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Mathematical relations to analyze aircraft performance for trajectory planning

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Abstract

We revise the derivation of the Bréguet-Coffin endurance and range formulas in the context of Newton's Second Law of Motion. We believe that the formulas we derived constitute valuable tools for analyzing aircraft performance and for analysing aircraft trajectory planning.
We point out that, in principle, the momentum of the mass of air required to burn the fuel that is taken in at rest and ejected at the airplane speed, should be taken into account since this mass is about 14.7 times that of the fuel burned. We give the exact solutions to the equations for the fuel consumption, obtained with this consideration, when the aircraft is in level flight. We consider two modes of flight: one with a constant angle of attack (as in the Bréguet-Coffin case), and one at constant speed. Comparison of the endurances and the ranges obtained with and without the correction terms show that some of the added terms can actually be neglected.
We then solve exactly the fuel consumption equations in which these terms are neglected, for climbing airplanes in three different modes of motion – flight at a constant angle of attack, flights at a constant speed, and flights at a constant Mach number. With the help of these solutions, we derive formulas for the speed, the altitude and the power required as a function of time, and as a function of the altitude. The behaviour of the solutions obtained is exhibited through many sample calculations that involve the model CP-1 airplane described in Anderson's book "Introduction to Flight".

Résumé

Nous avons étudié la fonction dérivée des formules de Bréguet-Coffin utilisées pour calculer l'autonomie et la distance franchissable, à la lumière de la deuxième loi du mouvement de Newton. Nous sommes d'avis que les formules ainsi élaborées constituent de précieux outils pour analyser les performances d'un aéronef et la planification des trajectoires d'aéronefs.
Nous signalons que, en principe, il faudrait tenir compte du mouvement de la masse d'air nécessaire pour brûler le carburant, laquelle est prélevée au repos et éjectée à la vitesse de l'avion, puisque cette masse est environ 14,7 fois plus grande que celle du carburant brûlé. En tenant compte de ce facteur, nous donnons des solutions exactes aux équations de consommation de carburant lorsque l'aéronef vole en palier. Nous prenons en considération deux modes de vol : un dont l'angle d'attaque est constant (comme dans le scénario de Bréguet-Coffin) et un à vitesse constante. La comparaison des valeurs d'autonomie et de distance franchissable obtenues au moyen des formules de correction et sans elles, démontre que certaines des formules ajoutées peuvent, en fait, être omises.
Nous avons ensuite résolu avec exactitude les équations de consommation de carburant où les formules en question étaient omises, pour des avions en montée, en fonction de trois modes de mouvement : des vols à un angle d'attaque constant, des vols à vitesse constante et des vols à un nombre de Mach constant. Grâce aux solutions ainsi obtenues, nous avons élaboré des formules pour calculer la vitesse, l'altitude et la puissance nécessaires en fonction du temps de même qu'en fonction de l'altitude. Le comportement des solutions obtenues est présenté à l'aide de nombreux exemples de calculs effectués au moyen du modèle d'avion CP1 décrit dans le livre de M. Anderson, *Introduction to Flight*.

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Nomenclature
(Essentially the same as in Anderson [1])

a(h) = speed of sound in air at altitude h = $\sqrt{\gamma R T(h)}$. At sea level, a(0) = 340.3029 m/s
a_1 = absolute value of the slope of the temperature as a function of altitude, before 11 km $= 6.5\times 10^{-3}$
AFR = air fuel ratio (about 14.7)
AR = aspect ratio = $\frac{b^2}{S}$
b = wingspan
c = the specific fuel consumption in Newton per Watt-second, that is in m^{-1}
C_D = global drag coefficient for the aircraft
C_{D0} = global drag coefficient at zero lift
C_L = global lift coefficient for the aircraft
D = drag
e = Oswald's efficiency factor
g = gravitational constant = 9.8 m/s^2
h = altitude of airplane
L = lift
M = Mach number = $\frac{V}{a(h)}$
P(t) = power of the engine in Watt at instant t
R = specific gas constant for air = 287.058 J/(kg K)
S = wing area
t = time variable
T_s = temperature at sea level = 288.16 K
T(h) = temperature at altitude h
v3 = vertical component of airplane velocity
V_∞ = airplane speed with respect to the undisturbed air in front of it
W(t) = weight of the airplane at time t
W_1 = weight of the airplane without fuel
W_f = total weight of fuel at the time of departure
$W_0 = W_1 + W_f$ = total weight of the airplane at departure

γ = ratio of the constant pressure specific heat to the constant volume specific heat = $\frac{c_p}{c_V}$ = 1.4 for air
η = propeller efficiency
ρ_s = air density at sea level = 1.225 kg/m^3
$\rho_\infty(h)$ = density of undisturbed air in front of airplane, at altitude h

Basic formulas

$C_D = C_{D,0} + \dfrac{C_L^2}{\pi e AR}$ (Drag polar)

$D = \frac{1}{2}\rho_\infty S C_D V_\infty^2$

$L = \frac{1}{2}\rho_\infty S C_L V_\infty^2$

1. INTRODUCTION

The Bréguet's range and endurance formulas are well known to everyone interested in airplane performances. Notwithstanding their name, it seems (see Anderson [1]) that their derivation was first published in a 1920 NACA report by Coffin [2]. Such formulas can play an important role in the design of airplanes, the analysis of their performances, and can be very useful in the path planning of their missions. It is our purpose in this article to derive more general formulas to calculate the amount of fuel used in flights on trajectories with a constant angle of ascension.

We recall that the Bréguet's equations pertain to level trajectories, for which the angle of attack is kept constant, that is, for which the lift and drag coefficients are constant. In such a situation, the speed of the aircraft decreases during the flight, as its weight decreases with the consumption of fuel. In the present study, we have obtained the corresponding formulas for flights for which it is the speed of the aircraft that is kept constant. Our main contribution however is the derivation of formulas for trajectories that climb at a constant angle. We treat three different modes of motion. The first one is the same as for the Bréguet's equations; the angle of attack is kept constant. The second situation is when the speed of the aircraft is kept constant and the third one is when its Mach number is kept constant. The formulas obtained are valid for any motion in which the vertical component of the velocity remains at a constant inclination with respect to the horizontal plane. In particular, they hold for rectilinear and helical trajectories. In all cases, we derive exact formulas for the fuel consumption of the airplane, the altitude and the speed.

In this study, we consider propeller driven airplanes. Thus, the power available for the motion $P_A = \eta\, P_p$, where η is the propeller efficiency and P_p is the power produced by the motor. If a particular motion requires the power P_R, and c is the specific fuel consumption for the motor, then the fuel rate of use is given by

$$\frac{dW}{dt} = -c\,P_P = -\frac{c}{\eta}P_R \qquad (1.1)$$

Furthermore, for simplicity purposes, we consider that the motion occurs below 11 km, so that the rate of variation of the temperature with the altitude a_1 is constant, being given by:

$$T(h) = T_s - a_1\, h. \qquad \text{with } a_1 = 6.5 \times 10^{-3} \tag{1.2}$$

Our results can easily be extended to situation in which the airplane traverses zones of the atmosphere with different temperature variations. It then suffices to connect the solutions valid within each zone, so that they are continuous at the boundary layers between the zones.

For trajectories that climb or descend at a fixed angle θ with respect to the horizontal plane, the following relation always holds for the altitude h:

$$\frac{dh}{dt} = V_\infty \sin(\theta) \tag{1.3}$$

so that

$$\frac{d\rho_\infty}{dt} = -4.2433\, a_1 \sin(\theta) \frac{V_\infty \rho_\infty}{T(h)} \tag{1.4}$$

since

$$\rho(h) = \rho_s \left[\frac{T(h)}{T_s}\right]^{4.2433} \tag{1.5}$$

The following transversal equilibrium condition is always satisfied:

$$L = W \cos(\theta), \tag{1.6}$$

which states that the lift cancels the component of the weight perpendicular to the trajectory. The sum of longitudinal forces that act along the motion of the airplane is:

$$T_R - D - W \sin(\theta) \tag{1.7}$$

where T_R is the thrust produced by the propeller and D is the drag. In all the cases we study, we consider that the airplane has enough power to follow the prescribed trajectory.

In order to illustrate the behavior of the solutions obtained, we perform explicit calculations for the hypothetical airplane CP-1 described by Anderson in Section 6 of his book [1] This is an airplane similar to the Cessna Skylane that has the following characteristic parameters.

$W_1 = 9{,}454.43$ N	$W_f = 1{,}343.31$N	$W_0 = 10{,}797.74$ N
$b = 10.9118$ m	$S = 16.1653\ m^2$	$AR = 7.3656$
$C_{D0} = 0.025$	$e = 0.8$	$h = 0.8$
$c = 7.4475\times10^{-7}$		

2. BRÉGUET'S EQUATIONS AND THE CONSERVATION OF MOMENTUM

In most textbooks, Bréguet's equations are introduced under a section called "Level un-accelerated flight". Their derivation then follows essentially that of Coffin [2]. It is considered that, in this mode of motion, the angle of attack is kept constant, so that the lift coefficient C_L and the drag coefficient C_D are both constant. We review briefly this derivation in the next section.

2.1 Derivation of Bréguet's equations

Given, Eq.(1.6), with $\theta = 0$, it is then required that the thrust of the motor just cancels the drag as:

$$T_R = D. \tag{2.1}$$

When this equation is multiplied by the speed V_∞, its left hand side becomes the power required P_R for the motion. The value so obtained for P_R is then substituted into Eq.(1.1), the drag D is expanded as $\frac{1}{2}\rho_\infty S C_D V_\infty^2$, and everywhere in the resulting equation, V_∞ is written as

$$V_\infty = \left[\frac{2W}{\rho_\infty S C_L}\right]^{1/2} \tag{2.2}$$

that follows from Eq.(1.6). This yields the following very simple equation for the weight of the airplane is then obtained:

$$\frac{dW}{dt} = -2k\,W^{3/2} \qquad \text{with} \qquad k = \frac{c\,C_D}{\eta\sqrt{2\rho_\infty S C_L^3}} \qquad \text{being a constant.} \tag{2.3}$$

This differential equation is readily solved to yield:

$$W(t)^{-1/2} - W_0^{-1/2} = k\,t. \tag{2.4}$$

Upon setting $W(t) = W_1$, the empty weight of the airplane, this equation yields the endurance $t = E_B$ as:

$$E_B = \frac{1}{k}\left[W_1^{-1/2} - W_0^{-1/2}\right] \tag{2.5}$$

The value of W(t), gotten from Eq.(2.4), can be substituted into Eq.(2.2) to yield an expression for the speed V_∞ as a function of time. When this $V_\infty(t)$ is integrated over time from 0 to E_B, one obtains the range:

$$R_B = \frac{\eta C_L}{c C_D} \ln\left[\frac{W_0}{W_1}\right]. \tag{2.6}$$

From Eq.(2.5), one can see that the endurance E_B is maximum when the ratio $\frac{C_L^{3/2}}{C_D}$ is minimum. Upon using the drag polar equation that relates C_D to C_L, it is readily computed that this happens when

$$C_L = \sqrt{3\pi e A R C_{D0}}\ . \tag{2.7}$$

Similarly, from Eq.(2.6), one can see that the range R_B is maximum when the ratio C_L/C_D is maximum. Again, using the drag polar one can calculate that this happens when

$$C_L = \sqrt{\pi e A R C_{D0}}\ . \tag{2.8}$$

2.2 Example of calculation

For Anderson's model airplane CP-1, the values of the parameters that correspond to maximum endurance are:

$C_L = 1.1783$	$C_D = 0.1000$
$V(0) = 30.42$ m/s	$V(E_B) = 28.47$ m/s
$E_B = 57{,}150.18$ s = 15.88 h	$R_B = 1{,}681.54$ km

and those for maximum range are:

$C_L = 0.6803$	$C_D = 0.0500$
$V(0) = 40.04$ m/s	$V(E_B) = 37.46$ m/s
$E_B = 50{,}142.58$ s = 13.93 h	$R_B = 1{,}941.68$ km

2.3 On Newton's second law

Newton's second law states that the rate of change of momentum equals the force that acts on the system. It may appear paradoxical that, in the above description of level flight, one takes care of cancelling out all the forces at play, according to Eqs (1.6) and (2.1), but that the momentum does not appear to be conserved. Actually, according to this description, both the mass and the speed of the airplane decrease so that there is a global decrease in its momentum. For example, in the first calculation shown above, the momentum of the aircraft at the beginning and that at the end of the flight are:

$$\frac{W_0}{g} V(0) = 33{,}519.75 \text{ kg m/s} \quad \text{and} \quad \frac{W_1}{g} V(E_B) = 27{,}463.41 \text{ kg m/s}.$$

There is then a loss of 6,056.34 kg m/s of momentum. The first explanation that comes to mind is that it is the lost fuel that has carried away that momentum. However, since the initial total mass of fuel is 137.07 kg, which is initially moving at the plane speed of 30.42 m/s, the momentum of this fuel was 4,169.74 kg m/s. There remains a loss of 1,886.60 kg m/s of momentum unaccounted for. This is the equivalent of a mass of 62 kg moving at the average airplane speed of 30.42 m/s, which is about half the amount of fuel more that is lost than that was there in the first place.

As this discussion shows, a more careful analysis of the momentum is warranted. The fact is, as was pointed out by Ramsey [7], that the following formulas

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} \qquad \text{and even} \qquad \frac{d(m\mathbf{v})}{dt} = \mathbf{F}, \tag{2.9}$$

in which **F** is the force acting on a body of mass m and velocity **v**, are not necessarily true when the mass is changing. Ramsey gives three examples of motion that show that in one case, the first of Eqs (2.9) holds, in the second case, the second equation holds, and in the third case, none of them holds and an additional term is required in the equation. In all cases, it is nevertheless true that the definition of force corresponds to the rate of change of momentum. He concludes that the only safe procedure to obtain the true equation of motion is to deduce it by computing the change in the momentum before and after a time interval δt; and taking the limit as δt goes to zero. Here is the analysis that then results.

For the airplane with an internal combustion engine, air is taken from the outside to serve for the combustion of the fuel. Initially, this air is essentially at rest but once it enters the plane, it moves with it at its velocity. Once the combustion has occurred, its residues, which are moving at the speed of the plane, are ejected outside, essentially leaving the plane at its velocity. Let then M be the mass of the airplane at time t, and let δm_a be the mass of the air that is taken in during the interval of time [t, t+ δt]. Since the velocity of the air is null, and that of the airplane is **v**, the total momentum of this mass of air and of the airplane with its fuel, at time t, is

$$M\,\mathbf{v} + \delta m_a \times \mathbf{0} \tag{2.10}$$

During the interval of time of duration δt, there is an amount of fuel δm_f that reacts with the air intake and burned. The result of the combustion is a gas of mass $(\delta m_a + \delta m_f)$. This gas is then ejected out of the plane, essentially at the same velocity as the plane. At the end of this time interval, the velocity of the airplane is $(\mathbf{v} + d\mathbf{v})$. The average velocity of the mass of combustion gas (air and fuel), which is $(\delta m_a + \delta m_f)$, can be represented by $\mathbf{v} + \varepsilon\, d\mathbf{v}$, where $0 < \varepsilon < 1$. Thus, the final momentum of the combustion gas and the airplane, which is now moving at the velocity $(\mathbf{v} + d\mathbf{v})$, is:

$$(M - \delta m_f)(\mathbf{v} + d\mathbf{v}) + (\delta m_a + \delta m_f)(\mathbf{v} + \varepsilon\, d\mathbf{v},) \tag{2.11}$$

Newton's law states that the difference between the expressions in Eqs (2.11) and (2.10) is equal to **F** δt. Upon dividing this equation by δt and taking the limit as δt → 0, one obtains the equation:

$$M\frac{d\mathbf{v}}{dt}+\left[\frac{dm_a}{dt}\right]\mathbf{v}=\mathbf{F} \qquad (2.12).$$

The amount of air intake is proportional to the amount of fuel to burn. Let AFR be the air to fuel ratio used (that is about 14.7 for gasoline or diesel fuel [5]), then

$$\frac{dm_a}{dt}=AFR\frac{dm_f}{dt}=-AFR\frac{dM}{dt} \qquad (2.13)$$

Thus, Eq.(2.12) is here:

$$M\frac{d\mathbf{v}}{dt}-AFR\left[\frac{dM}{dt}\right]\mathbf{v}=\mathbf{F}. \qquad (2.14)$$

For the general climbing trajectories considered in this article, the transversal forces are in equilibrium, according to Eq. (1.6). Eqs (2.14) and (1.7) then state that, for the longitudinal motion:

$$M\frac{dV_\infty}{dt}-AFR\left[\frac{dM}{dt}\right]V_\infty=T_R - D - W\sin(\theta). \qquad (2.15)$$

Note that the terms on the left hand side of this equation are normally not included (see for example Anderson [1] and Stengel [8]), as it is considered that all the forces at play should simply cancel out.

Upon multiplying this equation by $-\frac{c}{\eta}V_\infty$, one obtains the following equation for the weight of the airplane:

$$-\frac{c\,W\,V_\infty}{\eta g}\left[\frac{dV_\infty}{dt}\right]+\frac{c\,(AFR)\,V_\infty^2}{\eta g}\left[\frac{dW}{dt}\right]=\left[\frac{dW}{dt}\right]+\frac{cV_\infty}{\eta}\left[D+W\sin(\theta)\right] \qquad (2.16)$$

2.4 Bréguet's level flight formulas with correction terms

Let us consider the same mode of motion that leads to Bréguet's formulas, namely level flight with the angle of attack kept constant. Eq.(2.2) states that $V_\infty=\text{const.}\times W^{1/2}$, so it can be used to compute $\frac{dV_\infty}{dt}$, and to write V_∞ in terms of W in Eq.(2.16). Doing so yields the following simple equation:

$$\left[1-aW\right]\frac{dW}{dt}=-2\,k\,W^{3/2} \quad \text{with } a=\frac{c\left[2(AFR)-1\right]}{\eta\, g\, \rho_\infty S C_L} \text{ being a constant} \quad (2.17)$$

and k being the constant defined in Eq.(2.3). This equation is readily solved to yield:

$$W(t)^{-1/2} - W_0^{-1/2} - a\,[W_0^{1/2} - W(t)^{1/2}] = k\,t. \quad (2.18).$$

Upon setting $W(t) = W_1$ in this equation, one obtains the endurance as:

$$E = E_B - \frac{a}{k}\left[W_0^{1/2} - W_1^{1/2}\right]. \quad (2.19)$$

where E_B is the value of the endurance according to Bréguet's formula. Clearly, E is somewhat shorter than E_B. Eq.(2.18) yields for W:

$$W(t) = \frac{1}{4a^2}\left[A + kt - \sqrt{(A+kt)^2 - 4a}\right]^2 \quad \text{with } A = W_0^{-1/2} + aW_0^{1/2} \quad (2.20)$$

Upon substituting this value in Eq.(2.2) for the speed, one obtains

$$V_\infty(t) = B\left[A + kt - \sqrt{(A+kt)^2 - 4a}\right] \quad (2.21)$$

with $B = \dfrac{\eta g}{c\left[2(AFR)-1\right]}\sqrt{\dfrac{\rho_\infty S C_L}{2}}$ being a constant.

Upon integrating the speed, one obtains the distance traveled as:

$$X(t) = B\left[X_1(t) - X_1(0)\right] \quad \text{with}$$

$$X_1(t) = A\,t + \frac{k\,t^2}{2} - \frac{(A+k\,t)}{2\,k}\sqrt{(A+k\,t)^2 - 4a} + \frac{2\,a}{k}\ln\left[(A+k\,t) + \sqrt{(A+k\,t)^2 - 4a}\right] \quad (2.22)$$

We have calculated the endurance, the final speed and the range for Anderson's model CP-1 airplane that flies at the same constant angle of attack so as to maximize the endurance. The results obtained are as follows.

$C_L = 1.1783$	$C_D = 0.1000$
$V_\infty(0) = 30.42$ m/s	$V_\infty(E_B) = 28.47$ m/s
$E_B = 57{,}083.42$ s = 15.86 h	$R_B = 1{,}679.58$ km

With the parameters that maximize the range, we have found

$C_L = 0.6803$	$C_D = 0.5000$

$V_\infty(0) = 40.04$ m/s $\qquad V_\infty(E_B) = 37.46$ m/s
$E_B = 50041.12$ s $= 13.90$ h $\qquad R_B = 1937.74$ km

The maximum endurance found is 1 minute and 6.76 s smaller than the one given by Bréguet's formula. This represents a correction of only about 0.1 % over the whole flying time. The final speed is essentially the same as with Bréguet's formula, and the range is shorter by 1,965.8 m. The difference in maximum range with that given by Bréguet's formula is only 3.94 km, which represents a relative difference of 0.2 % over the whole range. Therefore, the corrections to Bréguet's formulas can be considered negligible given all the perturbations that would occur in an actual flight, due to winds and inhomogeneities of the atmosphere. It is nevertheless worthwhile knowing that the actual values for the endurance and the range are somewhat smaller than those given by Bréguet's formulas.

According to Eq.(2,15), the force that acts on the plane is

$$F(t) = \frac{[2(AFR)-1]cC_D}{g\eta\rho_\infty SC_L^2}\,\frac{W(t)^2}{[1-aW(t)]}. \tag{2.23}$$

From Eqs. (1.1) and (2.17), one obtains for the power required for the motion

$$P_R(t) = \sqrt{\frac{2}{\rho_\infty SC_L^3}}\,\frac{C_D W(t)^{3/2}}{[1-aW(t)]} \tag{2.24}$$

Figure 1 shows the graph of the speed as a function of time, Figure 2 shows the weight of fuel as a function of time, and Figure 3 shows the graph of the power required as a function of time. This graph shows that the power required is maximum at the beginning of the flight; the initial power required being 27,913.50 Watts. At the end of the flight, the power required is 22,866.55 Watts.

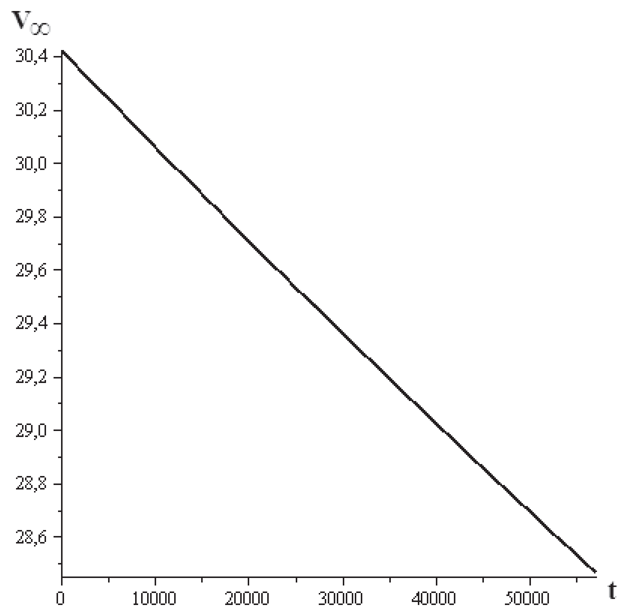

Figure 1: Speed as a function of time, for the CP-1 plane in horizontal flight, with constant angle of attack, when maximizing the endurance.

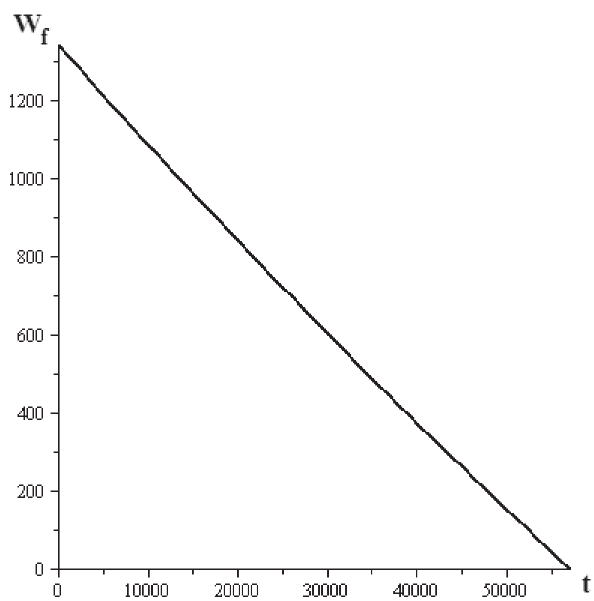

Figure 2: Weight of fuel for this airplane as a function of time, in the flight described in Figure 1.

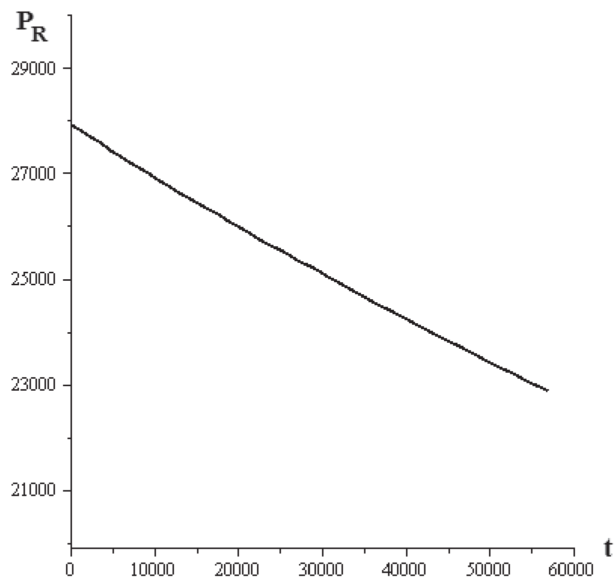

Figure 3: Power required in Watts for the CP-1 plane in the flight described in Figure 1.

2.4.1 On the conservation of momentum

The initial momentum of the plane is $P_{0\ plane}$ = 33,519 kg m/s and its final momentum is $P_{f\ plane}$ = 27,463.41 kg m/s. The momentum of the mass of combustion gases is approximately equal to its total mass times the average of the plane speed: P_{gases} = $(AFR+1)\frac{W_f}{g}V_{av}$ = 63,319.75 kg m/s. It is easy to check that the following equation for the conservation of momentum:

$$\left[P_{f\ plane} + P_{gases}\right] - P_{0\ plane} = \int_0^E F(t)\,dt \qquad (2.25)$$

is satisfied within 60 kg m/s, which is the discrepancy that results from the approximation we made in calculating P_{gases}.

3. HORIZONTAL FLIGHT AT CONSTANT SPEED

Let us now consider a level flight in which the speed is kept constant. Eq.(2.16) then becomes

$$\frac{c(AFR)V_\infty^2}{\eta g}\left[\frac{dW}{dt}\right] = \left[\frac{dW}{dt}\right] + \frac{cV_\infty D}{\eta} \qquad (3.1)$$

Eq.(2.2) determines the value of C_L as

$$C_L = \frac{2W}{\rho_\infty S V_\infty^2} \tag{3.2}$$

Upon expressing D in terms of C_D and then expressing C_D in terms of C_L, according to the drag polar equation, the fuel consumption Eq. (3.1) takes the form

$$\frac{dW}{dt} = -k_1 - k_2 W^2 \tag{3.3}$$

in which the constants k_1 and k_2 are defined as follows.

$$k_1 = \frac{c g \rho_\infty S C_{D0} V_\infty^3}{2\left[\eta g - c\,AFR\,V_\infty^2\right]} \quad \text{and} \quad k_2 = \frac{2cg}{\pi e\,AR\,\rho_\infty S V_\infty \left[\eta g - c\,AFR\,V_\infty^2\right]} \tag{3.4}$$

Eq. (3.3) is readily solved to yield:

$$W(t) = \frac{W_0 - \sqrt{\frac{k_1}{k_2}}\tan\left(\sqrt{k_1 k_2}\; t\right)}{1 + W_0\sqrt{\frac{k_2}{k_1}}\tan\left(\sqrt{k_1 k_2}\; t\right)}. \tag{3.5}$$

In this mode of flight, the endurance E_V, obtained by setting $W(E_V) = W_1$ in Eq.(3.5), is

$$E_V = \frac{1}{\sqrt{k_1 k_2}}\left\{\tan^{-1}\left[\frac{\sqrt{k_1 k_2}\left[W_0 - W_1\right]}{k_2\, W_0 W_1 + k_1}\right]\right\} \tag{3.6}$$

and the range R_V is $$R_V = V_\infty E_V. \tag{3.7}$$

We were not able, in this case, to obtain an exact formula for the speed V_∞ at which the endurance E_V and the range R_V are maximum. These values can, however, easily be gotten from the graphs showing the dependences of E_V and R_V on the speed V_∞. Figure 4 shows these graphs for the CP-1 plane described in Anderson [1]. A simple binary search can be used to determine that E_V is maximum at the speed V_∞ = 29.38 m/s while the range is maximum at the speed V_∞ = 38.69 m/s. For Anderson's model airplane CP-1, the values of the parameters that correspond to maximum endurance are:

V_∞ = 29.38 m/s

$C_L(0)$ = 1.2631	$C_D(0)$ = 0.1112
$C_L(E_V)$ = 1,1059	$C_D(E_V)$ = 0.05366
E_V = 57,049.79 s = 15.85 h	R_V = 1,676.35 km

and those for maximum range are:

$V_\infty = 38.69$ m/s
$C_L(0) = 0.7284$ $\quad$ $C_D(0) = 0.04697$
$C_L(E_V) = 0.6378$ $\quad$ $C_D(E_V) = 0.04697$
$E_V = 50{,}039.79$ s = 13.90 h $\quad$ $R_V = 1{,}936.19$ km

We note that maximum endurance at constant speed is smaller by 33.63 s than that at a constant angle of attack. This represents a difference of 0.06 % over the whole flying time. Similarly, the maximum range is 1.55 km shorter than when flying at constant angle of attack, which represents a difference of 0.08 % over the whole range.

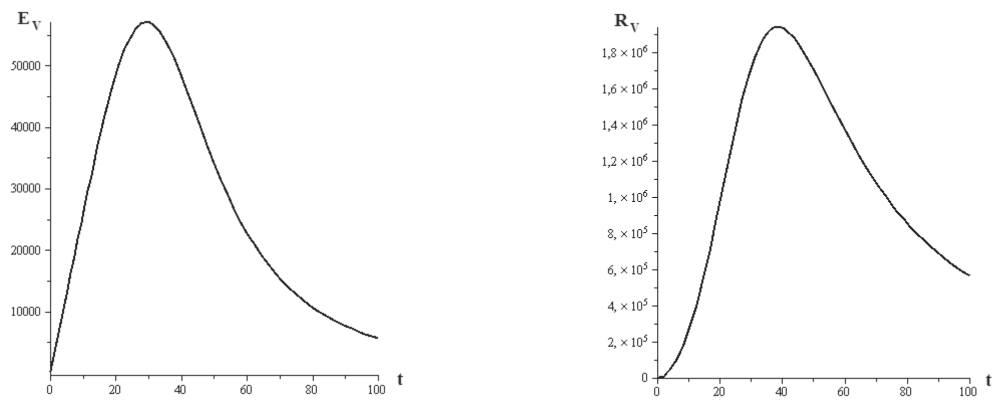

Figure 4: On the left-hand-side: graph of the endurance E_V for level flight at constant speed V_∞ as a function of the speed. On the right-hand-side: the range R_V as a function of V_∞.

According to Eqs.(1.1) and (3.3), the power required is

$$P_R = \frac{\eta}{c}\left[k_1 + k_2 W^2\right] \tag{3.8}$$

The curve for P_R as a function of time is very similar to that obtained for the flight at constant angle of attack. The power required is maximum at the beginning of the trajectory with $P_R(0) = 27{,}961.72$ Watts. At the end of the flight, the power required is 22,904.29 Watts. These values are slightly higher than those for the flight at constant angle of attack, by between 37 and 50 Watts, which represents about 0.17 % of the power required.

3.1 Approximate solution

We repeated the same calculations as above, while neglecting the term on the left-hand-side of Eq.(3.1). The values of the parameters that correspond to maximum endurance for the CP-1 airplane are:

V_∞ = 29.42 m/s
$C_L(0)$ = 1.2600 $C_D(0)$ = 0.1108
$C_L(E_V)$ = 1,1032 $C_D(E_V)$ = 0.09075
E_V = 57,118.69 s = 15.87 h R_V = 1,680.43 km

and those for maximum range are:

V_∞ = 38.75 m/s
$C_L(0)$ = 0.7263 $C_D(0)$ = 0.05349
$C_L(E_V)$ = 0.6359 $C_D(E_V)$ = 0.04685
E_V = 50,070.93 s = 13.91 h R_V= 1,940.25 km

We note that the difference in the values found, compared to those calculated the exact solutions, are only of about 0.12 % in the endurance and 0.24 % in the range.

4. CLIMBING AT A CONSTANT ANGLE OF ATTACK

Let us now consider ascending flights on trajectories tat make a constant angle θ with the horizontal. The first case that we study is when the angle of attack is kept constant so that the lift and drag coefficients C_L and C_D are constant. This is then the same mode of flight that leads to the Bréguet-Coffin equations for horizontal trajectories. Upon expressing the lift L in terms of the lift coefficient in Eq.(1.6), one obtains

$$V_\infty = k\left[\frac{W}{\rho_\infty}\right]^{1/2} \quad \text{with} \quad k = \left[\frac{2\cos(\theta)}{SC_L}\right]^{1/2} \text{ being a constant.} \tag{4.1}$$

The differentiation of V_∞ with respect to t yields

$$\frac{dV_\infty}{dt} = \frac{k}{2\sqrt{\rho_\infty W}}\left[\frac{dW}{dt}\right] + 2.12165\,k^2 a_1 \sin(\theta)\left[\frac{W}{\rho_\infty T}\right]. \tag{4.2}$$

so that Eq.(2.16) can be written as

$$\left[1 - k_1\frac{W}{\rho_\infty}\right]\left[\frac{dW}{dt}\right] = -k_2\left[\frac{W^{3/2}}{\rho_\infty^{1/2}}\right] - k_3\left[\frac{W^{5/2}}{\rho_\infty^{3/2}T}\right] \quad \text{with} \tag{4.3}$$

$$k_1 = \frac{c\,(AFR - 1/2)\,k^2}{\eta g} \qquad k_2 = \frac{c\,k}{\eta}\left[\frac{C_D}{C_L}\cos(\theta) + \sin(\theta)\right] \qquad k_3 = \frac{2.12165\,c\,a_1\,k^3\sin(\theta)}{\eta g}$$

being constant. Although Eq.(4.3) is not exactly soluble, it becomes soluble when the second term of the factor $\left[1 - k_1 \frac{W}{\rho_\infty}\right]$ in front of $\frac{dW}{dt}$, on its left-hand-side, is neglected. This term is equal to $\frac{c[2(AFR)-1]\cos(\theta)}{\eta g \rho_\infty C_L}\left[\frac{W}{S}\right]$. In it, the factor that precedes the wing loading (W/S) is roughly the same for all airplanes; its value being smaller than 3.24×10^{-6}. For the CP-1 airplane, the whole term is then smaller than 2.16×10^{-3}. It is then relatively small compared to "1". We also note that this term corresponds to the one that is dropped in the usual derivation of the Bréguet-Coffin formula for level flight. As we have shown in Section 2.4, its effect is indeed negligible, so that we shall solve Eq.(4.3) without it. In order to do so, we change variables from t to $z(t) = T[h(t)]$. This yields the following equation:

$$\frac{dW}{dz} = KW + \frac{k_4 W^2}{z^{5.2433}} \tag{4.4}$$

with $K = \frac{c}{\eta a_1}\left[\frac{C_D}{C_L}\cot(\theta) + 1\right]$ and $k_4 = \frac{2.12165\, c\, k^2\, T_0^{4.22433}}{\eta g \rho_\infty(0)}$ being constant.

We note that the second term on the right-hand-side of Eq.(4.4) is small compared to the first term. However, Eq.(4.4) is a Riccati equation that we can solve exactly even while keeping all the terms; so we shall do so. Thus obtaining the exact solution will allow us to examine whether that second term can be dropped without losing much precision. As explained in Ince [4], the Riccati equation can be transformed into a second order linear differential equation by a change of variable, which in the present case is:

$$\frac{u'}{u} = -\frac{k_4 W}{z^{5.2433}}\,. \tag{4.5}$$

Upon doing so, one obtains the following simple equation:

$$u'' - \left[K + \frac{5.2433}{z}\right]u' = 0\,. \tag{4.6}$$

Its solution is:

$$u'(z) = z^{-5.2433}\, e^{K z} \tag{4.7}$$

which, when integrated with respect to z, yields:

$$u(z) = y(z) + \Gamma \qquad \text{with } y(z) = -\frac{e^{Kz}}{4.2433\, z^{4.2433}}\ {}_1F_1(1, -3.2433, -Kz) \tag{4.8}$$

in which ${}_1F_1$ is the confluent hypergeometric function (see Section 9.2 of [3]). Upon substituting u and u' into Eq.(4.5), and imposing the initial condition: $z_0 = T(h_0)$ and $W(z_0) = W_0$, one obtains:

$$W = \frac{e^{Kz}}{k_4\left[\Gamma - y(z)\right]} \qquad \text{with} \qquad \Gamma = y(z_0) + \frac{e^{a z_0}}{k_4 W_0}. \tag{4.9}$$

When W is plotted as a function of the altitude h, for the CP-1 airplane that climbs at 10^0, from an altitude of 0 to 10,000 m, one obtains the graph shown in Figure 5.

Now, if the second term on the right-hand-side of Eq.(4.4) is neglected, the solution obtained W_a is:

$$W_a = W_0\, e^{K(z-z_0)}. \tag{4.10}$$

The graph of W_a as a function of the altitude is essentially not discernable from that for W. Figure 6 shows the graph of the difference W - W_a as a function of the altitude, for the flight described above. At the end of the flight, W_a is found to be larger by 0.16 kg of fuel. We repeated the same calculation for a climbing flight at 1^0, from 0 to 10,000m and found a similar result. The final difference between the two, in that case, is of about 0.15 kg of fuel. These differences can then be considered inconsequential, and we conclude that W_a can be used instead of W.

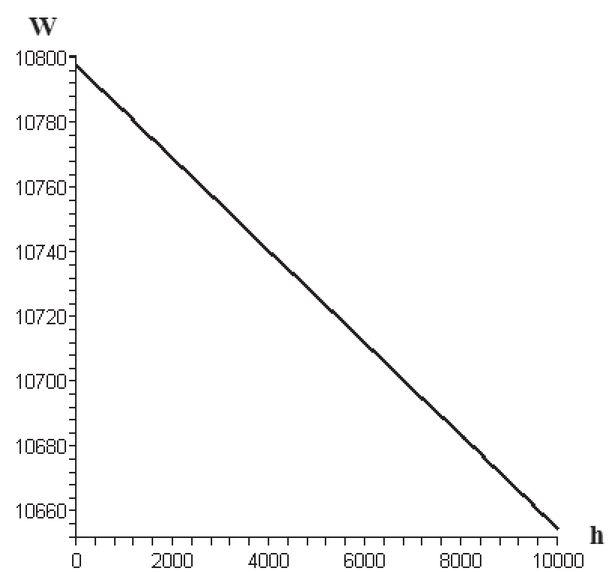

Figure 5: Weight of the airplane as a function of the altitude, in an ascension at 10^0, from 0 to 10,000 m.

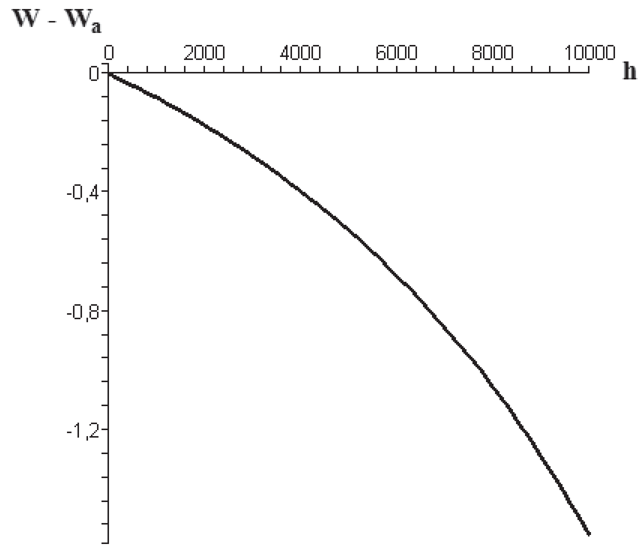

Figure 6: Difference between the values of the weight of the airplane with the second term of Eq.(4.4) W, and its approximation without this term, as a function of the altitude. In this graph, the airplane was in a climbing flight at 10^0, from 0 to 10,000 m.

4.1 Altitude

Eqs (4.8) and (4.5) yield the following equation for the altitude h:

$$\frac{dh}{dt} = V_\infty(0)\ \sin(\theta)\, T_0^{2.12165}\ T(h)^{-2.12165}\ \exp\left\{\frac{K[T(h)-T_0]}{2}\right\} \tag{4.11}$$

This is a separable equation in h and t so that it is solvable by a simple integration. It can be re-written as

$$T^{2.12165}\ \exp\left\{-\frac{KT}{2}\right\} dT = -B\,dt \quad \text{with} \tag{4.12}$$

with $$B = a_1\ V_\infty(0)\sin(\theta)\, T_0^{2.12165}\ \exp\left\{-\frac{KT_0}{2}\right\} \qquad \text{being constant.}$$

Upon integrating this equation (we used the mathematical software Maple [6] to perform most calculations and draw the graphs in this article), one obtains:

$$\frac{1}{B}\{\, y(T_0) - y(T)\} = t \tag{4.13}$$

with $$y(x) = \frac{x^{3.12165}}{3.12165}\ \exp\left\{-\frac{K\,x}{2}\right\}\ {}_1F_1\left(1,\, 4.12165,\, \frac{Kx}{2}\right) \tag{4.14}$$

Eq.(4.13) is not readily solved for h in terms of t. However, one can obtain a graph for h(t) by plotting t as a function of h, and reversing that graph. Figure 7 represents the graph so produced for a climb at 10^0, from 0 to 10,000 m that starts at the speed V_∞ = 39.73 m/s with a full tank of fuel. One notices that the slope of this curve varies very slowly, so that it can be very closely approximated by an arc of parabola. Upon matching the values at the altitudes of 5,000 m and 10,000 m, one finds that the following formula:

$$h = a\,t + b\,t^2 \quad \text{with} \quad a = 6.4358 \ \text{ and } b = 2.0923\times10^{-3} \qquad (4.15)$$

Let $t_a(h)$ represent the time gotten from Eq.(4.15), the difference between t_a and the real value of t, from Eq.(4.13) is shown in the graph of Figure 8. The maximum value of this difference is about 8 s, which represents a relative difference of about 0.71% of the whole flight duration of 1,135.00 s. For a climb at 1^0, from 0 to 10,000 m, and $V_\infty(0)$ = 40.04 m/s, which lasts 11,298.49 s, this difference is about 75s. This represents the same relative difference as in the flight at 10^0. Note that the parameters K and B, which appear in the solution, contain the angle θ so that the values of a and b depend on this angle.

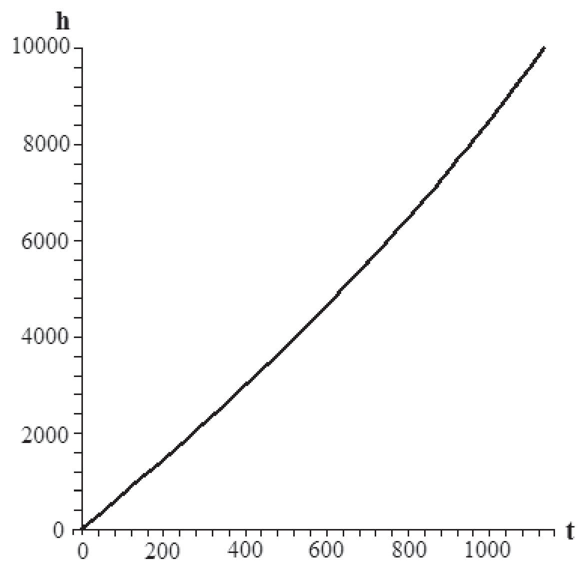

Figure 7: Altitude as a function of time, for a climb at 10^0, from 0 to 10,000 m.

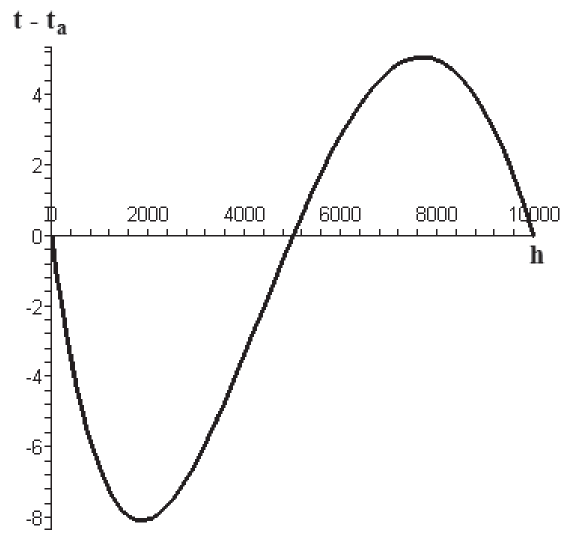

Figure 8: Graph of the difference between the real value of the time function t(h) and its approximation $t_a(h)$.

4.2 Fuel consumption

Upon substituting the value of h(t) from Eq.(4.15) into Eq.(4.10), one obtains an explicit formula for the weight of the airplane as a function of time. The graph of Figure 9 shows the fuel consumption for the CP-1 plane that starts with a full tank of gas at sea level and climbs up to 10,000 m, at an angle of 10^0, at the initial speed of 39.73 m/s. Figure 5 shows a similar climb that starts at 29.41 m/s and climbs at 1^0.

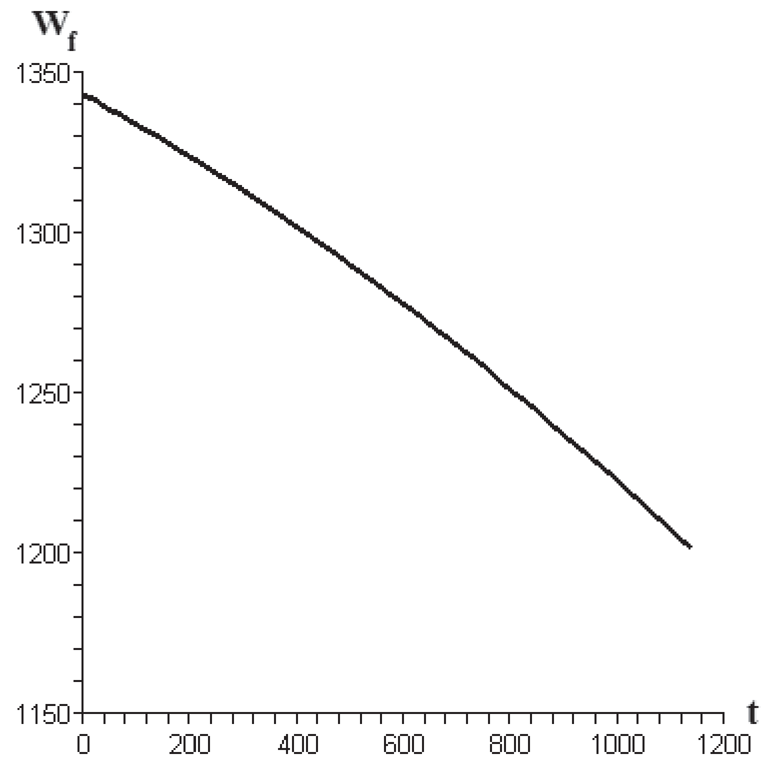

Figure 9: Fuel consumption as a function of time, in a climb at 10^0 from sea level to 10 km, with constant angle of attack, and initial speed of 39.73 m/s.

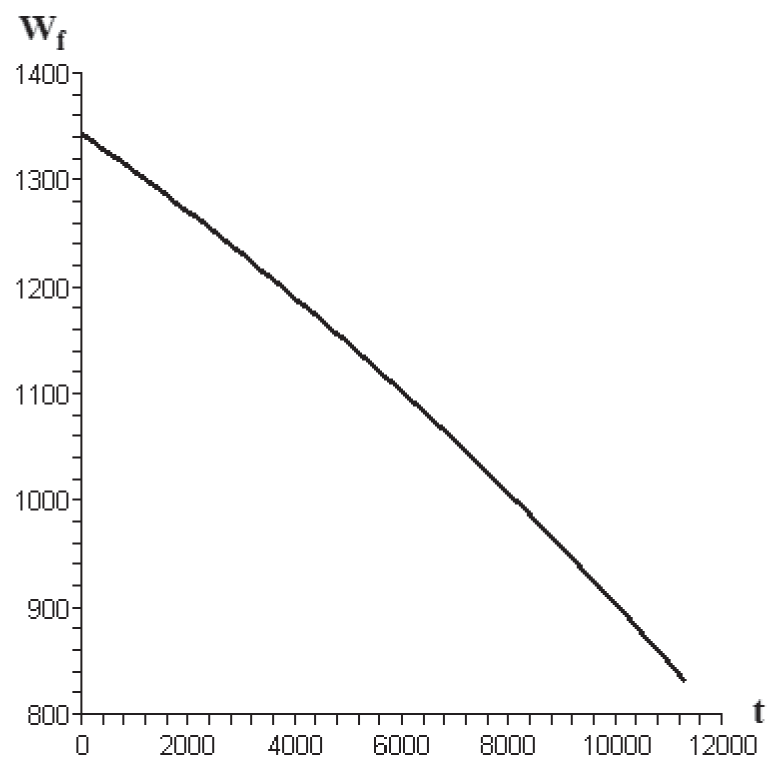

Figure 10: Weight of fuel as a function of time, in a climb at 1^0, from sea level to 10 km, at constant angle of attack with initial speed of 29.41 m/s.

Eqs.(1.1) and (4.4) give the power required for the flight at constant angle of attack. With the approximations made, this is

$$P_R = \frac{\eta}{c} k_2 \sqrt{\frac{W^3}{\rho_\infty}} . \tag{4.16}$$

Figure 11 shows the graph of P_R as a function of the altitude h in the climbing flight at 10^0. P_R is maximum at the highest altitude, its value at that point being 177,999.1 Watts.

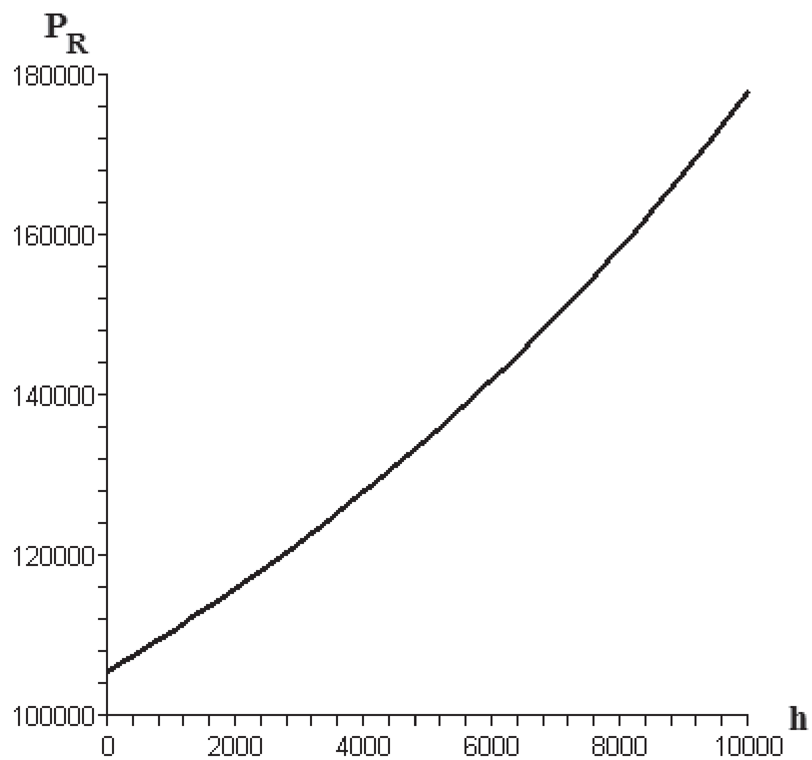

Figure 11: Power required P_R as a function of the altitude h, in a climbing flight at 10^0, with constant angle of attack, for the CP-1 airplane.

4.3 Sample flights

We calculated the parameters for two different climbs, one in which the airplane trajectory is at an angle of 10^0 with the horizontal, and one in which it is at 1^0, while it climbs from 0 to 10,000 m. For each climb, we considered three different initial speeds. One of them was the speed that corresponded to the angle of attack that optimizes the fuel consumption; for the climb at 10^0, this is 39.73 m/s and for that at 1^0, this is 40.04. The two other speeds were taken to be 10 m/s lower, and 10 m/s higher, than those speeds.

For the climb at 10^0:

Initial Speed = 30.00 m/s	**Final Speed** = 51.25 m/s	**Av. Speed** = 38.30 m/s
Duration = 1,503.44 s = 25.06 min	**Fuel left** = 1,195.12 N	**Max. Power P_R** = 140,673.4 W

Initial Speed = 39.73 m/s	**Final Speed** = 67.89 m/s	**Av. Speed** = 50.73 m/s
Duration = 1,135.00 s = 18.92 min	**Fuel left** = 1,201.83 N	**Max. Power P_R** = 177,999.1 W

Initial Speed = 50.00 m/s	**Final Speed** = 85.42 m/s	**Av. Speed** = 63.84 m/s
Duration = 902.02 s = 15.03 min	**Fuel left** = 1,197.38 N	**Max. Power P_R** = 230,936.2 W

For the climb at 1^0:

Initial Speed = 30.00 m/s	**Final Speed** = 50.20 m/s	**Av. Speed** = 37.94 m/s
Duration = 10,389.69 s = 4.19 h.	**Fuel left** = 763.23 N	**Max. Power P_R** = 53,094.4 W

Initial Speed = 40.04 m/s	**Final Speed** = 67.21 m/s	**Av. Speed** = 50.71 m/s
Duration = 11,298.49s = 3.14 h.	**Fuel left** = 832.03 N	**Max. Power P_R** = 62,873.1W

Initial Speed = 50.00 m/s	**Final Speed** = 83.78 m/s	**Av. Speed** = 63.28 m/s
Duration = 9,054.88 s = 2.52 h.	**Fuel left** = 791.61 N	**Max. Power P_R** = 84,395.9W

5. CLIMBING AT CONSTANT SPEED

Let us consider an ascending flight, at a constant ascending angle of θ, during which the speed of the airplane is kept constant at V_∞. The rate of climb is then constant:

$$h' = v_3 = V_\infty \sin(\theta). \tag{5.1}$$

so that the altitude and the temperature are simple linear functions of time with

$$h(t) = h_0 + v_3\, t \qquad\qquad T(t) = T_0 - a_1 v_3\, t. \tag{5.2}$$

Eq.(1.3) is now interpreted as specifying C_L as:

$$C_L = \frac{2W\cos(\theta)}{\rho_\infty V_\infty^2}. \tag{5.3}$$

When this value is substituted into Eq.(2.16), in which D is expanded, one obtains for W the Riccatti equation:

$$\frac{dW}{dt} = -\left[\alpha T^{4.2433} + \beta W + \delta T^{-4.2433} W^2\right] \quad \text{with} \tag{5.4}$$

$$\alpha = \frac{c\, g\, \rho_0\, S\, C_{D0} V_\infty^3}{2T_0^{4.2433}\, G} \qquad \beta = \frac{c\, g\, v_3}{G}$$

$$\delta = \frac{2\, c\, g \cos^2(\theta) T_0^{4.2433}}{\pi\, e\, AR\, \rho_0 S\, V_\infty G} \qquad G = \left[\eta g - c(AFR)V_\infty^2\right]$$

being constant. In this case, we don't need to make any approximation in order to make the equation soluble; it is already so as such. It is a Riccati equation that we transform into a second order linear differential equation by the following change of variable from W to u(t):

$$\frac{u'}{u} = \delta T^{-4.2433}\, W \tag{5.5}$$

Eq.(5.4) then becomes:

$$u'' + \left[\beta - \frac{4.2433\, a_1\, v_3}{T}\right] u' + \alpha\delta\, u = 0 \tag{5.6}.$$

The change of independent variable from t to z:

$$z = \frac{\beta}{a_1 v_3} T(t) \tag{5.7}$$

transforms Eq.(5.6) into

$$\frac{d^2u}{dz^2} + \left[\frac{b}{z} - 1\right]\frac{du}{dz} + Au = 0 \tag{5.7}$$

with $\quad b = 4.2433, \quad$ and $\quad A = \dfrac{\alpha\delta}{\beta^2} \quad$ being constant.

This is a confluent hypergeometric equation that can readily be solved to yield:

$$u(z) = \exp\left\{\left[1 - \sqrt{1 - 4A}\right]\frac{z}{2}\right\} \left[c_1 y_1(z) + c_2 y_2(z)\right]. \qquad (5.8)$$

Its general solution will have a different expression according to the particular value of A. The two cases to be considered are as follows.

Case 1: A = 1/4,

$$y_1(z) = z^{(1-b)/2} \; J_{b-1}\left(\sqrt{2\,b\,z}\right) \qquad (5.9)$$

$$y_2(z) = z^{(1-b)/2} \; Y_{b-1}\left(\sqrt{2\,b\,z}\right) \qquad (5.10)$$

where J_{b-1} and Y_{b-1} are Bessel functions. Their derivatives are:

$$y_1'(z) = -\sqrt{\frac{b}{2}}\, z^{-b/2} \; J_b\left(\sqrt{2\,b\,z}\right) \qquad (5.11)$$

$$y_2'(z) = -\sqrt{\frac{b}{2}}\, z^{-b/2} \; Y_b\left(\sqrt{2\,b\,z}\right) \qquad (5.12)$$

Case 2: A ≠ 1/4,

$$y_1(z) = {}_1F_1\left(k_1,\, b,\, \sqrt{1 - 4A}\; z\right) \qquad \text{with} \qquad k_1 = \frac{b}{2}\left[1 - \frac{1}{\sqrt{1 - 4A}}\right] \qquad (5.13)$$

$$y_2(z) = z^{1-b} \; {}_1F_1\left(k_1 + 1 - b,\, 2 - b,\, \sqrt{1 - 4A}\; z\right) \qquad (5.14)$$

The derivatives of these functions are

$$y_1'(z) = \sqrt{1 - 4A}\left[\frac{k_1}{b}\right] {}_1F_1\left(k_1 + 1,\, b + 1,\, \sqrt{1 - 4A}\; z\right) \qquad (5.15)$$

$$y_2'(z) = \frac{(1 - b)\; y_2(z)}{z} + \sqrt{1 - 4A}\;\frac{(k_1 + 1 - b)}{(2 - b)}\, z^{1-b} \; {}_1F_1\left(k_1 + 2 - b,\, 3 - b,\, \sqrt{1 - 4A}\; z\right) \qquad (5.16)$$

Upon substituting the expression for u(z) in Eq. (5.5), on obtains:

$$W(t) = -\frac{\beta\, T^b}{\delta}\left[\frac{\left(1 - \sqrt{1 - 4A}\right)}{2} + \frac{y_1'(z) + \Gamma\; y_2'(z)}{y_1(z) + \Gamma\; y_2(z)}\right] \qquad (5.17)$$

in which
$$\Gamma = -\frac{By_1(z_0) + y_1'(z_0)}{By_2(z_0) + y_2'(z_0)} \tag{5.18}$$

$$B = \frac{\delta W_0}{\beta T_0^b} + \frac{1}{2} - \frac{\sqrt{1-4A}}{2} \tag{5.19}$$

The graph of Figure 12 shows the fuel consumption for the CP-1 plane that starts with a full tank of gas at sea level and climbs up to 10,000 m, at an angle of 10^0, at the speed of 50.73 m/s, which is the average speed for the similar climbing flight done at constant angle of attack. Figure 13 shows a similar climb at 1^0, at the constant speed of 50.71 m/s, which is the average speed of the corresponding flight done at a constant angle of attack, described in Section 4.

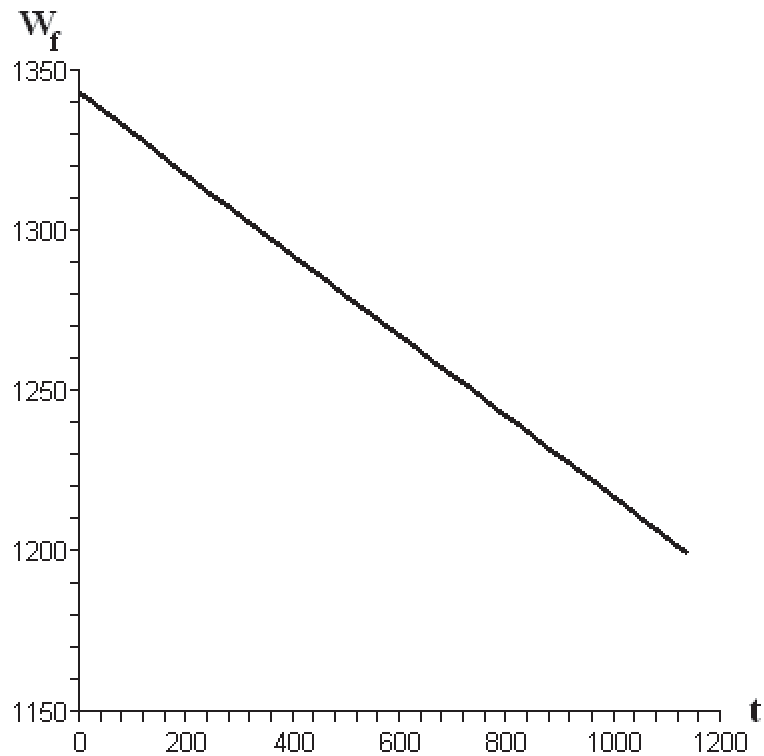

Figure 12: Fuel consumption as a function of time, in a climb of the CP-1 airplane at 10^0, from sea level to 10 km of altitude, with constant speed of 50.73 m/s.

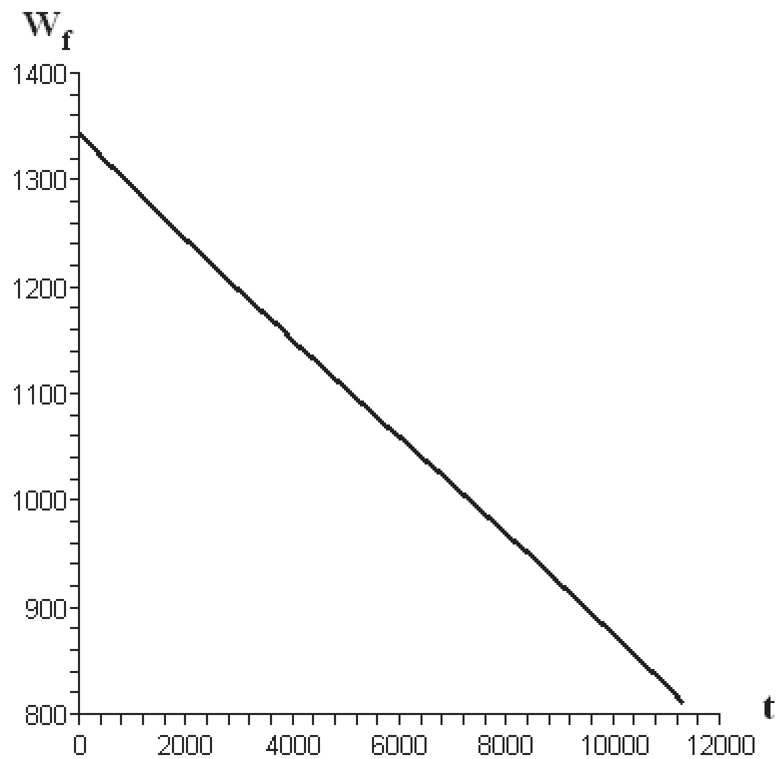

Figure 13: Weight of fuel as a function of time, in a climb at 1^0, from sea level to 10 km, at the constant speed of 50.71 m/s.

The power required for the flight at constant speed can be gotten from Eqs. (1.1) and (5.4). Figure 14 shows the graph of P_R as a function of the altitude h for the flight at 10^0. P_R is maximum at the lowest altitude, its value at that point being 140,316.3 Watts, and its value at the highest point being 140,123.7 Watts. We note that for some other constant speeds, such as at 63.84 m/s, the maximum power required is at the beginning instead of the end of the flight.

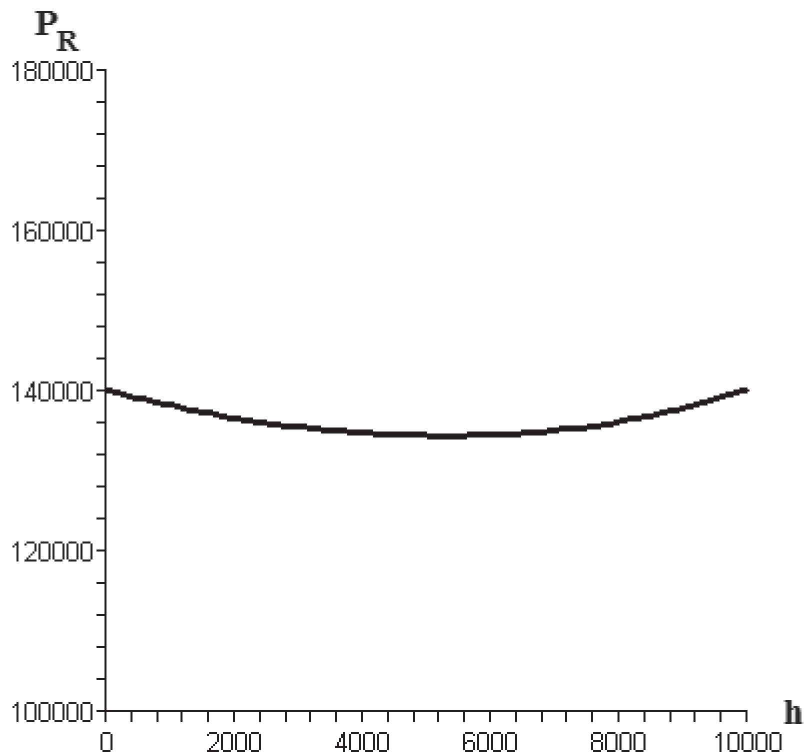

Figure 14: Power required P_R as a function of the altitude h, in a climbing flight at 10^0, with constant speed, for the CP-1 airplane.

5.1 Sample flights

We calculated the performances of the CP-1 airplane in sample climbing flights at the two different angles of 10^0 and 1^0, in which it starts with a full tank of gas and climbs from 0 to 10,000 m with a constant speed. For each angle, we considered three speeds, which were the average speeds for the flight at constant angle of attack, described in Section 4.3.

For the climb at 10^0:

Speed = 38.30 m/s		
Duration = 1,503.60 s = 25.06 min	**Fuel left** = 1,191.86 N	**Max. Power P_R** = 122,083.2 W

Speed = 50.73 m/s		
Duration = 1,135.00 s = 18.92 min.	**Fuel left** = 1,199.30 N	**Max. Power P_R** = 140,316.3 W

Speed = 63.84 m/s		
Duration = 902.06 s	**Fuel left** = 1,194.81 N	**Max. Power P_R** = 194,865.5 W

= 15.03 min		

For the climb at 1^0:

Initial Speed = 37.94 m/s		
Duration = 15,102.45 s = 4.20 h.	**Fuel left** = 735.19 N	**Max. Power P_R** = 55,587.0 W

Initial Speed = 50.71 m/s		
Duration = 11,299.29 s = 3.14 h.	**Fuel left** = 811.94 N	**Max. Power P_R** = 54,565.0 W

Initial Speed = 63.28 m/s		
Duration = 9,054.79 s = 2.52 h.	**Fuel left** = 772.55 N	**Max. Power P_R** = 85,162.9 W

6. CLIMBING AT CONSTANT MACH NUMBER

Let us finally consider an ascending flight, at a constant angle θ, during which the Mach number M is kept constant. Since the speed of sound varies as the square root of the temperature of the air, the speed of the aircraft will vary as

$$V_\infty(h) = k\,T(h)^{1/2} \qquad \text{with} \quad k = M\sqrt{\gamma R} \text{ being a constant.} \qquad (6.1).$$

6.1 Altitude

The rate of change of the altitude is

$$\frac{dh}{dt} = V_\infty \sin(\theta) = k \sin(\theta)\sqrt{T_0 - a_1(h - h_0)}\,. \qquad (6.2)$$

This is a separable equation that is readily solved for h to yield

$$h(t) = h_0 + V_\infty(h_0)\sin(\theta)\,t - \frac{k^2 a_1}{4}\sin^2(\theta)\,t^2\,. \qquad (6.3)$$

For the CP-1 airplane that starts with the speed of 30.42 m/s at sea level and climbs at an angle of 10^0, the altitude varies as the parabola shown in Figure 15.

Note that Eqs (6.1) and (6.2) imply that

$$\frac{dV_\infty}{dt} = -\frac{k\,a_1}{2}T(h)^{-1/2}\left[\frac{dh}{dt}\right] = -\frac{1}{2}k^2\,a_1\,\sin(\theta) = \text{a constant.} \qquad (6.4)$$

Thus, the speed increases at a constant rate, so that it is a linear function of time.

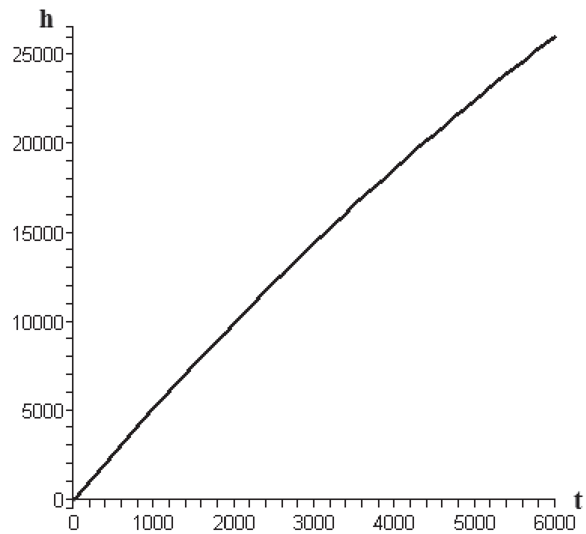

Figure 15: Altitude as a function of time for a climb at 10^0 with a constant Mach number of 0.09.

6.2 Fuel consumption

Upon substituting Eq.(6.4) in Eq.(2.16), one obtains

$$\left[1-\frac{c\,AFR\,V^2}{\eta\,g}\right]\left[\frac{dW}{dt}\right] = -\alpha\,T^{5.7433} - \beta\,T^{0.5}\,W - \delta\,T^{-4.7433}\,W^2 \quad \text{with} \qquad (6.5)$$

$$\alpha = \frac{c\,S\,C_{D0}\,\rho_\infty(0)\,k^3}{2\,\eta\,T_0^{4.2433}} \qquad \beta = \frac{c\,k\,\sin(\theta)}{\eta}\left[1+\frac{k^2 a_1}{2\,g}\right] \qquad \delta = \frac{2\,c\,\cos^2(\theta)\,T_0^{4.2433}}{\eta\,k\,\pi\,e\,AR\,S\,\rho_\infty(0)}$$

being three constants. Eq.(1.6) yields $V^2 = \dfrac{2\,W\cos(\theta)}{\rho_\infty S\,C_L}$ and upon substituting this value in the second term of the factor in front of dW/dt in Eq.(6.5), one sees that this term is essentially the same one that was shown to be negligible in Eq.(3.3). We shall then neglect it also here. The resulting equation is again a Riccati equation. Again, we transform it into a second order linear differential equation by the change of variable from W to u(t):

$$\frac{u'}{u} = \delta T^{-4.7433} W \tag{6.6}$$

Eq.(6.5) then becomes the following linear differential equation for u:

$$u'' + \left[\beta T^{1/2} - \frac{4.7433\, k\, a_1 \sin(\theta)}{T^{1/2}} \right] u' + \alpha \delta T u = 0\,. \tag{6.7}$$

The change of variable from t to z:

$$z = \frac{\beta}{a_1\, k \sin(\theta)} T[h(t)] \tag{6.8}$$

transforms Eq.(6.7) into exactly the same confluent hypergeometric equation as Eq. (5.26), with the constants:

$$b = 5.2433 \qquad \text{and} \qquad A = \frac{\alpha\delta}{\beta^2}\,. \tag{6.9}$$

Its solutions are the same ones as those described in Eqs (5.8) to (5.16) and the formula for the weight of the airplane is the same one as given in Eqs.(5.17) to (5.19).

The graph of Figure 16 shows the fuel consumption for the CP-1 plane that starts with a full tank of gas at sea level and climbs up to 10,000 m, at an angle of 10^0, while keeping a constant Mach number. Its initial speed is 50.73 m/s, which is the average speed for the similar climbing flight done at constant angle of attack described in Section 4; this corresponds to a Mach number of 0.1491. Figure 17 shows a similar climb at 1^0, at the initial speed of 50.71 m/s, which is the average speed of the corresponding flight done at a constant angle of attack. This corresponds to a Mach number of 0.1490.

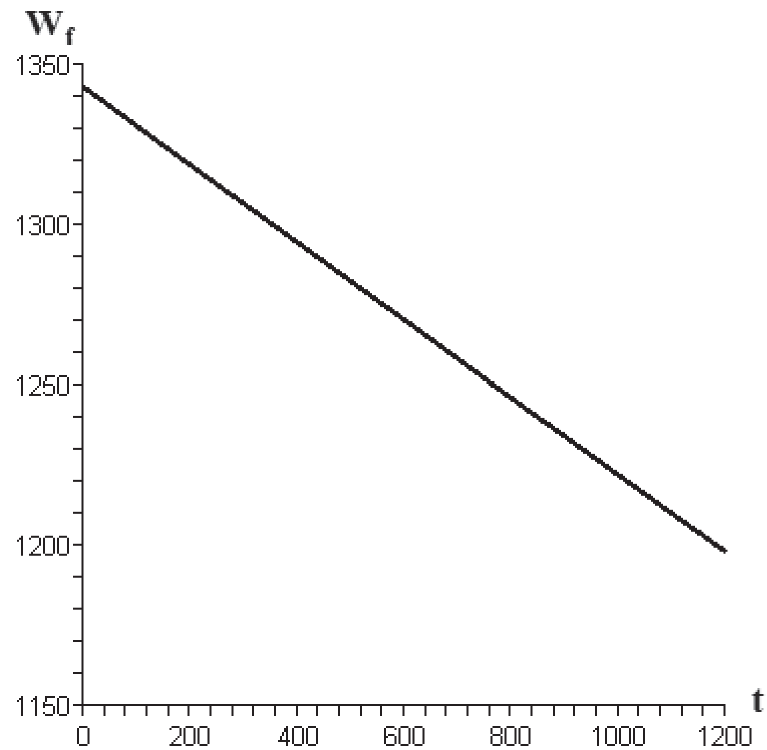

Figure 16: Fuel consumption as a function of time, in a climb of the CP-1 airplane at 10^0, from sea level to 10 km of altitude, with constant Mach number of 0.1491.

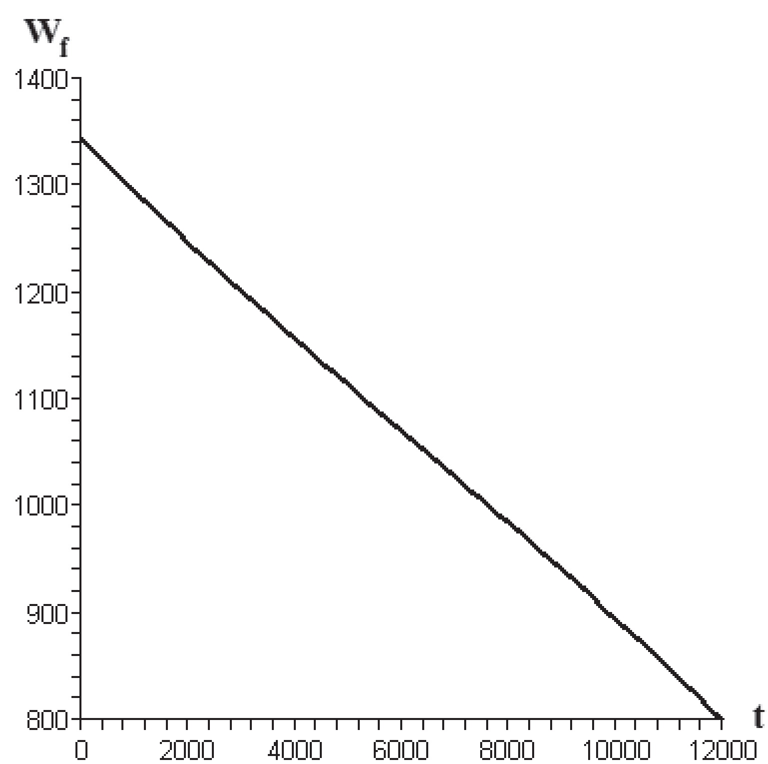

Figure 17: Weight of fuel as a function of time, in a climb at 1^0, from sea level to 10 km, at the constant speed of 50.71 m/s.

The power required for the flight at constant Mach number can be gotten from Eqs. (1.1) and (6.5). With the approximation made, this yields:

$$P_R = \frac{\eta}{c}\left[\alpha\, T^{5.7433} + \beta\, T^{0.5}\, W + \delta\, T^{-4.7433}\, W^2\right] \tag{6.10}$$

Figure 18 shows the graph of P_R as a function of the altitude h in the flight at 10^0 described above. P_R is maximum at the lowest altitude, its value at that point being 139,907.4 Watts.

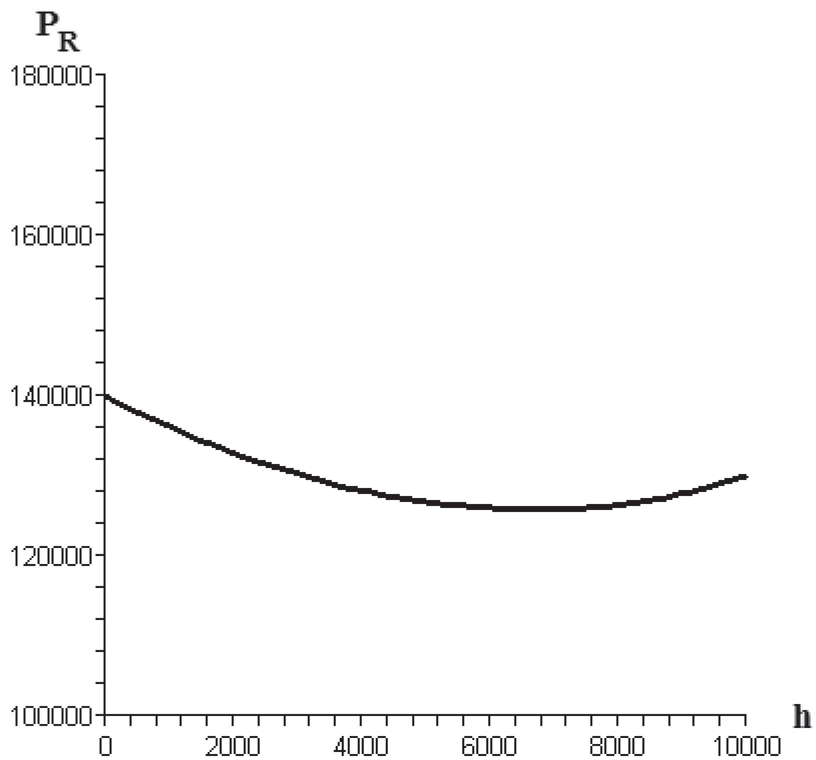

Figure 18: Power required P_R as a function of the altitude h, in a climbing flight at 10^0, with constant Mach number, for the CP-1 airplane.

6.3 Sample flights

Again, we calculated the performances of the CP-1 airplane in the same sample climbing flights as considered before, at the two different angles of 10^0 and 1^0, in which it starts with a full tank of gas and climbs from 0 to 10,000 m with a constant Mach number. For each angle, we considered three initial speeds, which were the average speeds for the flight at constant angle of attack, described in Section 4.3.

For the climb at 10^0:

Mach Num. = 0.1125		
Initial Speed = 38.30 m/s	**Final Speed** = 33.70 m/s	**Av. Speed** = 36.00 m/s
Duration = 1,599.55 s = 26.66 min.	**Fuel left** = 1,186.94 N	**Max. Power P_R** = 101,948.4 W

Mach Num. = 0.1491		
Initial Speed = 50.74 m/s	**Final Speed** = 44.65 m/s	**Av. Speed** = 47.69 m/s
Duration =1,207.43 s = 20.12 min.	**Fuel left** = 1,197.96 N	**Max. Power P_R** = 139,907.4 W

Mach Num. = 0.1876		
Initial Speed = 63.84 m/s	**Final Speed** = 56.18 m/s	**Av. Speed** = 60.01 m/s
Duration = 959.63 s = 15.99 min.	**Fuel left** = 1,196.08 N	**Max. Power P_R** = 194,328.9 W

For the climb at 1^0:

Mach Num. = 0.1115		
Initial Speed = 37.94 m/s	**Final Speed** = 33.39 m/s	**Av. Speed** = 35.66 m/s
Duration = 16,066.28 s = 4.46 h.	**Fuel left** = 687.60 N	**Max. Power P_R** = 37,440.7 W

Mach Num. = 0.1490		
Initial Speed = 50.71 m/s	**Final Speed** = 44.63 m/s	**Av. Speed** = 47.67 m/s
Duration = 12,020.41 s = 3.34 h.	**Fuel left** = 798.85 N	**Max. Power P_R** = 54,402.8 W

Mach Num. = 0.1860		
Initial Speed = 63.28 m/s	**Final Speed** = 55.69 m/s	**Av. Speed** = 59.48 m/s
Duration = 9,632.66 s = 2.68 h.	**Fuel left** = 783.62 N	**Max. Power P_R** = 84,752.4 W

7. DESCENDING FLIGHT

The formulas we have obtained for climbing flights are also valid for descending flights, for which the angle of climb θ is negative, but only when the airplane flies at an angle larger than the glide angle. When its angle of descend is equal or smaller than the minimum glide angle, then it can descend without expanding any power.

The minimum glide angle is the angle at which the power required for the flight can be set null by appropriately choosing the flight parameters. The explicit formulas we found in the preceding sections for the power required in three different modes of flight can be used to calculate this angle. Note that when using these formulas for descending flights, care must be taken to prevent the expression for the power to become negative since they obviously become invalid in that range of values. Upon setting the value of P_R to zero, one obtains the minimum glide angles θ_{min} as the smallest angle at which the following equations can hold.
For the motion at constant angle of attack:

$$\tan(\theta) = \frac{C_D}{C_L}\left[1 + \frac{4.2433\, a_1\, V^2}{2\, g\, T}\right]^{-1} \qquad (7.1)$$

For the motion at constant speed:

$$\tan(\theta) = \frac{C_D}{C_L}. \qquad (7.2)$$

For the motion at constant Mach number:

$$\tan(\theta) = \frac{C_D}{C_L}\left[1 + \frac{k^2\, a_1}{2\, g}\right]^{-1} \qquad (7.3)$$

We note that the dive angle θ is made larger when the lift coefficient C_L is decreased. Thus, the airplane can glide at any steeper angle than the minimum glide angle, provided the angle of attack is adjusted appropriately.

8. CONCLUSION

The derivation, we have presented, of the equations for the fuel consumption of propeller driven planes, which takes into account the conservation of momentum, has a definite pedagogical value. It can also be used, as we showed, to justify the neglect of some terms that have a minimal effect on the overall results.

We believe that the formulas we have provided for the dynamical parameters of airplanes in certain modes of flight constitute an important contribution to the analysis of airplane performances. These formulas are applicable to all trajectories with a constant angle of climb, whatever their form; rectilinear, helical or other. The modes of motion we considered are:

- flight with at a constant angle of attack,
- flight with at a constant speed,
- flight with at a constant Mach number.

Although we were not able to provide formulas for the values of the parameters that produce the optimal performances in all of these flight modes, the formulas we obtained allow for an easy determination of these parameters. In particular, they can be used for the construction of graphs from which the optimal parameters can be evaluated. The examples of calculations that we provide, for flights at three particular speeds, clearly indicate that in all cases, less power will be required at the lowest speed, less fuel is required at the middle speed, and a faster flight is obtained at the highest speed at the cost of a relatively small increase in fuel use. Thus, our formulas can easily be used to tune the flight parameters according to the needs of the mission. It is very important to be able to do this process because, as we showed, a small change in flight parameters can result in a large difference in airplane performances. For example, a small change in the speed can result in a large difference in the endurance and the range, as the graphs of Figure 4 show.

We believe that the formulas we provide for the power required P_R are also of great value. Figure 20 shows P_R as a function of altitude, for the three modes of motion, superposed. From this graph, one can see that in order to save fuel, the airplane should start its climb at a constant angle of attack, and remain in this mode up to about 4,000 m. At this altitude, it should then change to the constant Mach number mode and continue its ascension in this mode.

The derivation of the corresponding formulas for jet powered airplanes should now be straightforward.

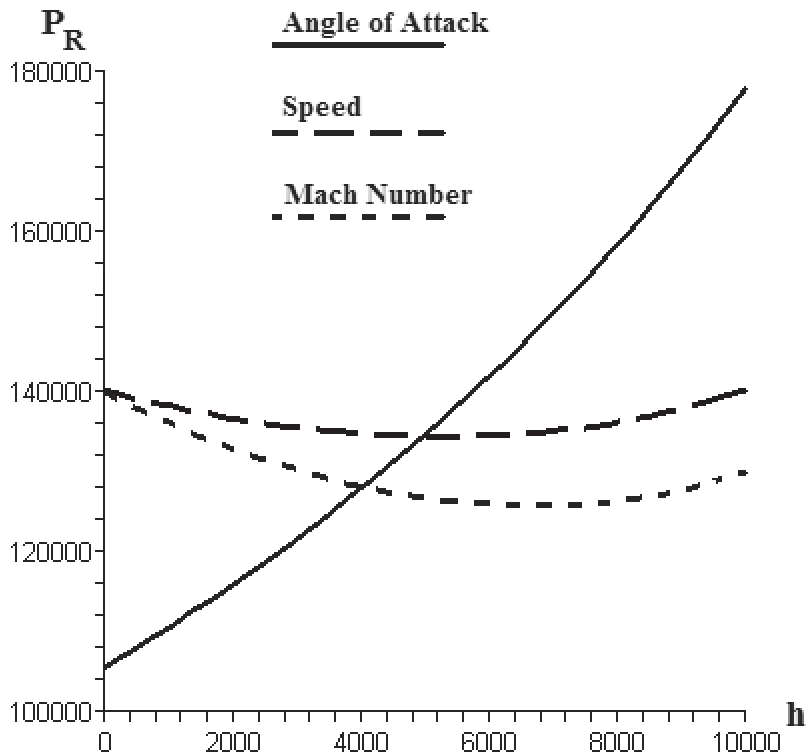

Figure 20: Curves of the power required by the CP-1 airplane, in climbing at 10^0 from 0 to 10,000 m. The curves for the three modes of motion studied are superposed: that for the climb at constant angle of attack, at constant speed, and at constant Mach number.

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We revise the derivation of the Br□guet-Coffin endurance and range formulas in the context of Newton□s Second Law of Motion. We believe that the formulas we derived constitute valuable tools for analyzing aircraft performance and for analysing aircraft trajectory planning.

We point out that, in principle, the momentum of the mass of air required to burn the fuel that is taken in at rest and ejected at the airplane speed, should be taken into account since this mass is about 14.7 times that of the fuel burned. We give the exact solutions to the equations for the fuel consumption, obtained with this consideration, when the aircraft is in level flight. We consider two modes of flight: one with a constant angle of attack (as in the Br□guet-Coffin case), and one at constant speed. Comparison of the endurances and the ranges obtained with and without the correction terms show that some of the added terms can actually be neglected.

We then solve exactly the fuel consumption equations in which these terms are neglected, for climbing airplanes in three different modes of motion □flight at a constant angle of attack, flights at a constant speed, and flights at a constant Mach number. With the help of these solutions, we derive formulas for the speed, the altitude and the power required as a function of time, and as a function of the altitude. The behaviour of the solutions obtained is exhibited through many sample calculations that involve the model CP-1 airplane described in Anderson□s book □Introduction to Flight□.

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