux décideurs des données quantitatives qui leur permettront d'évaluer la robustesse du rendement se rangeant à de petites variations des poids proposés. À la différence de l'approche de Brereton, la méthode proposée peut servir à analyser la sensibilité d'un classement d'un sous ensemble d'options. Les analystes peuvent en outre l'appliquer facilement au moyen des logiciels dont ils disposent déjà.

L'équipe de soutien des acquisitions (ESA) s'est servie de la méthode pour réaliser une analyse d'options pour le compte du Bureau de projet – Bâtiment de combat de surface du Canada (BP BCSC). L'ESA a démontré que le classement global des options établi par BP BCSC est peu sensible à des variations modérées des poids et n'indique d'aucune manière qu'il faille ajuster ce classement.

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Executive summary

Sensitivity analysis of additive weighted scoring methods

Bohdan L. Kaluzny, R. H. A. David Shaw; DRDC CORA TR 2009–002; Defence R&D Canada – CORA; April 2009.

Background: Within any large organization, decision-makers need to strike an appropriate balance between competing objectives. Fortunately, complex decisions can be facilitated through the application of multi-criteria decision analysis. The general form of a multi-criteria decision analysis problem is the evaluation of a number of options against a number of criteria. A simple, commonly-used approach is to apply an additive weighted scoring rule where the criteria are rated per option and relative weights are applied to the criteria ratings. The options are scored by taking the sum of the products (weight times rating) over all criteria. The ordering of options from highest scoring to lowest scoring provides a final ranking. When the criteria weights are subjectively chosen they should be analyzed to determine the sensitivity of the final ranking to changes in these weights.

Summary of principal results: This paper provides a methodology to answer the central questions:

How typical are the chosen weights?
How sensitive is the final ranking to changes in these weights?

Each ranking of options corresponds to a well-defined, continuous region within the space of the set of all possible weight allocations. We use high-dimensional computational geometry, namely polyhedral theory, to define three classes of sensitivity measures: volume, distance, and representativity. Volume is a global measure of sensitivity. We consider the volume of each ranking region relative to the volume of the entire weight space. The larger the region, the more robust (in the mathematical sense) is the selection of the corresponding ranking. The distance from a given set of weights to the nearest region boundary comprises a second, more local measure of sensitivity. Distance can be measured as the Euclidean distance between points or as a user-defined function of the weights (such as minimum percent change) subject to constraints. A third measure of sensitivity determines the degree to which the selected criteria weights represent the ranking weight-space they occupy, where the centroid point of the region is most representative and the region's boundary the least.

The majority of approaches proposed in scientific literature measure the amount of change to a single or pair of weights required to alter the final ranking. Some methods implicitly attempt to classify the weight-space. Using polyhedral computation we classify the weight space exactly and determine the three measures of sensitivity in general. The primary strength and novelty is that it can be used to analyze a change in the ranking of *any* subset of the options. Furthermore the methodology is flexible (weights can be subjected to customized constraints) and easy to apply in practice using software available to researchers.

Significance of results: The sensitivity methodology presented should be of use to Centre for Operational Research and Analysis scientists asked to evaluate the sensitivity of a ranking of options subject to changes to chosen criteria weights. The Acquisition Support Team applied the methodology on an options analysis problem faced by Project Management Office Canadian Surface Combatant (PMO CSC). AST demonstrated that the overall ranking of options presented by PMO CSC is stable against moderate variations in the weights and has not produced any evidence for fine-tuning; however, AST advised the sponsor that using a different consensus ranking technique reveals some contention as to the ordering of the third and fourth options.

Sommaire

Sensitivity analysis of additive weighted scoring methods

Bohdan L. Kaluzny, R. H. A. David Shaw; DRDC CORA TR 2009–002; R & D pour la défense Canada – CARO; avril 2009.

Considérations générales: Au sein des grandes organisations, les décideurs doivent souvent concilier des objectifs divergents. Heureusement, l'analyse décisionnelle à plusieurs variables facilite la prise de décisions dans ces circonstances. De façon générale, cette méthode consiste à évaluer diverses options en fonction d'un certain nombre de critères. Le plus souvent, on appliquera une règle de notation pondérée selon laquelle on attribue une note et un poids relatif à chacun des critères pour chaque option. On évalue les options en faisant la somme des produits (poids x note) sur l'ensemble des critères, puis on classe ces options par ordre décroissant du score obtenu. Si les poids attribués aux critères sont choisis de manière subjective, on doit les analyser pour déterminer le degré de sensibilité du classement final aux variations de ces poids.

Principaux résultats: Cet article propose une méthode qui permettra de répondre aux questions fondamentales suivantes :

Les poids choisis sont ils ceux qu'on utilise le plus souvent ?
Quel est le degré de sensibilité du classement final aux variations de ces poids ?

Chaque classement d'options correspond à une région continue bien définie dans l'espace de l'ensemble de toutes les pondérations possibles. À l'aide de la géométrie algorithmique en grandes dimensions, et notamment de la théorie polyédrique, nous définissons trois types de mesures de sensibilité : volume, distance et représentativité. Le volume est une mesure de sensibilité globale. Nous considérons le volume de chaque région de classement par rapport au volume de l'espace-poids total. Plus la région est grande, moins le classement correspondant sera sensible. La distance entre un ensemble donné de poids et la frontière la plus proche de la région constitue la deuxième mesure de sensibilité, qui a un caractère plus local. Cette distance peut être mesurée comme la distance euclidienne entre des points ou calculée comme une fonction ? définie par l'utilisateur ? des poids (p. ex. variation relative minimum) soumise à des contraintes. Le troisième indice de sensibilité sert à déterminer dans quelle mesure les poids choisis sont représentatifs de l'espace poids qu'ils occupent, le centroïde de la région correspondant au poids le plus représentatif et la frontière de la région correspondant aux poids les moins représentatifs. La plupart des méthodes proposées dans les ouvrages scientifiques servent à déterminer le degré de sensibilité du classement final à la variation d'un ou de deux poids. Certaines méthodes tentent implicitement d'ordonner l'espace poids. Au moyen du calcul polyédrique, nous ordonnons l'espace poids avec exactitude et déterminons les trois mesures de sensibilité. Le principal avantage de cette méthode, et en même temps son aspect inédit, est qu'elle peut servir à analyser la variation d'un classement d'un sous ensemble d'options. Elle est en outre flexible (les poids peuvent être soumis à des contraintes définies par l'utilisateur) et les chercheurs peuvent l'appliquer facilement au moyen des logiciels dont ils disposent déjà.

Portée des résultats: La méthode d'analyse de sensibilité décrite dans cet article sera utile aux spécialistes du Centre d'analyse et de recherche opérationnelle qui sont chargés de mesurer la sensibilité d'un classement d'options aux variations des poids choisis attribués aux critères. L'équipe de soutien des acquisitions s'est servie de la méthode pour réaliser une analyse d'options pour le compte du Bureau de projet – Bâtiment de combat de surface du Canada (BP BCSC). L'équipe a démontré que le classement global des options établi par BP BCSC est peu sensible à des variations modérées des poids et n'indique d'aucune manière qu'il faille ajuster ce classement ; elle a toutefois avisé le commanditaire que si l'on utilise une autre méthode de classement par consensus, il y a divergence dans le classement des troisième et quatrième options.

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1 Introduction

1.1 Background

Within the Assistant Deputy Minister (Materiel) (ADM(Mat)) organization, decision-makers are frequently called upon to strike an appropriate balance between competing objectives. In acquiring a new system, for example, a decision-maker might wish to identify the option that maximizes the capability provided to the Canadian Forces (CF) while simultaneously minimizing the cost of the system over the course of its life cycle. Adequately defining the nebulous concept of "capability" requires, in turn, the assessment of the relative importance of such objectives as minimizing the system weight, maximizing system reliability, etc. Fortunately, complex decisions such as these can be facilitated through the application of multi-criteria decision analysis (MCDA).

MCDA is a remarkably complex area of study with a rich and extensive literature (see [1]). The general form of a multi-criteria decision analysis problem is the evaluation of a number of options against a number of criteria. One commonly used, mathematically simple approach is to apply an additive weighted scoring rule where the criteria are rated per option and relative weights are applied to the criteria. The options are scored by taking the sum of the products (weight times rating) over all criteria. Given n options and m criteria, let v_{ij} be the rating of option j relative to criterion i. Let W_i be the weight allocated to criteria i. The combined rating (or score) of an option j is defined as the sum of the products $v_{ij}W_i$ over all criteria i. The option with the highest score is deemed the "winner." Furthermore, the ordering of options from highest scoring to lowest provides a final consensus ranking or preference order. When the weights are subjectively chosen they should be analyzed to determine the sensitivity of the final ranking to changes in these weights. This paper provides a methodology to answer the central questions:

How typical are the chosen weights?
How sensitive is the final ranking to changes in these weights?

A similar question was studied by Brereton [2] who asked:

"*What W_i's are required so that the relative ranking of the* [options] *is altered*?"

Motivated by an actual staff check of a consensus ranking of options evaluated by Project Management Office (PMO) Canadian Surface Combatant (CSC) performed by Directorate Materiel Group Operational Research (DMGOR) Acquisition Support Team (AST) at the request of Director General Major Projects Delivery (Land & Sea) (DGMPD (L&S)), this paper generates three geometrically based measures of sensitivity of the final consensus ranking to changes to the allocation of weights to individual criteria.

1.2 Previous work

Several approaches to sensitivity analysis of additive scoring methods have been proposed in the open literature. A majority of these measure the amount of change to a single weight or a pair of weights required in order to change the ordering of the options. For example Morrice *et al.* [3] proposed a one-at-a-time sensitivity analysis on each weight while holding the ratio of the remaining weights constant. Triantaphyllou and Sanchez [4] focused their efforts on determining the most critical criterion, defined as the criterion whose smallest change in weight resulted in a different ordering of options. Triantaphyllou and Sanchez also provided a good recap of many of the other methods for sensitivity analysis of additive weighted scoring methods. We highlight some of the more interesting approaches in what follows. Gehrlein and Fishburn [5] approached the problem using probabilities. They assessed the sensitivity of a ranking of options by computing the probability that two randomly selected score vectors would yield the same rank ordering. Their analysis was limited to a maximum of four options. Evans [6] explored a linear programming-like sensitivity analysis. He used high-dimensional geometry to determine the "maximum confidence sphere" around the baseline weights for which changes to the weight allocation would yield the same ranking of options. Butler *et al.* [7] presented a Monte Carlo simulation approach which allowed for simultaneous variation of all the weights. The method randomly samples from the entire weight space to implicitly classify the weights by the rank orderings they produce.

Brereton had the insight to view the W_i's as describing a point $W = (W_1, \ldots, W_m)$ in an m-dimensional space. His stated question can then be viewed geometrically as determining the minimum distance of the selected point W to a point $\bar{W} = (\bar{W}_1, \ldots, \bar{W}_m)$ which alters the final ranking of options. Brereton defined two distance measures of sensitivity. The first, as mentioned, determines the minimum change to the weights required in order to alter the ranking. The second determines the minimum percent change of the weights to change the solution. The latter was defined in order to take into account relative weightings and also in an attempt to overcome difficulties that might be encountered with the former, namely interpreting weights set to zero. Methods for computing both measures were provided, formulating them as nonlinear problems that were solved using Lagrange multipliers and differentiation.

1.3 Objective

Many of the documented methods attempt to implicitly classify the high-dimensional weight-space. We take the same approach as Brereton and Evans and utilize geometric intuition. However we use high-dimensional computational geometry, namely polyhedral theory, to define three measures of sensitivity classes: volume, distance, and representativity. Using polyhedra we classify the weight space exactly. While the methods of Brereton and most others are limited to the comparison of pairs of options, our method is general. Its primary strength and novelty is that it can be used to analyze any subset of n options. Furthermore the methodologies are flexible and easy to apply in practice using software available to researchers.

1.4 Scope

While there are many techniques that can be applied to generate a consensus ranking of a number of distinct options, additive weighted scoring methods are of particular interest to DMGOR AST, as Public Works and Government Services Canada (PWGSC) makes extensive use of weighted scoring methods in assessing the merits of vendor bids. Given the prevalence of evaluations conducted using these techniques, there is a requirement for rigorous methods to evaluate the sensitivity of the final ranking of options to changes in the allocation of the criteria weights. This report consequently develops three such measures of sensitivity and provides an example of this application to a real world example.

1.5 Outline

This paper is structured into four main sections. In the following section, the geometric intuition is described and the sensitivity measures and methodology are formally defined. In Section 3 the proposed sensitivity analysis methodology is applied on a decision problem faced by PMO CSC. Finally, Section 4 concludes by summarizing the sensitivity analysis methodology proposed and its application to PMO CSC. Annex A demonstrates the use of software tools to perform the sensitivity analysis. Annex B presents alternative definitions for the centre of a polytope. Annex C discusses the mathematical relationship between the coefficient of representativity and the fraction of the volume contained within the scaled polytope.

2 Methodology

The additive weighted scoring rule is formally defined as follows. Given n options and m criteria, let v_{ij} be the rating of option j relative to criterion i. Let W_i be the weight allocated to criterion i normalized such that $\sum_{i=1}^{m} W_i = 100$. The combined rating (or score) of an option j is evaluated as $S_j = \sum_{i=1}^{m} v_{ij} W_i$. The option with the highest score is deemed the winner. The ordering of options from highest scoring to lowest provides a final consensus ranking.

We assume that the option ratings, v_{ij}'s, have been determined, although it should be noted that when quantitative comparisons are not possible finding appropriate relative values for the ratings is a problem in itself (see [8]). Nonetheless we assume that the ratings are fixed and that the weight allocation is variable. Varying the allocation of weights across the set of criteria may impact the conclusion generated. To avoid potential pitfalls and gaming when the weights are subjectively chosen, sensitivity analysis should be performed to determine how much variation is required to produce a change. We therefore require measures of the sensitivity of our consensus ranking to changes in the overall allocation of criteria weights.

For the remainder of this paper we label options and denote a ranking of options by using angled brackets with the most preferred option listed first, second most preferred listed second, etc. Options that are tied in a ranking will be grouped and separated by the symbol '$-$'. For example, a ranking of options A, B, C and D denoted by $\langle BAC-D \rangle$ indicates that option B is most preferred, followed by option A, and finally options C and D are tied for last place. Ranking $\langle BAD-C \rangle$ is equivalent as there is no order for tied options. This follows the notational conventions established by Emond and Mason [9].

2.1 Brereton's approach

Brereton treated the W_i's as describing a point $W = (W_1, ..., W_m)$ in an m-dimensional space. He then analyzed the sensitivity of W geometrically by determining the minimum distance of the selected point W to a point $\bar{W} = (\bar{W}_1, ..., \bar{W}_m)$ which results in an altered final ranking of two options.

Brereton defined two measures of sensitivity based on distance. The first determines the minimum change to the weights in order to alter the ranking. The second determines the minimum percent change of the weights to change the solution. The latter was defined in an attempt to overcome difficulties that might be encountered with the former, namely interpreting zero weights and misproportioned weight changes. Methods for computing both measures were provided, formulating them as nonlinear problems, (1) and (2), respectively.

$$
\begin{aligned}
\text{Minimize } \psi = & \sum_{i=1}^{m} (\bar{W}_i - W_i)^2 \\
\text{subject to } \quad & \sum_{i=1}^{m} \bar{W}_i = 100 \\
& \sum_{i=1}^{m} (v_{i1} - v_{i2}) \cdot \bar{W}_i = 0,
\end{aligned}
\tag{1}
$$

and

$$\text{minimize } \psi = \sum_{i=1}^{m} \frac{(\bar{W}_i - W_i)^2}{W_i} \quad (2)$$

$$\text{subject to} \quad \sum_{i=1}^{m} \bar{W}_i = 100$$

$$\sum_{i=1}^{m} (v_{i1} - v_{i2}) \cdot \bar{W}_i = 0.$$

Brereton provided a complicated method using Lagrange multipliers and differentiation to solve for $\bar{W}_i$. As noted in [2], in order to avoid negative weights the method may have to be iteratively applied and will "terminate eventually." It is not clear how to generalize the methodology to analyze the ranking of more than two options simultaneously.

We define an easy to apply methodology designed to analyze the sensitivity of a ranking for any subset of n options.

2.2 Geometric intuition

We are able to develop effective sensitivity measures by converting our ranking problem into an equivalent problem in geometry. To motivate our development of measures of sensitivity and to facilitate visualization, let us consider a simple, illustrative problem instance—determining the sensitivity of a ranking of three options, X, Y and Z, to variations in the allocation of weight to each of three criteria.

Figure 1 shows the combinations of weights that result in each of the six possible complete consensus rankings. Each ranking corresponds to a well-defined, continuous region within the larger triangle corresponding to the set of all possible weight allocations. The observant reader will note that we need only specify two of the three weights, as the total of all three weights must sum to 100.

With the geometrical analogue of Figure 1 in hand, two obvious measures of sensitivity come to mind. A global measure of sensitivity would be to consider the area of each region relative to the area of the entire triangle. The larger the region is, the more robust (in the mathematical sense) the selection of the corresponding consensus ranking. Phrased differently, if one were to select the criteria weights randomly, the probability of obtaining a particular consensus ranking is directly proportional to the size of the appropriate region. This is very similar to the approach taken by Gehrlein and Fishbrun [5] and by Butler *et al.* [7]. In the example of Figure 1, we would not be surprised to obtain a ranking of $\langle ZYX \rangle$ or even $\langle XZY \rangle$, but a finding of $\langle YXZ \rangle$ would be cause for some concern, as it occupies the smallest region within the triangle—one would effectively have to "fine-tune" the weights to obtain this result.

The distance from a given set of weights to the nearest region boundary comprises a second, more local measure of sensitivity. This is the measure that Brereton used [2]. Referring to Figure 1, let us say the star represents the particular allocation of criteria weights used in the baseline analysis. It is therefore clear that the "nearest" alternative solution is $\langle YXZ \rangle$. By determining the minimum

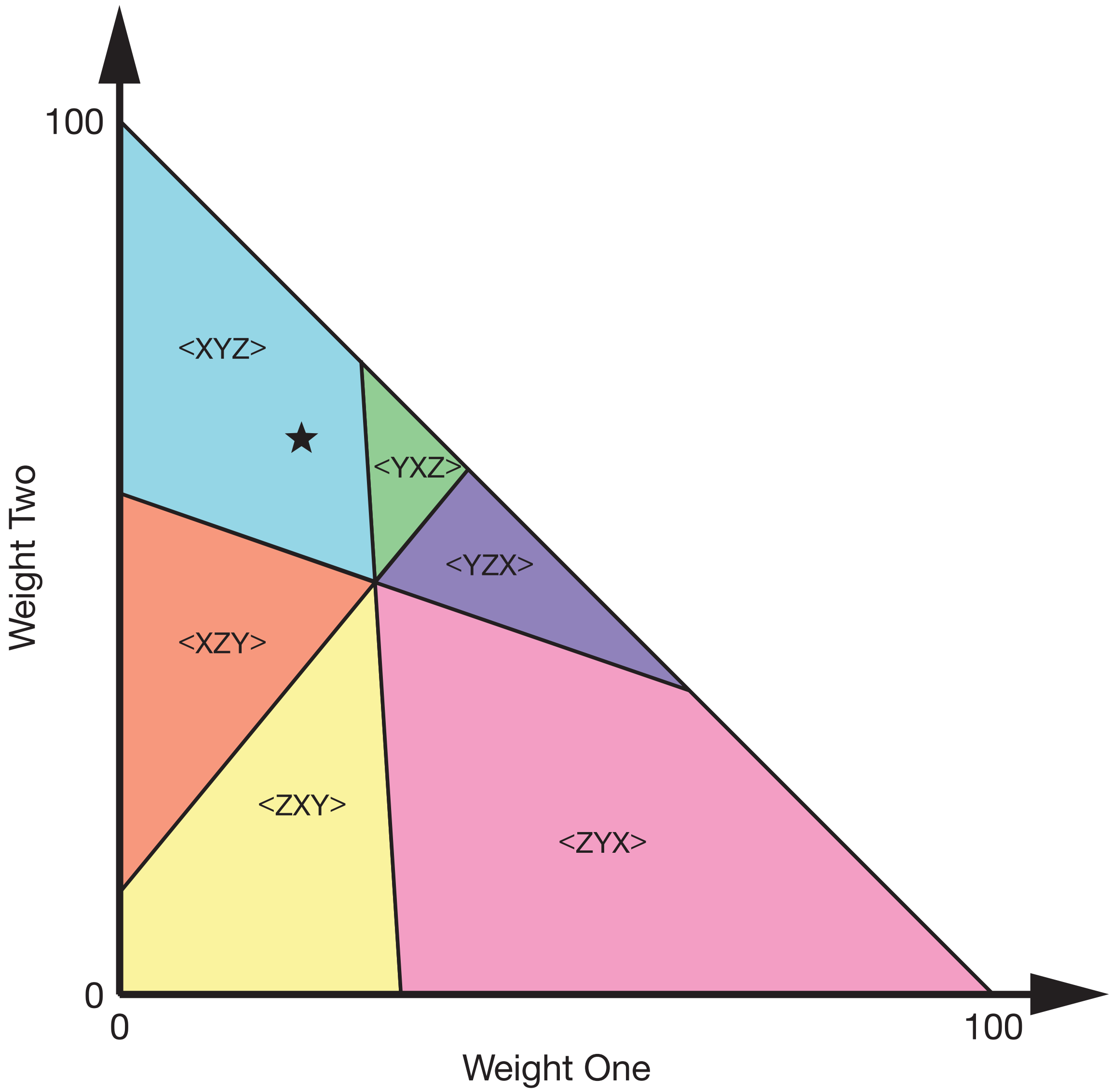

Figure 1: *A schematic representation of the regions of weight-space associated with each of the six complete consensus rankings of three options.*

percentage change of the weights required to alter the ranking of options we can determine whether or not the set of weights marked by the star is abnormally close to a region boundary.

A third measure of sensitivity determines the degree to which the selected criteria weights represent the consensus ranking weight-space they occupy. Intuitively the most representative weights would be the central point of the ranking region. Similarly, the least representative weights would be any point that lies infinitely close to a boundary with another ranking region. This sensitivity measure provides complementary information to the raw distance measure. In fact, two points may be equidistant from the centre of a ranking region but may not be equally representative of the region.

2.3 High-dimensional geometry

For more interesting instances the number of criteria may exceed three. While in these cases the high-dimensional geometry becomes difficult to visualize, the underlying principles remain the same and can be expressed algebraically. Let w_i be a variable representing the weight allocated to criterion i for $i = 1, \ldots, m$. The weight-space is bounded by the constraints

$$\sum_{i=1}^{m} w_i = 100 \qquad (3)$$
$$w_i \geq 0 \quad \text{for } i = 1, \ldots, m.$$

The constrained region defined by the linear equation and inequalities (3) is called a *convex polytope*. The study of convex polytopes is a rich and challenging field (refer to [10] and [11] for more information). Using established convex polytope theory we can define a methodology to analyze the weight-space for any number of options.

In general, a linear inequality defines a *half space* and a convex polytope P in dimension m is a bounded subset of $\mathbb{R}^m$—the set of all real number points in dimension m—which is the intersection of a finite set of *half spaces*. This is known as an H-representation of the polytope. A polytope also has a dual V-representation which represents the polytope as a finite set of points. The convex hull of a set of points $V = (v_1, v_2, \ldots, v_k)$, denoted $\mathrm{conv}(V)$, is the minimal convex set[1] containing V. The V-representation of a polytope P is the convex hull $\mathrm{conv}(V)$ of the set of all points in $P \in \mathbb{R}^m$. For a polytope P, the finite set $\mathrm{conv}(V)$ is referred to as the vertices of P. Computing the V-representation of polytope given its H-representation is a long-studied problem whose complexity is unknown. Nevertheless there are several algorithms and computer codes that work well in practice (see [12–15]).

For the remainder of this paper we will only deal with polytopes that are convex and shall omit 'convex'. For the simple problem previously presented, the weight-space is defined by a two-dimensional polytope (more specifically a polygon or triangle). While the polytope defined by inequalities (3) resides in m-dimensional space, there are only $m-1$ degrees of freedom as setting $m-1$ of the variables dictates the value of the remaining one. This is why our simple example in Figure 1 with three criteria weights can be drawn in (projected to) two dimensions.

The final score, S_j of option j is expressed as

$$S_j = \sum_{i=1}^{m} v_{ij} w_i. \qquad (4)$$

When S_j is greater than S_k option j is ranked higher than option k. When $S_j = S_k$ then the options are tied. The equation $S_j = S_k$, expressed as

$$\sum_{i=1}^{m} v_{ij} w_i - \sum_{i=1}^{m} v_{ik} w_i = 0, \qquad (5)$$

[1] A set is convex if for every pair of points within the set, every point on the straight line segment that joins them is also within the set.

defines a *hyperplane* (the high-dimensional generalization of a line in two dimensions and plane in three dimensions) that divides the weight-space in two: weights where $S_j > S_k$ and weights where $S_j < S_k$ (and trivially the weights where $S_j = S_k$). Each of these sub-divisions is also a polytope. For example, the weight-space for our simple example is defined by the polytope

$$\begin{aligned} \sum_{i=1}^{3} w_i &= 100 \\ w_i &\geq 0 \quad \text{for } i = 1, \ldots, 3. \end{aligned} \tag{6}$$

The hyperplane $S_Y = S_Z$ splits the region into two polytopes:

$$\begin{aligned} \sum_{i=1}^{3} w_i &= 100 \\ \sum_{i=1}^{3} v_{iY} w_i - \sum_{i=1}^{3} v_{iZ} w_i &\leq 0 \\ w_i &\geq 0 \quad \text{for } i = 1, \ldots, 3 \end{aligned} \tag{7}$$

and

$$\begin{aligned} \sum_{i=1}^{3} w_i &= 100 \\ \sum_{i=1}^{3} v_{iY} w_i - \sum_{i=1}^{3} v_{iZ} w_i &\geq 0 \\ w_i &\geq 0 \quad \text{for } i = 1, \ldots, 3. \end{aligned} \tag{8}$$

The interior of polytope (7) represents the weight-space of all rankings with option Z preferred to option Y, and the interior of polytope (8) the opposite.

Any ranking of n options can be represented by a polytope defined by adding $n-1$ inequalities to (3): for every pair (i, j) of adjacent options in the ranking, add the constraint $S_i \geq S_j$ if option i is preferred to option j, $S_i = S_j$ if the options are tied, otherwise add constraint $S_i \leq S_j$. For example ranking $\langle X Y - Z \rangle$ is defined by the interior of polytope bounded by inequalities (9-12). Inequality (10) encodes the constraint $S_X \geq S_Y$ and (11) encodes the constraint $S_Y = S_Z$

$$\sum_{i=1}^{3} w_i = 100 \tag{9}$$

$$\sum_{i=1}^{3} v_{iX} w_i - \sum_{i=1}^{3} v_{iY} w_i \geq 0 \tag{10}$$

$$\sum_{i=1}^{3} v_{iY} w_i - \sum_{i=1}^{3} v_{iZ} w_i = 0 \tag{11}$$

$$w_i \geq 0 \quad \text{for } i = 1, \ldots, 3. \tag{12}$$

The set of all weights resulting in rankings with a certain object ranked first can also be defined by a polytope. For example, the set of weights with option X ranked first is denoted by

$$\sum_{i=1}^{3} w_i = 100 \tag{13}$$

$$\sum_{i=1}^{3} v_{iX} w_i - \sum_{i=1}^{3} v_{iY} w_i \geq 0 \tag{14}$$

$$\sum_{i=1}^{3} v_{iX} w_i - \sum_{i=1}^{3} v_{iZ} w_i \geq 0 \tag{15}$$

$$w_i \geq 0 \quad \text{for } i = 1, \ldots, 3, \tag{16}$$

where inequality (14) dictates that X be ranked higher than Y and inequality (15) dictates that X be ranked higher than Z. In general, $n-1$ such inequalities are required expressing the preference of the top-ranked option to the others. Note that no inequalities dictate the relative preference of any options ranked below the top-ranked option.

2.3.1 Partitioning the weight space

In the general case, the entire weight-space is divided by $\binom{n}{2}$ hyperplanes. These hyperplanes partition the weight-space into a number of distinct polytopes of varying dimensionality, each of which corresponds to a specific ranking of the set of options under consideration. The total number of possible polytopes is quite large; the number Q_n of rankings of n objects is given in [9] as

$$Q_n \approx \frac{K^{n+1}(n!)}{2} \quad \text{where} \quad K = \frac{1}{\ln 2} = 1.44269\ldots \tag{17}$$

As an example, we refer to Figure 1 noting that $Q_3 = 13$. In particular, we see six two-dimensional polytopes (polygons) corresponding to those rankings that specify a complete ordering of the three options without ties, six one-dimensional polytopes (line segments) corresponding to rankings in which two of the options are tied with one another, and one zero-dimensional polytope (point) for which all three options are tied. The reduction in dimension associated with each tie is due directly to the inclusion of equalities in the definition of the polytope; i.e., if option A is tied with option B, then we include the constraint $S_A = S_B$ on the polytope in weight space that corresponds to any ranking in which these two options are tied.

2.4 Formal definition of sensitivity measures

We are now ready to detail the computation of the three measures of sensitivity proposed. In order to simplify the methodology explanation we note that any ranking region can be converted and represented by the inequalities of the standard form $Aw \leq b$ where A is a $\bar{n} \times m$ real matrix, b an $\bar{n}$ real vector, and w the vector of variables $(w_1, \ldots, w_m)^T$. The number of inequalities, $\bar{n}$, depends on the number of ties, imposed constraints, etc.

2.4.1 Volume sensitivity measure

One measure of sensitivity of weights would be to consider the volume of each ranking region relative to the volume of the entire weight-space. Volume is a global measure of sensitivity. The larger the region is, the more robust the selection of the corresponding consensus ranking. If one were to select the criteria weights randomly, the probability of obtaining a particular consensus ranking is directly proportional to the volume of the appropriate region. If the baseline weights lie in a very small ranking region this raises cause for some concern. It effectively means that one would have to "fine-tune" the weights to obtain this result. In such cases the decision makers should be made aware of the occurrence and decide whether the baseline weights are indeed appropriate.

The issue of rankings involving ties between options requires careful consideration. Referring once more to Figure 1, consider the line segment to the right of the star, which forms the boundary between the polytope associated with the ranking $\langle XYZ \rangle$ and $\langle YXZ \rangle$. Criteria weight allocations falling along this line correspond to the ranking $\langle X - YZ \rangle$. However, this line segment occupies no "volume" (i.e., its two-dimensional Lebesgue measure is zero); therefore, from purely a probabilistic standpoint it is impossible to randomly select weights that would result in this ranking. As seen above, ties in the ranking are directly related to constraint equations in the weight space that reduce the dimension of the corresponding polytope. One might therefore ask whether the constraint equations associated with a tie correspond to an actual relationship between criteria weights, which we shall denote as a *strong* tie, in direct parallel to the usage of Emond and Mason [9], or is merely a spurious relation that has arisen "accidentally", which we denote a *chance* tie. For strong ties, once we have determined through consultation with subject matter experts / senior decision makers that any tied options truly are equivalent, we accept the constraint as legitimate and evaluate the volumes within the induced lower dimensional polytope. Returning to the example of Figure 1, if we accept that X and Y are truly equal, then we evaluate the volume measure by comparing the length of the line segment associated with $\langle X - YZ \rangle$ to the total length of the segments $\langle X - YZ \rangle$ and $\langle ZX - Y \rangle$. Conversely, if we determine the tie has arisen by chance, then we report the possible fine-tuning. In this case it may also be useful to specify the outward normals of the hyperplanes bounding the polytope associated with the ranking with ties. These normals correspond to the directions along which infinitesimal movements in weight-space will break the degeneracy in the ranked options.

In two dimensions, computing the area of the regions is simple. As the dimension increases computing the volume of a polytope is computationally intractable (#P-Hard[2] [16]). The computational time required grows exponentially as the number of criteria increases. Nonetheless, efficient programs exist to estimate or exactly solve practical-sized instances with up to 30 criteria (see [12, 17, 18]).

2.4.1.1 Critical value: Volume sensitivity measure

We wish to develop a test to determine when the volume associated with a given ranking is so small as to raise concerns about robustness and/or possible fine-tuning of weights. To this end, we need to establish a critical value for the volume measure that will identify the smallest acceptable volume in weight-space associated with a robust ranking.

[2] #P, "number P", is a complexity class in computational complexity theory. It is the set of counting problems associated with the set of decision problems solvable in polynomial time by a non-deterministic Turing machine.

Given the connection between the volume measure and probability outlined above, if all $n!$ rankings of n options[3] were equally likely, each should occupy $\frac{1}{n!}$ of the total volume of the space of feasible criteria weight allocations. It would therefore seem reasonable to define the critical value of the volume measure to be of the form $\frac{1}{\gamma n!}$, where $\gamma > 1$ is a factor of our choosing. While the specific choice of γ is arbitrary, in the discussion that follows we attempt to motivate a reasonable choice for the value of γ.

Consider an urn containing a very large number of balls, each marked with one of the $n!$ complete rankings. The exact number of balls is unknown, as is the proportion associated with each ranking. We conduct a number of trials in which we draw a single ball from the urn, note its ranking, and replace it. For a given ranking of interest, we denote a successful trial as one in which we draw a ball associated with the specified ranking, otherwise the trial is unsuccessful. We now ask how many consecutive unsuccessful trials t would be required for us to be 95% confident in the conclusion that the true probability of selecting a ball with the specified ranking from the urn is less than $\frac{1}{n!}$. This is equivalent to solving for t in the equation

$$\left(1 - \frac{1}{n!}\right)^t = \frac{1}{20}. \tag{18}$$

Noting that $e^3 = 20.0855\ldots \approx 20$, we have

$$t \log\left(1 - \frac{1}{n!}\right) \approx -3, \tag{19}$$

and since for $n!$ large, $\log(1 - \frac{1}{n!}) \approx -\frac{1}{n!}$, we find

$$t \approx 3n!. \tag{20}$$

A related calculation considers the case where the probability of drawing a ball associated with a given ranking is exactly $\frac{1}{3n!}$. After drawing $n!$ balls from the urn with replacement, using the Poisson distribution we determine that we can be $\frac{4}{3}\exp(-\frac{1}{3}) \approx 95.5\%$ confident that we will draw at most one ball marked with the specified ranking. Alternatively phrased, we could state that if we drew one ball with the specified ranking in $n!$ trials, we would be 95.5% confident that the actual probability of drawing a ball associated with the specified ranking exceeds $\frac{1}{3n!}$.

Given that both of these confidence limits involve the same factor of three, we select $\gamma = 3$; our threshold for the volume sensitivity measure is therefore $\frac{1}{3n!}$. We consider those rankings whose polytopes occupy volume fractions smaller than this fraction to be suspect—there is a greater likelihood that fine-tuning of weights has gone on to generate these particular rankings of options.

2.4.2 Distance sensitivity measure

A local measure of sensitivity considers the distance from the given set of criteria weights to the boundary of adjacent ranking regions. This measure determines the minimum amount of change required to the weights in order to alter the ranking of the options. Computing the distance of a point

[3] When considering strong ties, n is taken to the number of distinct clusters of equivalent options.

$W = (W_1, W_2, ..., W_m)^T$ to a polytope P requires finding a point $\bar{W} = (\bar{W}_1, \bar{W}_2, ..., \bar{W}_m)^T \in P$ such that $\sqrt{\sum_{i=1}^m (\bar{W}_i - W_i)^2}$ is minimized. This can be solved with the Quadratic Program (in standard form)

$$\text{Minimize } Q = \frac{1}{2}\left(\sum_{i=1}^{m} W_i + \sum_{i=1}^{m} \bar{W}_i^2\right) \quad (21)$$
$$\text{subject to } \bar{W} \in P.$$

The actual minimal distance is $\sqrt{\sum_{i=1}^m W_i^2 - 2Q}$. Quadratic Programs of the form (21) can be solved quickly [19] using any of a slew of available software programs (see [20]) such as CPLEX 11.1 [21]. Unlike Breteton's method, the formulation (21) explicitly omits negative weights.

Some caution must be exercised when computing $\bar{W}$. As noted by Brereton, the weights $\bar{W}$ may have undesirable properties. For example $\bar{W}$ may assign a weight of zero to a criterion or the changes from W might be grossly mis-proportioned. Hence Brereton suggested finding the *minimum percent change solution*, $\bar{W} = (\bar{W}_1, \bar{W}_2, ..., \bar{W}_m)^T \in P$ such that $\sqrt{\sum_{i=1}^m \frac{(\bar{W}_i - W_i)^2}{W_i}}$ is minimized. This objective function can also be modeled using a quadratic program similar to (21). Unfortunately Brereton's second distant measure of sensitivity may also generate unrealistic zero-weights or lead to mis-proportioned weights. A more robust mechanism for avoiding undesirable properties is to add further constraints to (21) on the $\bar{W}_i$'s, such as $l_i \leq \bar{W}_i \leq u_i$ for each $i = 1, ..., m$ where the l_i's and u_i's are instance-specific parameters. For example, the decision makers might strongly believe that a certain criterion should hold between 10% and 20% of the total weight.

2.4.2.1 Critical value: Distance sensitivity measure

We wish to set a critical value to allow us to test the robustness of a given set of weights with respect to the distance sensitivity measure. In the discussion that follows we define a comparator that allows us to relate the computed distance measure of a given set of weights to the maximal distance measure for any set of weights lying in the same ranking region.

The Chebyshev sphere of a polytope P is the largest Euclidean sphere that lies inside P. Evans [6] used the notion of a "maximum confidence" sphere, noting that in some senses the centre of the Chebyshev sphere is the most resilient to weight changes in all directions. The Chebyshev centre is a point that is furthest from the boundaries of P—the radius of the sphere is the maximal distance that a point in P can be from the boundary of P.

Given an H-representation of a polytope,

$$P = \left\{ \vec{x} \in \mathbb{R}^d \;\middle|\; \sum_{j=1}^{d} a_{ij} x_j \leq b_i,\ i = 1, 2, \ldots, \bar{n} \right\}, \quad (22)$$

we can compute the radius r of the Chebyshev sphere of P by solving the linear program

$$\text{maximize } r \quad (23)$$
$$\text{subject to } \sum_{j=1}^{d} a_{ij} x_j + r\sqrt{\sum_{j=1}^{d} a_{ij}^2} \leq b_i \quad \text{for all } i = 1, ..., \bar{n}$$

for r and $x = (x_1, ..., x_d)^T$. The optimal value of r is the radius of the largest sphere that fits inside P (for a proof see Boyd *et al.* [22]).

For our application, we are given P as

$$P = \left\{ \vec{w} \in \mathbb{R}^m \;\middle|\; \sum_{j=1}^{m} w_j = 100, \sum_{j=1}^{m} a_{ij} w_j \leq b_i,\ i = 1, 2, \ldots, \bar{n} \right\}, \tag{24}$$

where the inequalities include nonnegativity constraints on the weights. A little caution is required in formulating the linear program (23) as our P is not full-dimensional. The equality $\sum_{j=1}^{m} w_j = 100$ constrains P to lie in an $m-1$-dimensional hyperplane in $\mathbb{R}^m$. If we simply replace the equality by two inequalities $\sum_{j=1}^{m} w_j \leq 100$ and $\sum_{j=1}^{m} -w_j \leq -100$, then the formulated linear program would find the largest m-dimensional sphere within P—a sphere with radius zero. Instead, we wish to find the largest $m-1$-dimensional sphere lying in the hyperplane $\sum_{j=1}^{m} w_j = 100$ that is bounded by the inequalities of (24). This can be accomplished by a simple transformation. We first find the components of the vectors $\vec{a}_i = (a_{i1}, ..., a_{im})$, for $i = 1, ..., \bar{n}$, that are orthogonal to the normal of the equality constraint, namely the vector $\vec{e}$ of all ones. We solve $\vec{a}_i = k_i \vec{e} + \vec{y}_i,\ \vec{y}_i \cdot \vec{e} = 0$ for some scalars k_i and vectors $\vec{y}_i = (y_{i1}, ..., y_{im})$ and formulate the following linear program:

$$\text{maximize } r \tag{25}$$

$$\text{subject to } \sum_{j=1}^{m} a_{ij} w_j + r \sqrt{\sum_{j=1}^{m} y_{ij}^2} \leq b_i \quad \text{for all } i = 1, ..., \bar{n},$$

$$\sum_{j=1}^{m} w_j = 1. \tag{26}$$

This formulation finds the largest m-dimensional sphere bounded by inequalities, which at the intersection of $\sum_{j=1}^{m} w_j = 100$ define P. Note that the linear program is unbounded in the directions parallel to $\vec{e}$, however the sphere is constrained to have its centre lie in the hyperplane $\sum_{j=1}^{m} w_j = 100$. The intersection of the m-dimensional sphere with this hyperplane yields the desired $m-1$-dimensional Chebyshev sphere of P.[4]

There are two distinct distances that we should compare against the radius r of the Chebyshev sphere. The first, which we shall denote $d_{\leftrightarrow}$, is the distance from the baseline weights W to the nearest adjacent polytope corresponding to a different ranking of the options. The other distance to be considered is the distance from the baseline weights W to the nearest boundary of the polytope associated with the baseline ranking, which we denote as d_b. From the definition of the two distance measures, we observe that $d_b \leq d_{\leftrightarrow}$.

The second distance measure is of interest as it includes the hyperplanes $w_j \geq 0\ \forall j$. While the output ranking will not be affected if our baseline weight allocation is very close to one of these boundaries, it would tend to indicate that the corresponding criteria weight w_j is effectively too small to contribute significantly to the ranking of the options. As such, it provides complementary information to Brereton's minimum percent change solution—Brereton attempts to ensure that all

[4]Should additional, user-defined linearities further reduce the dimension of P, the vectors $\vec{y}_i$ need to be computed in similar fashion—perpendicular to the normals of any linear equations.

weights contribute to the evaluation, while this measure can suggest that a particular weight is not significant and could conceivably be removed from the evaluation.

With these two distances in hand, we compute the ratios $\delta_{\leftrightarrow} = \frac{d_{\leftrightarrow}}{r}$ and $\delta_b = \frac{d_b}{r}$. By definition, $\delta_b \leq 1$; however, as $d_{\leftrightarrow}$ only considers the closest distance to a subset of the boundary of the polytope, namely those facets that are adjacent to polytopes associated with other rankings, it is possible for $\delta_{\leftrightarrow} > 1$. While the choice of critical values on these ratios is arbitrary, we select critical values $\hat{\delta}_{\leftrightarrow} = \hat{\delta}_b = 0.05$; i.e., if either of the two distance measures is less than 5% of the Chebyshev radius r, we will consider the allocation of criteria weights to be "too close" to a boundary of the polytope associated with the baseline ranking. Specifically, for $\delta_{\leftrightarrow} < 0.05$ a small change in the allocation of criteria weights will lead to a change in the final ranking, while in the case where $\delta_b \neq \delta_{\leftrightarrow}$ and $\delta_b < 0.05$ we determine that one or more of the criteria weights do not significantly contribute to the evaluation.

2.4.3 Representativity sensitivity measure

Another local measure of sensitivity computes the representativity of the chosen weights $W = (W_1, W_2, ..., W_m)^T$. This notion is based on the intuition that the centre of the polytope is the most representative and points on the boundary the least. Mathematically, the centre of a polytope is not defined except under conditions of regularity. Nevertheless, various notions try to capture the essence of a "centre" of a polytope. See Annex B for further discussion.

For the remainder of the paper we focus on the geometric centre, also known as the centroid or generalized barycentre of the polytope. It is defined as the average of the vertices of P given by the V-representation of the polytope. Solymosi [23] used the notion of this centroid for determining the most robust weights for a particular ranking of options. Barycentres are heavily used in computer graphics modelling as representative points of both convex and non-convex shapes. Computing the centroid of a polytope given by a set of inequalities (H-representation) is computationally intractable (#P-Hard [24]) but can be accomplished by converting to the dual V-representation. The centroid of a polytope given its V-representation is simply the average of all the vertices.

Given a polytope centroid we can compute a measure of representativity of the baseline weights.[5] Define α to be the coefficient of representativity that lies between zero and one. Let $Aw \leq b$ be the halfspace representation of a polytope P representing a ranking, C the centroid of P and W the selected baseline weights. The polytope P' defined by $Aw' \leq b - AC$ is polytope P translated such that the centroid coincides with the origin of the space. Let $W' = W - C$ be the translated version of W. To determine the coefficient of representativity of W we find the largest value for α such that $AW' \leq (1-\alpha)(b - AC)$ with $0 \leq \alpha \leq 1$. This is a one-dimensional linear program. Note that when $\alpha = 1$ then $W' = (0, 0, ...0)$ indicating that the selected weights correspond to the centroid and are most representative of the ranking region. When α approaches 0 then the chosen weights are least representative. Figure 2 illustrates the coefficient of representativity on an example ranking region. At the centroid of the region α is 1 and at the outer border α is 0. The inner polygon contains all points where $\alpha \geq 0.35$.

[5] This measure of representativity requires that we select a polytope centre—in this case the polytope barycentre. However, the computation is independent of which centre is chosen.

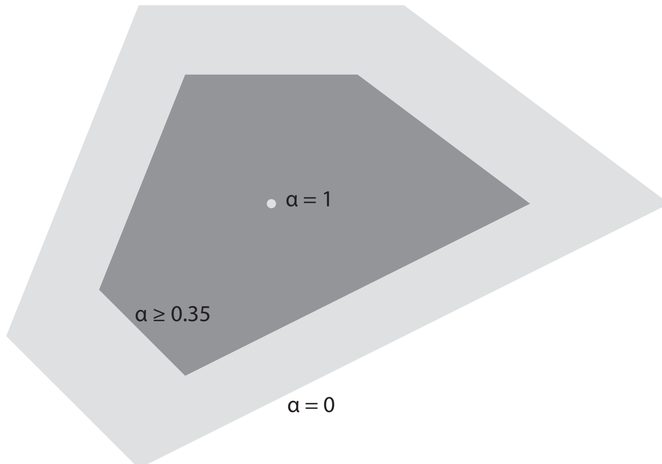

Figure 2: *Illustrating the coefficient of representativity of a ranking region.*

We can also quantify all weights that are representative of a ranking region to a desired degree. For example, the set of all weights that have a coefficient of representativity greater than or equal to 0.35 of a ranking defined by $Aw \leq b$ is the polytope $Aw' \leq (1-0.35)(b-AC)$ with $w' = w - C$.

The coefficient of representativity can also be interpreted using volume. Given a particular α, the weight region defined by the polytope $A'w \leq (1-\alpha)b'$ has a volume that is $(1-\alpha)^{m-1}$ times the volume of $A'w \leq b'$ (a proof is provided in Annex C).

2.4.3.1 Critical value: Representativity sensitivity measure

As with the two previous sensitivity measures, we wish to set a critical value to allow for testing of the robustness of a given set of weights with respect to the representativity measure. As with the previous two cases, the specific choice of the threshold value is arbitrary; In the discussion that follows, we will motivate two possible methods for setting a reasonable value for the critical value $\hat{\alpha}$.

Physically speaking, if we were to express our baseline weight vector as a convex combination of vectors corresponding to the centroid and the extremal points of the polytope associated with the ranking, then for a given α at most $100\alpha\%$ of the centroid vector can contribute. Using statistical hypothesis testing as a benchmark, we set a threshold on the contribution due to the centroid as 5%; i.e., $\hat{\alpha}_{\text{conv}} = 0.05$

A potential alternative would be to determine $\hat{\alpha}_{\text{vol}}$ such that the polytope associated with $\alpha \geq \hat{\alpha}_{\text{vol}}$ contains a given fraction (say 0.95) of volume of the full polytope. Given our earlier result for the volume fraction contained within $\alpha \geq \hat{\alpha}$ of a d-dimensional polytope, we must solve

$$(1-\hat{\alpha})^d = 0.95, \tag{27}$$

obtaining

$$\hat{\alpha} = 1 - \sqrt[d]{0.95}. \tag{28}$$

As the dimensionality d of the polytope increases, the critical value $\hat{\alpha}$ tends towards zero. In particular, for $d \geq 2$, the critical value derived via the volume fraction becomes less restrictive than the simple test above; i.e., $\hat{\alpha}_{\text{vol}} < \hat{\alpha}_{\text{conv}} = 0.05$.

While both critical values are reasonable, we prefer to base the test on the convex combination based threshold $\hat{\alpha}_{\text{conv}} = 0.05$ as it is more restrictive in general and operates directly on the vector corresponding to our baseline weight allocation. Weight vectors for which the representativity is $\alpha < 0.05$ are deemed to be atypically close to the boundary of the polytope.

2.5 Discussion

In theory all three of the sensitivity measures may require computations that are known to be computationally intractable as the dimension (number of criteria) is increased; however, practical instances are most often solvable using state-of-the-art software packages. For volume and distance measures, since equation (17) can be extremely large, analysts can choose which ranking regions to evaluate. For example one may just decide to measure the magnitude of change required in order for a particular option to ascend the order and be ranked first (find the nearest $\bar{W}$ in the polytope defining the region with the desired option as ranked first). For the representativity measure, advanced polytope transformations are required. Despite the potential computational limitations, simple to use software is available to perform these computations quickly on realistic-sized instances.[6] The analysis of one such realistic instance is presented in the following section.

One significant benefit for modeling the ranking regions as polytopes is that the above three sensitivity measures can be computed while imposing additional global constraints on the criteria weights. This was touched upon in Section 2.4.2. In general, any constraints on any linear combination of a subset of the criteria weights can be introduced and added to the polytope definition (3). These can be minimum and maximum bounds on individual weights (even fixed weights), uniform weight allocation to a subset of the criteria, weight relationships between criteria (e.g., w_j should be at least double that of w_i), etc. We can then apply the proposed sensitivity analysis easily while respecting such instance-specific constraints. The three thresholds defined may have to be adjusted by analysts performing the sensitivity analysis to ensure that they are appropriate for the specific problem instance.

2.6 The inverse problem

While the methodology presented in this paper is intended to facilitate sensitivity analysis on a pre-existing set of weights, it is theoretically possible to adapt the process to the generation of a set of criteria weights that will produce a specified ranking of options. However, we must be extremely careful in applying this reverse approach—considerable potential for analytical dishonesty exists. Generally speaking, once an evaluation of options using an additive weighted scoring system has begun, the weights should be treated as fixed and the methodology applied in the "forward direction" for sensitivity analysis only. On the other hand, the application of this approach in the "reverse

[6] DMGOR AST developed a *Mathematica* program called PolySense to fully automate the sensitivity analysis [25]. Problems with up to 30 criteria have been analyzed.

direction" could, when used carefully, help to incorporate all available information about the end users' preferences into the design of an additive weighted scoring function.

As an illustration of a scenario where reverse engineering weights might be of utility, let us consider the case where we wish to develop a set of weights to be used to evaluate candidate systems for a future land combat vehicle. Both tracked and wheeled solutions are expected amongst the bid proposals. Various low level factors, such as maximum road and cross-country speeds, grade climbing ability, obstacle avoidance, reliability and operating costs, could enter into the decision to select a wheeled option over a tracked option or vice verso. Weights can be applied to these low level criteria using any number of "bottom-up" techniques. However, the project staff may also indicate a wholistic preference—all else being equal, given a "typical" tracked vehicle and a "typical" wheeled vehicle, the tracked vehicle is preferred.

Once criteria scores associated with "typical" wheeled and tracked options have been determined—admittedly a non-trivial exercise—the H-representation of the polytope representing the desired ranking region (i.e., tracked preferred to wheeled) can be constructed. The centroid of this polytope is the set of weights that is most representative of the preferred ranking region. The vertices of this polytope represent extremal weights that are the least representative—very close to weights that produce other rankings.[7] Any set of weights that is a convex combination of the extreme points of the polytope is also a set of weights that guarantees that the preferred ranking is obtained. Furthermore, a linear objective on the weights can be optimized (this is a linear program where the constraints are the inequalities defining the ranking region's polytope). The weights generated via this wholistic "top-down" approach may then be compared with those resulting from the "bottom-up" approach. In the event that inconsistencies are observed between the preferences indicated by each set of weights, further analysis should be undertaken to identify the source of the inconsistency and to better resolve the true preference structure of the project staff. In some senses, the true strength of using the process developed here in the "reverse direction" to generate a set of weights that will return a desired output ranking is as a method of sensitivity analysis during the development of an additive weighted evaluation measure.

[7] Note that a vertex $v \in V$ of P lies on the boundary of P and not in the interior. The true extremal weight $\bar{v}$ associated with v which generates the ranking corresponding to P with centroid C is computed as $\bar{v} = \beta v + (1 - \beta)C$ for β infinitely close to 1.

3 Example

During the options analysis period of a major Crown acquisition, the project management office evaluates feasible implementation options against select criteria. The evaluation methodology needs to be sound and defensible. At the request of DGMPD (L&S), DMGOR AST performed a sensitivity analysis on the options analysis performed by PMO CSC.

The goal of the CSC project is to recapitalize the Navy's entire fleet of surface combatants. The first CSC flight will replace the IROQUOIS Class destroyers with three warships. The project seeks to balance the competing objectives of minimizing overall costs (both acquisition and life-cycle) and risks to deliver warships as per the Statement of Requirements (SOR). During the options analysis period, the PMO identified and compared four recapitalization options as shown in Table 1.

Table 1: *The four options being ranked.*

Option	Description
A	Upgrading and life extension of the IROQUOIS class for an additional 15 years.
B	Lease or buy existing platforms from a foreign country.
C	Purchase an existing foreign design and build in Canada.
D	Design and build warships in Canada.

3.1 Baseline solution

To assess the feasibility of each recapitalization option, PMO CSC isolated thirteen evaluation criteria and assigned to them weights indicative of their relative importance (Table 2):

For each criterion, the four recapitalization options were ranked in order of preference. PMO CSC employed the Borda count[8], a consensus ranking method where each option is credited with a certain number of points (rating) corresponding to the position in which it is ranked. For each criterion under consideration, 4 points were awarded to the option ranked first, 3 points to the one ranked second, 2 points for the third place option, and 1 point for the option ranked last (in case of ties the points awarded is the average of the points allocated to the rank positions that the tied options occupy). The point allocation is presented in the last four columns of Table 2. An additive weighted scoring rule was then applied: the points awarded to each option were scaled by the weights assigned to each criterion and summed. Using this technique, Options D, C, A and B received 321, 267, 211 and 201 points, respectively. In other words, the option to design and build new warships in Canada was most preferred, followed by purchasing a foreign design, followed by upgrading and extending the life of the IROQUOIS, and finally leasing/buying existing warships from a foreign country. This ranking is mathematically denoted as $\langle DCAB \rangle$.

[8]Named after French mathematician and naval captain Jean de Borda (1733-1799).

Table 2: *The thirteen criteria for which the four options are ranked, criteria weights, the input ranking of options and corresponding Borda point allocation.*

	Weight			Point Allocation			
Criterion	Symbol	Value	Evaluation	A	B	C	D
Cost: Sail-away	w_1	15	$\langle ACBD \rangle$	4	2	3	1
Cost: In Service Support & Ops	w_2	8	$\langle DCBA \rangle$	1	2	3	4
Capability Upgrades	w_3	8	$\langle DCBA \rangle$	1	2	3	4
Growth Margin	w_4	9	$\langle DCBA \rangle$	1	2	3	4
Environmental Compliance	w_5	8	$\langle DCBA \rangle$	1	2	3	4
Crewing & Training (Facilities)	w_6	8	$\langle DCBA \rangle$	1	2	3	4
Operations & Doctrine	w_7	7	$\langle DACB \rangle$	3	1	2	4
Schedule (Design/Build/Upgrade)	w_8	10	$\langle ABCD \rangle$	4	3	2	1
Sustainment & Obsolescence	w_9	9	$\langle DCBA \rangle$	1	2	3	4
Design Origin	w_{10}	7	$\langle DCAB \rangle$	2	1	3	4
Economic Benefits	w_{11}	2	$\langle DCAB \rangle$	2	1	3	4
Infrastructure Requirements	w_{12}	2	$\langle AB\text{-}C\text{-}D \rangle$	4	2	2	2
Acquisition Risk	w_{13}	7	$\langle DBAC \rangle$	2	3	1	4
Overall Ranking			$\langle DCAB \rangle$	211	201	267	321

3.2 Sensitivity analysis

AST conducted a variety of sensitivity analyses on the PMO CSC ranking of recapitalization options. Although there are 13 criteria in total, there are only seven distinct rankings of options against these criteria. For example, the criteria Design Origin and Economic Benefits can be aggregated as the ranking of the four options is the same for both—the allocation of weight between these two criteria does not impact the overall consensus ranking provided the total weight assigned to the combination of these two criteria remains constant. Table 3 lists the weight aggregation by ranking that results in seven master weights summing up to 100. Our problem therefore has an equivalent geometrical representation in six dimensions.

Table 3: *Aggregate weights of criteria rankings.*

Weights			
New	Definition	Value	Evaluation
W_1	$w_2 + w_3 + w_4 + w_5 + w_6 + w_9$	50	$\langle DCBA \rangle$
W_2	$w_{10} + w_{11}$	9	$\langle DCAB \rangle$
W_3	w_1	15	$\langle ACBD \rangle$
W_4	w_7	7	$\langle DACB \rangle$
W_5	w_8	10	$\langle ABCD \rangle$
W_6	w_{12}	2	$\langle AB\text{-}C\text{-}D \rangle$
W_7	w_{13}	7	$\langle DBAC \rangle$

As a first, simple test of the impact of weight allocations on the final consensus ranking, all 13 criteria weights were set to be of equal importance. No change was observed in the consensus ranking $\langle DCAB \rangle$.

3.2.1 Volume, distance and representativity measures

AST mapped out the size and location of the regions associated with each of the 24 possible (complete) consensus rankings of the four options. In order to evaluate the robustness of the PMO's Borda consensus ranking $\langle DCAB \rangle$, we determined the volume of each region and compared them to the volume of the entire space of possible allocations of criteria weights. Only 16 of the 24 possible consensus ranking regions have a non-zero volume. The volume and the corresponding rankings are tabulated in Table 4. In this case, the Borda consensus ranking $\langle DCAB \rangle$ is the third most likely result, occurring for 11.25% of the possible weight allocations. Given that this percentage is large compared to the threshold value of $\frac{1}{3 \cdot 4!} \approx 1.39\%$ defined in Section 2.4.1.1, we can also be relatively confident that little "fine-tuning" of the criteria weights has gone on to generate a given conclusion.

Table 4: *The fraction of the volume in weight space associated with each of the 24 possible Borda consensus rankings.*

Final Ranking	Fraction of Volume
$\langle DACB \rangle$	0.2637
$\langle ADCB \rangle$	0.2273
$\langle DCAB \rangle$	0.1125
$\langle DABC \rangle$	0.0961
$\langle ADBC \rangle$	0.0824
$\langle ACDB \rangle$	0.0820
$\langle ACBD \rangle$	0.0412
$\langle ABCD \rangle$	0.0310
$\langle ABDC \rangle$	0.0241
$\langle DBAC \rangle$	0.0180
$\langle DCBA \rangle$	0.0159
$\langle DBCA \rangle$	0.0031
$\langle CDAB \rangle$	0.0014
$\langle CADB \rangle$	0.0014
$\langle BADC \rangle$	1.1×10^{-5}
$\langle BDAC \rangle$	7.3×10^{-6}
$\langle CDBA \rangle$	—
$\langle CABD \rangle$	—
$\langle CBDA \rangle$	—
$\langle CBAD \rangle$	—
$\langle BDCA \rangle$	—
$\langle BCDA \rangle$	—
$\langle BCAD \rangle$	—
$\langle BACD \rangle$	—

AST also computed the fraction of the total volume associated with the ranking regions where each of the possible options occupy the first position in the consensus ranking. As Table 5 demonstrates, the weight space is almost evenly split volume-wise with weightings where option D as the winner versus weightings where option A as the winner. This is interesting as the baseline weight selection produces a ranking where option A is ranked third.

Table 5: *The fraction of the total volume associated with each of the possible options occupying the first position in the final Borda consensus ranking.*

Winner	Fraction
D	0.5092
A	0.4880
C	0.0028
B	1.8×10^{-5}

With our results for the volume sensitivity measure in hand, we turn our attention to the two more locally oriented sensitivity measures—representativity and distance. To determine the representativity of our baseline set of weights, we first compute the centroid of the polytope associated with the ranking $\langle DCAB \rangle$, obtaining $W_C = (\frac{5450}{207}, \frac{4670}{161}, \frac{42415}{2898}, \frac{1325}{138}, \frac{310}{69}, \frac{4150}{483}, \frac{1520}{207})^T$. In comparison to the centroid, our baseline weight vector have a coefficient of representativity $\alpha = 0.1527$, which exceeds the critical value $\hat{\alpha} = 0.05$ defined in Section 2.4.3.1. The subset of the original polytope containing all points with a coefficient of representativity of at least 0.1527 accounts for 37.00% of the original volume. The representativity measure indicates that the weights have a slight, non-excessive bias to other ranking regions.

The distance sensitivity measure is the only measure not yet considered. For each of the 24 possible Borda consensus rankings Table 6 lists the adjusted weight vector that is a minimal distance from the baseline weight allocation. The nearest adjacent ranking is $\langle DCBA \rangle$ at a distance of $d_{\leftrightarrow} = 3.03$ from our baseline weights. We also determine the distance to the nearest boundary of the polytope, $d_b = 2.16$, which corresponds to a allocation of weights with $\bar{W}_6 = 0$. In order to determine whether these distances are atypically small compared to the overall size of the baseline ranking, we compute the Chebyshev radius of the polytope associated with $\langle DCAB \rangle$, obtaining $r = 9.38$. We may then determine $\delta_{\leftrightarrow} = 0.323$ and $\delta_b = 0.230$, both of which exceed the critical value $\hat{\delta} = 0.05$ defined in Section 2.4.2.1. We therefore conclude that all weights contribute significantly to the evaluation and that the baseline ranking is stable against small perturbations to the criteria weights.

Based on the three sensitivity measures applied, we conclude that the set of weights chosen by PMO CSC is a "typical" representative of the set of weights that result in the final ranking $\langle DCAB \rangle$.

3.2.2 Sensitivity to option ratings

The determination of the option ratings (v_{iA}, v_{iB}, v_{iC} and v_{iD} for $i = 1, \ldots, 13$) by employing the Borda count led AST to examine the sensitivity of final consensus ranking to changes to the rating generation scheme.

Table 6: *The distance within the six-dimensional hyperplane in weight space from the baseline allocation of weights to the nearest point of the polytope associated with each of the 24 possible Borda consensus rankings.*

Ranking	Distance	Weights						
		$\bar{W}_1$	$\bar{W}_2$	$\bar{W}_3$	$\bar{W}_4$	$\bar{W}_5$	$\bar{W}_6$	$\bar{W}_7$
$\langle DCAB \rangle$	0	50	9	15	7	10	2	7
$\langle DCBA \rangle$	3.035	51.71	8.87	13.95	5.95	9.87	0.95	8.70
$\langle CDAB \rangle$	12.84	48.54	7.54	23.0	2.38	14.85	3.69	0
$\langle DACB \rangle$	15.12	39.5	2.58	16.75	8.75	15.83	7.83	8.75
$\langle CADB \rangle$	17.39	41.75	3.50	23.25	4.25	18.25	7.5	1.5
$\langle ACDB \rangle$ $\langle ADCB \rangle$	17.61	40.34	2.72	22.18	5.27	18.34	8.11	3.04
$\langle DBCA \rangle$	19.56	45.50	0	10.5	2.5	17.50	3.50	20.50
$\langle CDBA \rangle$	20.02	60.71	0	21.43	0	17.85	0	0
$\langle DABC \rangle$ $\langle DBAC \rangle$	20.24	41.40	0	11.67	3.67	18.67	5.55	19.03
$\langle CABD \rangle$	23.37	43.08	0	27.69	0	29.23	0	0
$\langle ADBC \rangle$	24.65	36.56	0	16.96	0	24.29	7.56	14.62
$\langle ACBD \rangle$	26.42	38.8	0	25.4	0	27.6	5.2	3.0
$\langle ABCD \rangle$ $\langle ABDC \rangle$	28.97	34.23	0	19.46	0	30.34	4.28	11.67
$\langle BADC \rangle$ $\langle BDAC \rangle$	38.86	37.03	0	5.19	0	42.22	0	15.57
$\langle BACD \rangle$ $\langle BCAD \rangle$ $\langle BCDA \rangle$ $\langle BDCA \rangle$ $\langle CBAD \rangle$ $\langle CBDA \rangle$	44.81	50	0	0	0	50	0	0

PMO CSC's decision to use the Borda method implies that the criteria have no measurable quantitative properties that can be used to generate the option ratings. In such cases, the Borda method is one of many possible approaches. The popular Saaty's Analytical Hierarchy Process [26] is another method. However, there are several noted flaws that must be considered [27] as it attempts to quantify the strength of preference between options. As it is often impossible to compare the strength of preference of one criterion (individual voter) to the next [28], it may be argued that the only acceptable ranking exercise is ordinal. Ordinal ranking of objects captures **only** the preference order. However an ordering must somehow be translated into actual rating values.

The Borda method is often labeled as the most transparent of all ranking systems and it is used widely used by institutions around the world. While it is considered an ordinal ranking method, the assignment of ratings to options based on individual rankings implies that the strength of preference between two options is proportional to the number of rankings between them. Furthermore, whether or not the relationship between ratings should be linear may be questioned. When this is not the case then other rating generating schemes, such as Hunter and Emond's [8] multiplicative relationship, should be considered.

AST used the Hunter-Emond method to assign relative ratings to the options. Using a minimum option rating of $1/8$ and a multiplicative factor[9] of 1.4883, the resulting consensus ranking using the baseline weights was $\langle DCBA \rangle$ indicating that the order of the last two options is sensitive to the relationship between ratings.

By design the Borda method favours options supported by a broad consensus among the voters rather than the option which is preferred by a majority of the individuals. Critics of the method deem this as a fatal flaw as it violates the Condorcet property which is taken as a fundamental axiom by many analysts (see [29]). The property, named after French philosopher Marquis de Condorcet (1743-1794), stipulates that the option which is preferred by a majority of the voters in a pair-wise comparison with all other options should be ranked first, labeled the "Condorcet winner", in the consensus solution. However there is no guarantee that there will always be an option satisfying the property. Worse, the Borda method may not satisfy the Condorcet principle even if the Condorcet winner exists.

To further complicate things, Arrow's Impossibility Theorem [28] states, in layman's terms, that there exists no means of ranking three or more alternatives in a manner that will always satisfy all commonly accepted measures of "fairness." While many techniques exist for ranking a set of alternatives, each of these suffers from a set of mathematical drawbacks that can generate misleading results if applied in the wrong situation. Most techniques yield identical rankings when there is strong consensus between the individual component rankings; however, they can yield substantially different results when the individual criterion rankings demonstrate little consensus.

One of many alternatives to the Borda method is the Kendall τ_x rank correlation consensus ranking method developed by Emond and Mason [9]. The τ_x ranking solution, an improvement over the Kendall τ_b method [30], is based on pair-wise comparison and is guaranteed to find the Condorcet

[9] Obtained by solving equation $\sum_{j=1}^{4} R^{j-1} \frac{1}{8} = 1$ for R.

winner (if it exists), albeit at the expense of potential rank reversal problems (see [31] for more information).

Using the identical set of criteria weights as in the original problem, AST used the Kendall tau-x rank correlation coefficient to generate an alternative consensus ranking: $\langle DCBA \rangle$. Even when setting all 13 criteria weights to be of equal importance no change is observed in the Kendall tau-x consensus ranking $\langle DCBA \rangle$. The ordering of the last two options in this consensus ranking indicates that leasing/buying warships from a foreign country is preferred to upgrading and extending the life of the IROQUOIS, which is contrary to the consensus obtained using the Borda count. It would therefore appear that there is only strong consensus on the ranking of the first and second place options. In light of these findings, a more descriptive and defensible ranking of the four recapitalization options is likely $\langle DCA - B \rangle$, where we report the last two options as tied. The ranking of option *D*, the Condorcet winner, by both methods indicates that in this case one of the potential pitfalls of the Borda method has been avoided.

4 Conclusion

Within any large organization, decision-makers need to strike an appropriate balance between competing objectives. Fortunately, complex decisions can be facilitated through the application of multi-criteria decision analysis. When the options are evaluated against criteria for which no measurable comparison is possible, subjective weights are chosen. In this case, to avoid potential pitfalls and gaming, sensitivity analysis should be performed. In order to perform such an analysis we require measures of sensitivity of the solution to changes in the overall allocation of criteria weights.

This paper generalizes a geometric sensitivity analysis approach suggested by Brereton [2]. Three different classes of sensitivity measures are proposed: volume, distance and representativity. The methodology can be used to analyze the subjective selection of criteria weights and provide wise decision makers with quantitative evidence to accept or reject the proposed weights. In contrast to Brereton's approach, the proposed methodology is designed to analyze the sensitivity of a ranking for any subset of options. Furthermore the methodology is easy to apply using existing software available to researchers. It is even possible to apply the methodology in the reverse direction—generating a set of weights to obtain a predetermined ranking of options; however, we advise that this only be done during the development and testing of a scoring function, and even then only conducted with extreme caution.

While the methodology for sensitivity analysis developed in this paper can provide significant new and useful insight into the impact of variations in the criteria weights on the generated ranking of options, a few cautionary words are in order. Even in the event that a given allocation of weights passes the three threshold tests proposed for each of the respective sensitivity measures, the final ranking of options should still be subjected to additional scrutiny—the tests developed here speak to the validity of the weights within the context of an additive weighted scoring approach, but not to the validity of using the additive weighted scoring approach itself. A full sensitivity analysis should also check for robustness to changes in the method used to generate the final ranking, look for obvious flaws in the formulation of the problem, etc. The methodology we have developed serves to inform judgment as to the validity and robustness of a given ranking of options, but should not be the only factor considered.

The Acquisition Support Team (AST) applied the methodology on an options analysis problem faced by Project Management Office Canadian Surface Combatant (PMO CSC). AST demonstrated that the overall ranking of options presented by PMO CSC is stable against moderate variations in the weights and has not produced any evidence for fine-tuning; however, AST advised the sponsor that using a different consensus ranking technique reveals some contention as to the ordering of the third and fourth options.

Breteton's final paragraphs nicely summarize the use of such sensitivity analyses:

> ''A final word of warning. The weights, which result from applying either method in actual cases, frequently appear to be slightly modified by the process. This indicates that the selection is often highly dependent on the assumed weights. In these circumstances, the decision maker may well come under considerable pressure to modify the

weights so that a system which is “only marginally” worse can be selected in an apparently rational manner.

Devious machinations of this sort are to be frowned upon. They are not honest. However, if you wish to show that your favorite system is really best, the methods described in this paper provide an efficient technique.”

- R.C. Brereton, June 1977.

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Annex A: Using polytope software

Using parts of the example presented in Section 3, we illustrate how to formulate and compute the sensitivity measures using available software[10]. As stated, the PMO CSC problem evaluates four options, A, B, C and D against 13 criteria that can be aggregated and analyzed using seven weights $w_1, w_2, \ldots, w_7$ as defined in Table 3. The baseline weight vector chosen by PMO CSC is $W = (50, 9, 15, 7, 10, 2, 7)^T$ which results in a ranking $\langle DCAB \rangle$. To compute the volume and representativity measures of sensitivity we show how to use freely available software *lrs* by Avis [12]. *lrs* is a state-of-the-art software program that converts the representation of an input polytope (V- $\leftrightarrow$ H-representations) and can compute the volume of a V-polytope. We also show how to use CPLEX 11.1 [21] to compute distance measures of sensitivity.

A.1 Computing volume

In order to compute the volume of the region describing $\langle DCAB \rangle$ we first define the associated polytope and the scoring equations for the four options. The entire weight space is described by

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 100 \tag{A.1}$$
$$w_i \geq 0 \quad \text{for } i = 1, \ldots, 7,$$

and the scoring equations of the options are

$$S_D = 4w_1 + 4w_2 + w_3 + 4w_4 + w_5 + 2w_6 + 4w_7, \tag{A.2}$$
$$S_C = 3w_1 + 3w_2 + 3w_3 + 2w_4 + 2w_5 + 2w_6 + w_7, \tag{A.3}$$
$$S_A = w_1 + 2w_2 + 4w_3 + 3w_4 + 4w_5 + 4w_6 + 2w_7, \tag{A.4}$$
$$S_B = 2w_1 + w_2 + 2w_3 + w_4 + 3w_5 + 2w_6 + 3w_7. \tag{A.5}$$

The polytope of ranking region $\langle DCAB \rangle$ is described by adding halfspaces (inequalities) to (A.1) that model $S_D \geq S_C$, $S_C \geq S_A$, and $S_A \geq S_B$:

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 = 100 \tag{A.6}$$
$$w_1 + w_2 - 2w_3 + 2w_4 - w_5 + 3w_7 \geq 0 \tag{A.7}$$
$$2w_1 + w_2 - w_3 - w_4 - 2w_5 - 2w_6 - w_7 \geq 0 \tag{A.8}$$
$$-w_1 + w_2 + 2w_3 + 2w_4 + w_5 + 2w_6 - w_7 \geq 0 \tag{A.9}$$
$$w_i \geq 0 \quad \text{for } i = 1, \ldots, 7.$$

Inequality (A.7) forces option D to receive a higher score than option C. Similarly, inequality (A.8) forces option C to receive a higher score than option A, and inequality (A.9) forces option A to receive a higher score than option B. *lrs* is first used to compute the V-representation of the polytope and then used again to compute the volume of the polytope.

The input file, named "DCAB.ine", to *lrs* is a text file that looks like the following:

[10] Just prior to final publication of the present paper, a *Mathematica* program named POLYSENSE was developed and documented in [25]. POLYSENSE completely automates the sensitivity analysis proposed. Nevertheless, this Annex gives the reader insight into using polyhedral software.

```
DCAB.ine
H-representation
linearity 1 1
begin
11  8 rational
100 -1 -1 -1 -1 -1 -1 -1
0 1 1 -2 2 -1 0 3
0 2 1 -1 -1 -2 -2 -1
0 -1 1 2 2 1 2 -1
0 1 0 0 0 0 0 0
0 0 1 0 0 0 0 0
0 0 0 1 0 0 0 0
0 0 0 0 1 0 0 0
0 0 0 0 0 1 0 0
0 0 0 0 0 0 1 0
0 0 0 0 0 0 0 1
end
```

The first line of the input specifies the problem name. The second line indicates that the input is an H-representation of a polytope. Line 3 indicates that there is one equality constraint and specifies that it appears first. The line following "‘begin" indicates that there are 11 constraints each specified by 8 rational inputs. The remaining lines each represent a constraint. An inequality of the form $a_1w_1 + a_2w_2 + ... + a_mw_m \geq b$ is listed as $-b$ a_1 a_2 ... a_m in the input file. All the nonnegative constraints on the w_i's are all explicitly listed. To invoke *lrs* we simply type the command:

```
./lrs DCBA.ine
```

The ouput looks like the following:

```
*Copyright (C) 1995,2006, David Avis   avis@cs.mcgill.ca
*Input taken from file DCAB.ine
DCAB
V-representation
begin
***** 8 rational
 1  50  50  0  0  0  0  0
 1  0  100  0  0  0  0  0
 1  0  200/3  100/3  0  0  0  0
 1  0  50  75/2  25/2  0  0  0
 1  0  50  0  50  0  0  0
 1  0  200/3  0  0  100/3  0  0
 1  0  60  20  0  20  0  0
 1  0  200/3  0  0  0  100/3  0
 1  0  400/7  200/7  0  0  100/7  0
```

```
 1  200/3  0  100/3  0  0  0  0
 1  100/3  0  125/3  25  0  0  0
 1  100/3  0  100/3  0  0  0  100/3
 1  100/3  0  140/3  0  0  0  20
 1  200/3  0  0  100/3  0  0  0
 1  100/3  0  0  200/3  0  0  0
 1  100/3  0  0  100/3  0  0  100/3
 1  50  0  0  0  50  0  0
 1  200/3  0  0  0  0  100/3  0
 1  50  0  0  0  0  50  0
 1  400/9  0  200/9  0  0  100/3  0
 1  400/9  0  0  0  0  100/3  200/9
 1  0  50  0  0  0  0  50
 1  0  50  40  0  0  0  10
end
*Totals: vertices=23 rays=0 bases=31 integer_vertices=8
*0.015u 0.000s 1844Kb 453 flts 0 swaps 0 blks-in 0 blks-out
```

In this case the output lists 7-dimensional coordinates of the 23 vertices of the polytope[11]. This is the V-representation of the ranking region $\langle DCAB \rangle$. In order to compute the volume of this polytope we simply copy the above output, save it to a text file called "DCAB.ext", and modify as follows:

```
DCAB.ext
V-representation
begin
23 8 rational
 1  50  50  0  0  0  0  0
 1  0  100  0  0  0  0  0
 1  0  200/3  100/3  0  0  0  0
 1  0  50  75/2  25/2  0  0  0
 1  0  50  0  50  0  0  0
 1  0  200/3  0  0  100/3  0  0
 1  0  60  20  0  20  0  0
 1  0  200/3  0  0  0  100/3  0
 1  0  400/7  200/7  0  0  100/7  0
 1  200/3  0  100/3  0  0  0  0
 1  100/3  0  125/3  25  0  0  0
 1  100/3  0  100/3  0  0  0  100/3
 1  100/3  0  140/3  0  0  0  20
 1  200/3  0  0  100/3  0  0  0
 1  100/3  0  0  200/3  0  0  0
 1  100/3  0  0  100/3  0  0  100/3
```

[11] *lrs* precedes each set of vertex coordinates with the number 1.

```
 1  50  0  0  0  50  0  0
 1  200/3  0  0  0  0  100/3  0
 1  50  0  0  0  0  50  0
 1  400/9  0  200/9  0  0  100/3  0
 1  400/9  0  0  0  0  100/3  200/9
 1  0  50  0  0  0  0  50
 1  0  50  40  0  0  0  10
end
volume
```

The only changes (other than deleting non-essential comment lines starting with a single "*") is replacing "*****" after the "begin" line by the number of vertices, 23, and adding the option "volume" after the "end" line. Re-running *lrs* on the above input as follows:

```
./lrs DCBA.ext
```

generates the output:

```
*Copyright (C) 1995,2006, David Avis   avis@cs.mcgill.ca
*Input taken from file DCAB.ext
DCAB
*volume
linearity 1  1
begin
***** 8 rational
-100  1  1  1  1  1  1  1
 0  0  0  1  0  0  0  0
 0  0  0  0  1  0  0  0
 0  0  0  0  0  1  0  0
-100  3  2  0  0 -1 -1  0
-100  0  2  3  3  2  3  0
 0  0  0  0  0  0  1  0
 0  1  0  0  0  0  0  0
 0  0  1  0  0  0  0  0
 300 -2 -2 -5 -1 -4 -3  0
 100 -1 -1 -1 -1 -1 -1  0
end
*Volume= 2391359375000/15309
*Totals: facets=10 bases=59 linearities=1 facets+linearities=11
*0.046u 0.000s 2048Kb 504 flts 0 swaps 0 blks-in 0 blks-out
```

The third to last line contains the computed volume, $\frac{2391359375000}{15309}$, of the polytope.

A.2 Computing distance

To compute the distance of the baseline weight to another ranking region we formulate a quadratic program and use CPLEX 11.1 [21]. For example, the distance of $W = (50, 9, 15, 7, 10, 2, 7)$ to the ranking region $\langle CDAB \rangle$ is formulated as described in Section 2.4.2. The CPLEX input file, "DistCDAB.qp", looks like the following:

```
Minimize
obj: -50w1 - 9w2 - 15w3 - 7w4 - 10w5 - 2w6 - 7w7
+ [w1 ^2 + w2 ^2 + w3 ^2 +  w4^2 + w5 ^2 + w6 ^2 + w7 ^2] /2
Subject To
-1w1 -1w2 +2w3 -2w4 +1w5      -3w7 >= 0
3w1 +2w2 -3w3 +1w4 -3w5 -2w6 +2w7 >= 0
-1w1 +1w2 +2w3 +2w4 +1w5 +2w6 -1w7 >= 0
w1 + w2 + w3 + w4 + w5 +w6 + w7 = 100
w1 >= 0
w2 >= 0
w3 >= 0
w4 >= 0
w5 >= 0
w6 >= 0
w7 >= 0
end
```

The objective function minimizes the distance of W from a point $w = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)$ that lies in the ranking region $\langle CDAB \rangle$ which is defined by the listed inequalities. CPLEX is then run on the input file:

```
CPLEX> read DistCDAB.qp

File type: lp
Problem 'DistCDAB.qp' read.
Read time =    0.02 sec.
CPLEX> optimize
Tried aggregator 1 time.
QP Presolve eliminated 7 rows and 0 columns.
Reduced QP has 4 rows, 7 columns, and 27 nonzeros.
Presolve time =    0.00 sec.
Parallel mode: none, using 1 thread for barrier
Number of nonzeros in lower triangle of A*A' = 6
Using Approximate Minimum Degree ordering
Total time for automatic ordering = 0.00 sec.
Summary statistics for Cholesky factor:
  Rows in Factor            = 4
```

```
  Integer space required    = 4
  Total non-zeros in factor = 10
  Total FP ops to factor    = 30

Barrier - Optimal:  Objective = -1.4215384605e+003
Solution time =    0.00 sec.  Iterations = 9

CPLEX> dis sol var -
Variable Name           Solution Value
w1                           48.538461
w2                            7.538461
w3                           23.000000
w4                            2.384615
w5                           14.846154
w6                            3.692308
w7                            0.000001
```

The solution variables describe the closest point to W in region $\langle CDAB \rangle$.

A.3 Computing representativity

The main difficulty of computing the representativity measure of the baseline weights is finding the centroid of the polytope representing the ranking region. Computing the centroid is easy when the vertices of the polytope are known. Since we have already used *lrs* to convert the H-representation of the polytope representing $\langle DCAB \rangle$ to its vertex representation, we only need to compute the average of all 23 vertices. This can be easily done in MS Excel or *Mathematica*. In this case the centroid is $C = (\frac{5450}{207}, \frac{4670}{161}, \frac{42415}{2898}, \frac{1325}{138}, \frac{310}{69}, \frac{4150}{483}, \frac{1520}{207})^T$. We convert the equations (A.6-A.9) of the polytope representing $\langle DCAB \rangle$ into an inequality description in standard form, $Aw \leq b$, namely:

$$2w_1 + 2w_2 + 5w_3 + w_4 + 4w_5 + 3w_6 \leq 300 \tag{A.10}$$

$$-3w_1 - 2w_2 + w_5 + w_6 \leq -100 \tag{A.11}$$

$$-2w_1 - 3w_3 - 3w_4 - 2w_5 - 3w_6 \leq -100 \tag{A.12}$$

$$-w_1 \leq 0 \tag{A.13}$$

$$-w_2 \leq 0 \tag{A.14}$$

$$-w_3 \leq 0 \tag{A.15}$$

$$-w_4 \leq 0 \tag{A.16}$$

$$-w_5 \leq 0 \tag{A.17}$$

$$-w_6 \leq 0 \tag{A.18}$$

$$w_1 + w_2 + w_3 + w_4 + w_5 + w_6 \leq 100. \tag{A.19}$$

To determine the coefficient of representativity of W we find the largest value for α such that $AW' \leq (1-\alpha)(b - AC)$, as per (A.20-A.29), with $0 \leq \alpha \leq 1$ and

$$W' = W - C = (23.67, -20.00, 0.36, -2.60, 5.51, -6.59, -0.34)^T.$$

$$
\begin{aligned}
8.80193 &\leq 62.8019(1-\alpha) && \text{(A.20)}\\
-32.087 &\leq 23.913(1-\alpha) && \text{(A.21)}\\
55.4865 &\leq 65.4865(1-\alpha) && \text{(A.22)}\\
-23.6715 &\leq 26.3285(1-\alpha) && \text{(A.23)}\\
20.0062 &\leq 29.0062(1-\alpha) && \text{(A.24)}\\
-0.364044 &\leq 14.636(1-\alpha) && \text{(A.25)}\\
2.60145 &\leq 9.60145(1-\alpha) && \text{(A.26)}\\
-5.50725 &\leq 4.49275(1-\alpha) && \text{(A.27)}\\
6.59213 &\leq 8.59213(1-\alpha). && \text{(A.28)}\\
0.342995 &\leq 7.343(1-\alpha). && \text{(A.29)}
\end{aligned}
$$

In this case inequality (A.22) is the most constraining with $1-\alpha = 0.8473$ meaning that the baseline weights have a coefficient of representativity of 0.1527 in the associated ranking region.

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Annex B: Centres of polytopes

Mathematically, the centre of a polytope is not defined except under conditions of regularity (generalized analog in any number of dimensions of polygon regularity). Nevertheless, various notions try to capture the essence of a "centre" of a polytope. These include the centre of mass, barycentre, Chebyshev centre, Steiner point, etc. Each centre has special attributes and varying computational complexity. The study of centres of polytopes is in itself a rich field motivated by interior-point methods for convex optimization (see Kaiser *et al.* [32]). For our application we have chosen to use the generalized barycentre, defined as the average of the vertices of P given by the V-representation of the polytope (see Section 2.4.3).

Another centre of interest is the Chebyshev centre of a polytope P—the centre of the largest Euclidean sphere that lies inside P. The radius of the Chebyshev sphere was used in Section 2.4.2 to define a critical value for the distance sensitivity measure. Let $\text{depth}(x,P)$ be the smallest distance from point $x \in P$ to the exterior of polytope P. A Chebyshev centre is also defined as any point of maximum depth in P: $x_{cheby} = \text{argmax depth}(x,P)$.

A Chebyshev centre, x_{cheby}, is easy to compute given an H-representation of a polytope P:

$$P = \left\{ \vec{w} \in \mathbb{R}^m \;\middle|\; \sum_{j=1}^{m} a_{ij} w_j \le b_i,\ i = 1,2,\ldots,\bar{n} \right\}. \tag{B.1}$$

We solve the linear program

$$\begin{aligned} &\text{maximize } r \\ &\text{subject to } \sum_{j=1}^{m} a_{ij} w_j + r\sqrt{\sum_{j=1}^{m} a_{ij}^2} \le b_i \quad \text{for all } i = 1,\ldots,\bar{n} \end{aligned} \tag{B.2}$$

for r and w_j. The optimal value of $r = r^*$ is the radius of the largest sphere that fits inside P, the corresponding solution $w = w^*$, or x_{cheby}, is the desired Chebyshev centre (for a proof see Boyd *et al.* [22]).[12]

A polytope may have multiple Chebyshev centres as the solutions $(r,\vec{w})$ to (B.2) are not necessarily unique. The set of all Chebyshev centres for a polytope P is itself a polytope, P_{cheby}, expressed as

$$\sum_{j=1}^{m} a_{ij} w_j + r^* \sqrt{\sum_{j=1}^{m} a_{ij}^2} \le b_i \quad \text{for all } i = 1,\ldots,\bar{n}. \tag{B.3}$$

Logically, the interior point furthest from the boundary of a polytope P is a prime candidate for being the most representative point of P. However, the representativity sensitivity measure defined in Section 2.4.3 is based on a single centre point C. To employ Chebyshev centres with representativity, one would have to assign a point $x_{cheby} \in P_{cheby}$ as the "most representative" centre—a notion that is counterintuitive when P_{cheby} contains multiple points; all points in P_{cheby} are in some

[12] For our application a revised linear programming formulation is required (see Section 2.4.2).

senses equally representative with respect to being furthest from the boundary of P. Nevertheless, the barycentre of P_{cheby} may be an ideal candidate.

The barycentre of the Chebyshev centre polytope, P_{cheby}, may be use in the design of a representative weight vector of a particular ranking region (as per the discussion of Section 2.6).

Annex C: Relationship between the coefficient of representativity and volume

In Section 2.4.3 we claim that given a particular $\alpha \in [0,1]$ the weight region defined by the polytope $A'w \leq (1-\alpha)b'$ has a volume that is $(1-\alpha)^{m-1}$ times the volume of $A'w \leq b'$ (where the polytope $A'w \leq b'$ has dimension $d = m-1$). This statement is based on the fact that all known algorithms for exact volume computation of a polytope decompose the polytope into *simplices*—the high-dimensional analogue of a triangle [17]. A *triangulation* of a d-dimensional polytope P is a set $\{\Delta_i : i = 1, \ldots, s\}$ of d-dimensional simplices such that $P = \cup_{i=1}^{s} \Delta_i$ and no distinct simplices have an interior point in common. Given a triangulation, the volume of P (denoted $\mathrm{Vol}(P)$) is simply the sum of the volumes of the simplices:

$$\mathrm{Vol}(P) = \sum_{i=1}^{s} \mathrm{Vol}(\Delta_i). \tag{C.1}$$

Let $\Delta(v_0, \ldots, v_d)$ denote the simplex in $\mathbb{R}^d$ with vertices $v_0, \ldots, v_d \in \mathbb{R}^d$. The volume, denoted $Vol(\Delta(v_0, \ldots, v_d))$, is computed using the following formula:

$$\mathrm{Vol}(\Delta(v_0, \ldots, v_d)) = \frac{|\det(v_1 - v_0, \ldots, v_d - v_0)|}{d!}. \tag{C.2}$$

Let V be the volume of polytope $P : A'w \leq b'$ whose centroid is the origin. When we scale P by $(1-\alpha)$ with $\alpha \in [0,1]$ to obtain P^{α}, we scale the vertices $v_1, \ldots, v_n$ of P: the vertices of P^{α} are $(1-\alpha)v_1, \ldots, (1-\alpha)v_n$. A decomposition of P into simplices is also a decomposition of P^{α} however with vertices scaled by $(1-\alpha)$. Hence the volume of a simplex of P^{α} is computed as

$$\mathrm{Vol}(\Delta((1-\alpha)v_0, \ldots, (1-\alpha)v_d)) = \frac{|\det((1-\alpha)v_1 - (1-\alpha)v_0, \ldots, (1-\alpha)v_d - (1-\alpha)v_0)|}{d!}, \tag{C.3}$$

$$= (1-\alpha)^d \frac{|\det(v_1 - v_0, \ldots, v_d - v_0)|}{d!}. \tag{C.4}$$

By equations (C.1) and (C.4) the volume $\mathrm{Vol}(P^{\alpha})$ is simply $(1-\alpha)^d V$.

List of symbols/abbreviations/acronyms/initialisms

AST	Acquisition Support Team
BP BCSC	Bureau de projet – Bâtiment de combat de surface du Canada
CF	Canadian Forces
CORA	Centre for Operational Research and Analysis
CSC	Canadian Surface Combatant
DGMPD (L&S)	Director General Major Projects Division Land & Sea
DMGOR	Directorate Materiel Group Operational Research
DND	Department of National Defence
DRDC	Defence Research and Development Canada
ESA	équipe de soutien des acquisitions
PMO	Project Management Office
PWGSC	Public Works and Government Services Canada
SOR	Statement of Requirements

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This paper generalizes a sensitivity analysis approach originally suggested by Brereton [Brereton, R.C. (1977), Weighting Factors and How to Fool Your Friends, (Staff Note ORD SN 1977/06) Operational Research Division, Ottawa, Canada] for additive weighted scoring methods. Using high-dimensional computational geometry, three different sensitivity measure classes are proposed: volume, distance and representativity. The methodology can be used to analyze the subjective selection of criteria weights and provide decision makers with quantitative evidence to evaluate the robustness of the output ranking to small variations in the proposed weights. In contrast to Brereton's approach, the proposed methodology can be used to analyze the sensitivity of a ranking for *any* subset of options. Furthermore the methodology is easy to apply using existing software available to analysts.

The Acquisition Support Team (AST) applied the methodology on an options analysis problem faced by Project Management Office Canadian Surface Combatant (PMO CSC). AST demonstrated that the overall ranking of options presented by PMO CSC is stable against moderate variations in the weights and has not produced any evidence for fine-tuning.

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